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EMPIRICAL DISTRIBUTIONS IN MULTIPLE PRODUCTION

AND THE UNCORRELATED JET MODEL

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A B S T R A C T

A method of calculating two and more-particle distributions from the experimental one-particle distributions is presented. The method is based on the uncorrelated jet model. It takes into account energy-momentum conservation, but no other correlations are included. The method is applied to find the distribution of azimuthal angles between pairs of transverse momenta. The result is similar to the results obtained from a Monte Carlo calculation based on the statistical model.

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An interesting problem in the analysis of the many-particles production at high energies is the study of two or more-particle correlations.

The comparison of the predictions of various models with experiment could possibly give an insight into the production mechanism, and allow to recognize at least some general features of the high-energy phenomena.

As an example, many models fall into the general class of the uncorrelated jet models (UJM)¹⁾. In this class of models the probability of any momentum for one particle in the final state of a production process depends as little as possible on the momenta of the remaining particles, the only correlation being given by the over-all conservation laws, usually by energy-momentum conservation.

Consequently all these models predict no correlation between the momenta of pairs of secondaries, apart from the correlation which is produced by the conservation laws. If no UJM can fit the experimental data, we may conclude that there are non-trivial correlations, i.e., correlations which are not explained by energy-momentum conservation.

As yet, very little is known about these correlations. As we shall discuss in more details below, one example of these correlations is the effect analyzed by Goldhaber, Goldhaber, Lee and Pais (GGLP)²⁾. They found that in the reaction $p\bar{p} \rightarrow 2\pi^+ + 2\pi^-$ at $E_{\text{cms}} = 2.1$ GeV, the average angle between the momenta of a pair of pions of equal charge was smaller than the average angle between the momenta of a pair of pions of different charge.

2.

In order to test the UJM it is necessary to get a way to compute, from the experimental one-particle distributions, the two and more-particle correlations which arise solely from energy-momentum conservation.

The aim of this note is the following :

- a) to give a relation between the primary and empirical one-particle distributions, i.e., between the functions which give the probability of any n particle final state in the UJM [see formula (1)], and the experimentally measured one-particle distributions ;
- b) to compute from the so-obtained primary distributions an interesting two-particle distribution : the distribution of the angle between the transverse momenta of two secondaries.

2. DEFINITIONS

Let us denote by x_i the set of numbers defining the state of a particle. Usually, once the kind of particle is known, x_i are the momenta (p_{ix}, p_{iy}, p_{iz}) of the particles.

We further denote by $F_i(x_i)$ the empirical one-particle distribution, i.e., the one-particle distribution which is measured experimentally ^{*)}. As we have already noticed in the Introduction and shall discuss in more detail later, the knowledge of all the $F_i(x_i)$ is enough to evaluate all the one-particle averages (namely averages which refer to only one particle, as the mean energy of a particle), but is not enough to determine two or more-particle averages (two-particle average means average over a quantity which refers to two particles at the same time, as the angle between the momenta of two particles).

A simple example will clarify this statement. Suppose we have a state, containing four pions, $2\pi^+$ and $2\pi^-$, whose momenta lie all in a plane. We make the assumption that the energies of all the pions have a common value p_0 . Let us assume further that the total momentum is zero (this is the correlation included in the UJM) and that each momentum has an isotropic distribution with respect to a reference

*) Actually in experiment only the distributions for groups of identical particles are measured ; e.g., one obtains a distribution of π^+ , a distribution of π^- , and so on. Whether or not each π^+ separately has the same distribution as the group of π^+ 's does not follow from the experimental histogram. Further we assume that this is the case, which simplifies considerably the analysis (see Appendix). The authors wish to thank Dr. R. Sosnowski for a discussion about this point.

direction in the plane. These assumptions fix the empirical distributions $F_i(x_i) = F_i(\hat{p}_i)$. All the one-particle distribution for the π^+ and π^- are equal and known.

We can nevertheless obtain the same one-particle distribution by grouping the particles two by two in an infinity of different ways, all compatible with energy-momentum conservation. Two illustrative cases are shown in Fig. 1. The isotropy of the one-particle distributions is compatible with the assumption that these configurations, at fixed Θ , are randomly distributed with respect to the z axis. In the first case the mean angle between the momenta of the two like particles is small, in the second case it is large.

We can now explain why the GGLP effect is due to non-trivial correlations. In fact the momentum distributions of all the pions in the quoted experiment $p\bar{p} \rightarrow 2\pi^+ + 2\pi^-$ were found to be almost symmetrical, thus it is a good approximation to take all the $F_i(x_i)$ equal. The fact that the mean angle between the momenta of two like and between the momenta of two unlike pions were found to be different means therefore that in this case the knowledge of the $F_i(x_i)$ is insufficient to determine the two-particle distributions, namely that there are non-trivial correlations.

From all what we said, it is clear that the two and more-particle distributions cannot be obtained from the $F_i(x_i)$ without further assumptions.

Perhaps the simplest reasonable assumption yielding all the averages, and not only the one-particle averages, in terms of the $F_i(x_i)$ is the UJM. In this model, by assuming that between the secondaries there are no correlations except those given by the conservation laws, one takes into account the correlations which arise from the energy-momentum conservation and uses the knowledge of the one-particle distributions.

In the UJM, the probability of finding a configuration $x_1, \dots, x_n$ for the n particles forming the final state is by definition

$$\prod_{i=1}^n \delta(X - \sum_{k=1}^n x_k) f_i(x_i) dx_i \quad (1)$$

Here the δ function secures the conservation laws (usually it is a fourfold δ for energy-momentum conservation). The functions $f_i(x_i)$, which shall be further called primary one-particle distributions, are chosen so as to yield the correct (i.e., experimentally measured) $F_i(x_i)$

$$F_i(x_i) = f_i(x_i) \int \delta(X - \sum_{k=1}^n x_k) \prod_{j \neq i} f_j(x_j) dx_j \quad i=1, 2, \dots, n. \quad (2)$$

Some models predict the functions $f_i(x_i)$; for instance, in the covariant statistical model we have

$$f_i(x_i) = 1/2E_i$$

(E_i is the total energy of the i^{th} particle).

When $f_i(x_i)$ is given by the model, formula (2) is a prediction of the model, which can be checked with experiment by comparing with the measured one-particle distributions.

We are however concerned here with a more general problem, namely whether it is possible to invert formula (2), obtaining in this way the functions $f_i(x_i)$ from the knowledge of the functions $F_i(x_i)$. In this way one could obtain the primary distributions $f_i(x_i)$ consistent with the empirical one-particle distributions $F_i(x_i)$.

As we shall see, some ambiguities arise in solving this problem.

3.

RELATION BETWEEN THE EMPIRICAL AND PRIMARY ONE-PARTICLE
DISTRIBUTIONS

Let us note first that the primary functions $f_i(x_i)$ may be completely unsimilar to the empirical functions $F_i(x_i)$. For instance, in the non-covariant statistical model (Fermi's version)³⁾, the f_i are constants, while (for high multiplicity) the F_i are essentially $F_i \propto \exp[-\beta(p_1^2 + m_1^2)^{\frac{1}{2}}]$ with the coefficient β strongly depending on the energy.

Secondly the relation (2), considered as an equation for the unknown functions $f_1, \dots, f_n$ has no unique solution. In particular it is easy to verify that the substitution

$$f_i(x_i) \rightarrow (\alpha_i e^{-\beta x_i} f_i(x_i)) \quad (3)$$

with

$$\prod \alpha_i = \frac{e^{+\beta X}}{C}$$

and α_i, β, C constants leaves the right-hand side of (2) unchanged; therefore, if the f_i are a set of solutions of (2), so is the set of functions (3). Beside this general ambiguity, there are some more solutions outside the class (3). In order to see that, let us choose a one-dimensional case, by assuming that x is just one variable, and rewrite the system of Eqs. (2) for $n = 2$ ^{*)}. Performing the trivial integration we obtain for $X = 0$

$$\begin{aligned} F_1(x_1) &= f_1(x_1) f_2(-x_1) \\ F_2(x_2) &= f_1(-x_2) f_2(x_2) \end{aligned} \quad (4)$$

*) We are grateful to Dr. J.D. Bessis for a discussion on this point.

This gives immediately the consistency condition

$$F_1(x) = F_2(-x) \quad (5)$$

If this condition is not fulfilled there are no solutions ; if it is fulfilled, both equations (4) are identical. We may therefore choose arbitrarily $f_1(x)$ and determine $f_2(x)$ from the equation. Thus the solution depends on an arbitrary function.

We shall discuss now an iterative method, which yields a solution to the system of Eq. (2). This solution is that in which the f_i are as nearly equal to the F_i as possible. In neither the ambiguity (3) nor in (4) the choice of the f_i has any influence on the two and more-particle averages, because in both cases the change affects the distribution (1) only for $X \neq \sum x_i$, when it is zero anyhow. We have no evidence whether non-trivially different solutions, i.e., solutions yielding different averages, are possible for one given set of the F_i . In any case, the solutions we shall be considering are very natural and seem worth while to study first.

4. ITERATIVE SOLUTION OF Eq. (2)

In order to solve Eq. (2), we propose the following iterative method. The zero order approximation is

$$f_i^{(0)}(x) = F_i(x) \quad (6)$$

From the k^{th} approximation one evaluates the $k + 1^{\text{st}}$ according to

$$F_i(x) = f_i^{(k+1)}(x) \int \delta\left(X - \sum_{j=1}^n x_j\right) \prod_{l \neq i} f_l^{(k)}(x_l) dx_l \quad (7)$$

It is easy to see that, when this procedure is convergent, i.e., when all the limits

$$f_i(x) = \lim_{k \rightarrow \infty} f_i^{(k)}(x) \quad (8)$$

exist then the limits satisfy Eq. (2).

Of course by this method we obtain only one solution out of the infinity of possible ones, as we have discussed in connection with the ambiguity problem.

We shall not discuss the convergence of (8); we would like, however, to make two remarks, showing that the method is particularly adapted to the study of high multiplicities.

In order to simplify the notation, we shall choose a one-dimensional case, in which the functions depend only on one variable x ; there is therefore only one δ function. Moreover we shall assume that all the functions $F_i(x)$ [and therefore also $f_i(x)$] are equal. The extension of this analysis to the general case is trivial.

Our first remark is that if the first three momenta of $F_i(x)$ remain bounded for $n \rightarrow \infty$ and if the constant β , defined in Eq. (3), is chosen so that the sum of the mean values of x coincides with its total value X

$$\sum_i \frac{\int x f_i(x) dx}{\int f_i(x) dx} = X \quad (9)$$

then for sufficiently large n and $|x - \tilde{x}_i| \ll ns$

$$f_i(x) = F_i(x) \exp \left[(x - \tilde{x}_i)^2 / s(n-1) \right] \quad (10)$$

where *)

$$\tilde{x}_i = \frac{\int dx x f_i(x)}{\int dx f_i(x)} \quad s = \frac{\int dx (x - \tilde{x}_i)^2 f_i(x)}{\int dx f_i(x)} \quad (11)$$

For values of x so large that the exponential is significantly larger than unity, the approximation (10) for the correction factor $\exp[(\tilde{x}_i - x)^2 / s(n-1)]$ is not valid, but there $F_i(x)$ is practically zero. Formula (10) shows that for $n \rightarrow \infty$ the zero order approximation is exact, and therefore that the higher order approximations are corrections for the finite number of particles.

*) The notation $\tilde{x}_i$ is used to stress the difference with the average $\bar{x}_i$, corresponding to the distribution (1).

Formula (10) follows easily from formula (2). It is enough to notice that the integral is the distribution function for a sum of $n - 1$ independent random variables, each with the distribution $f(x)$, and to apply the central limit theorem.

Our second remark is that when $f_i(x)$ is represented as (10) multiplied by a power series in $(\frac{1}{n-1})^{m/2}$, $m = 0, 1, \dots$, then for a large class of functions $F_i(x)$, the k^{th} iteration yields correctly $f_i(x)$ up to and including the terms of the order $(\frac{1}{n-1})^{k+\frac{1}{2}}$. The higher order terms are given, but their values are unreliable. The proof is easily obtained by applying the saddle point method to evaluate the integral.

We shall illustrate the situation by an explicit example in Section VI.

5. A MONTE CARLO METHOD

As we have already said, from (1) and (2) it follows that the one-particle distributions can be directly evaluated from the empirical distributions $F_i(x)$. For an arbitrary function of the parameters of one particle, $g(x)$, we have

$$\bar{g}(x) = \int F_i(x) g(x) dx \quad (12)$$

In order to compute the two and more-particle averages, however, one needs the primary functions $f_i(x)$, and, in general, a complicated integration.

This integration can be replaced, for practical purposes, by an average over Monte Carlo events; this replacement is, however, legitimate only if the events are composed of particles chosen at random from the primary distribution $f_i(x)$.

A method of generating such events has been proposed by Bardadin et al.⁴⁾; for sake of completeness, we shall describe here the method.

The first step consists in selecting at random from the experimental distributions $n-1$ particles. This corresponds to the zero order approximation (6) for the first $n-1$ particles.

$$f_i(x) = \bar{F}_i^{\text{exp}}(x) \quad i=1,2,\dots,n-1. \quad (13)$$

[We have introduced the superscript "exp" in order to distinguish the actual measured experimental distributions from the approximations for $F_i(x)$ obtained by inserting in (2) approximate values of $f_i(x)$].

Then, the n^{th} particle was added with the condition

$$x_n = X - \sum_{i=1}^{n-1} x_i$$

From the resulting events again a selection was made so that the value of $F_n(x)$ coincides with the experimental distribution

$$F_n(x) = F_n^{\text{exp}}(x) \quad (14)$$

Comparing with (7) we can see that this result corresponds to the first approximation for f_n , $f_n = f_n^{(1)}$. Thus this first step is already a little better than the simple zero iteration (6).

As a second step, from the events obtained in step one, the distributions $F_i^{(1)}(x)$, $i = 1, 2, \dots, n-1$, have been found. Since the final selection deforms the sample, the results are not exactly $F_i^{\text{exp}}(x)$, but some functions $F_i^{(1)}(x) = \varphi_i^{(1)}(x) \cdot F_i^{\text{exp}}(x)$. Then the calculation is repeated as in step (1) but taking

$$f_i(x) = \frac{F_i^{\text{exp}}(x)}{\varphi_i^{(1)}(x)} \quad (15)$$

The result is a little better than the first iteration, since it corresponds to the first iteration for the first $n-1$ particles but to the second iteration for the n^{th} particle.

This procedure should be continued until self-consistency is obtained, i.e., until the next step gives $\varphi_i^{(k+1)}(x) = 1$ for $i = 1, \dots, n-1$. The values obtained in this way for $f_i(x)$ are then a good representation of the true $f_i(x)$. In the case considered by Bardadin et al. (nucleon + 6 pions in the final state at $E_{\text{cms}} = 4.4$ GeV) it was shown that two steps were enough. The convergence of this procedure may be improved by using (10) as a starting distribution instead of (13).

6. THE GAUSSIAN MODEL *)

In order to illustrate the preceding discussions, and to derive the formulae which will be useful for the discussion of the correlation between transverse momenta, we shall discuss a simple model. We again assume a one-dimensional model, with only one x variable and only one δ function. Moreover we choose an especially simple form for $F_i(x)$ **)

$$F_i(x) = e^{-\alpha x^2} \quad (16)$$

In this simple case the system of equations (2) can be solved exactly for $X = 0$.

For $X \neq 0$ the system (2) has no solutions. This is most easily seen multiplying both sides of (2) by x_i , integrating over x_i [the left-hand side gives zero, because of the particular form (16)], and then summing over i . The result is

$$X \int \delta\left(X - \sum_{i=1}^n x_i\right) \prod f_j(x_j) dx_j = 0 \quad (17)$$

*) A one-dimensional Gaussian model has been used by Dr. L. Michejda (private communication) in order to test the convergence of the iteration scheme in the Monte Carlo method described above.

**) The normalization factors are systematically omitted.

Thus the equation system (2) with a Gaussian form for $F_i(x)$ is consistent only for $X = 0$.

We begin with the iteration method. The integral necessary for the first iteration is

$$\int \delta\left(\sum_{i=1}^n x_i\right) \exp\left[-a \sum_{j \neq i} x_j^2\right] dx_j^{n-1} \quad (18)$$

Replacing the δ function by its Fourier transform and inverting the order of integrations we have

$$(2\pi)^{-1} \int_{-\infty}^{+\infty} dt \left(\int_{-\infty}^{+\infty} dx e^{-a x^2 - i t x} \right)^{n-1} e^{-i t x_i} \quad (19)$$

performing the Gaussian integration we have finally

$$\exp\left[-a \cancel{x_i^2} / \cancel{n-1}\right] \quad (20)$$

Substituting into (7) we obtain

$$f_i^{(1)}(x) = \exp\left[-a_1 x_i^2\right] \quad \text{with} \quad a_1 = a \frac{n-2}{n-1} \quad (21)$$

Repeating this procedure we obtain in all approximations Gaussian distributions with the recurrence formula for the coefficients

$$a_k = a - (n-1)^{-1} \cdot a_{k-1} \quad (22)$$

For $k \rightarrow \infty$

$$f_i(x) = \exp\left[-a x^2 \frac{n-1}{n}\right] \quad (23)$$

The same result can be obtained directly by substituting into (2) the form $f_i(x) = \exp(-\alpha x^2)$ and solving for α .

It is seen that the successive iterations build up an expansion of the exponent into powers of $(n-1)^{-1}$. $f_i(x)$ in each approximation is represented by an infinite series in inverse powers of $n-1$, but for the k^{th} iteration only the k first terms are reliable. This result is in agreement with the general remark made in Section 4.

The terms with odd powers of $(n-1)^{-\frac{1}{2}}$ are absent in this example because the $O(t^3)$ term in the saddle point expansion of integral (19) vanishes. It is also easy to recognize in (21) the approximation (10) with $s = a^{-1}$.

The one-particle empirical distribution obtained by integrating formula (1) over $n-1$ variables is by assumption equal to $F(x)$, but the two-particle distribution obtained by integrating over $n-2$ variables is

$$P(x_1, x_2) = F(x_1)F(x_2) e^{\frac{a}{n}(x_1^2 + x_2^2) - \frac{a(n-1)}{n(n-2)}(x_1 + x_2)^2} \quad (24)$$

and is different from the product of the two one-particle distributions. Physically this means, as we have already discussed, that the two particles are correlated by the x conservation. Of course, the correcting factor tends to unity for $n \rightarrow \infty$.

A third method of obtaining the same result (23) is to use the sequence of approximation described in Section 5.

7. CORRELATIONS BETWEEN TRANSVERSE MOMENTA RESULTING
FROM ENERGY-MOMENTUM CONSERVATION

We shall now apply the previous consideration to the concrete physical problem of the correlation between the transverse momenta of the secondaries produced in a high-energy collision. Our aim is to estimate the correlation between two transverse momenta, which arises from the conservation laws.

In the thermodynamic (i.e., many particle) limit of the statistical model the transverse momenta ^{*}) (denoted further by $\hat{p}_T$) are uncorrelated, in the sense that

$$\overline{\hat{p}_{T_i} \cdot \hat{p}_{T_k}} = \overline{\cos \varphi_{ik}} = 0 \quad (25)$$

For a finite number of particles, because of momentum conservation, the transverse momenta of the secondaries have to compensate each other (the total transverse momentum is zero by assumption) and consequently the averages (25) tend to be negative. In particular for the trivial case of a two-particle final state $\overline{\cos \varphi_{12}} = -1$.

We shall study the two-particle transverse momentum distribution making the following two assumptions :

- 1) only the conservation of the total transverse momentum is relevant ;
- 2) the distribution of transverse momenta is Gaussian.

^{*}) By transverse momentum we mean, as usual, the two-dimensional vector (p_x, p_y) where z is along the beam direction.

Assumption 1) implies neglecting the correlations due to energy conservation and to the possible correlations between longitudinal and transverse momenta. This seems reasonable, since in actual cases the energy of a particle is determined mainly by its longitudinal momentum, at least when the mean energy per particle is sufficiently high. Moreover one might expect that energy conservation affects the moduli of the transverse momenta rather than their mutual orientation.

The second assumption is also reasonable. The Gaussian is perhaps not the best fit for the transverse momentum distributions, but it approximates them rather well, and it is often used as an empirical formula for the transverse momentum distribution. Moreover, as we have seen in Section 6, this is a particularly simple form, which makes it possible to compute the primary distribution in a rather simple way.

We have to make here a remark : the formula worked out in Section 6 was true only in the case in which the total transverse momentum is zero, $\hat{p}_T = \sum_{i=1}^n p_{Ti} = 0$. In general, it would be useful to work out a similar formula for the case in which one is considering only a subset of the final particles, and therefore the relevant $\hat{p}_T$ is different from zero (a typical case is the reaction $\pi^+ p \rightarrow p + n \pi^-$ in which one considers only the final pions).

In this general case one has to take for the empirical distribution a shifted Gaussian, viz.,

$$F_i(\hat{p}_T) = \exp \left[-a \hat{p}_T^2 - \hat{b} \cdot \hat{p}_T \right] \quad (26)$$

The equation is, in our two-dimensional case,

$$F_i(\hat{p}_{Ti}) = \int \delta^{(2)} \left(\hat{p}_T - \sum_{j \neq i}^n \hat{p}_{Tj} \right) f_i(\hat{p}_{Ti}) \prod_{k \neq i} f_k(\hat{p}_{Tk}) d^2 \hat{p}_{Tk} \quad (27)$$

A trivial modification of the calculation of Section 6 gives for the primary distribution

$$f_1(\hat{p}_T) = e^{-a \hat{p}_T^2 \frac{n-1}{n}} \quad (28)$$

with the consistency condition

$$|\hat{b}| = 2a \frac{n-1}{n} |\hat{p}_T| \quad (29)$$

The vectors $\hat{b}$ and $\hat{p}_T$ have moreover to be antiparallel. The two-particle distribution is

$$P(\hat{p}_{T1}, \hat{p}_{T2}) = e^{-\alpha(\hat{p}_{T1}^2 + \hat{p}_{T2}^2)} \exp \left[-\frac{\alpha}{n-2} (\hat{p}_{T1}^2 + \hat{p}_{T2}^2 + 2\hat{p}_{T1} \cdot \hat{p}_{T2}) + \frac{2\alpha}{n-1} \hat{p}_T \cdot (\hat{p}_{T1} + \hat{p}_{T2}) \right] \quad (30)$$

where

$$\alpha = a \frac{n-1}{n}$$

Let us put for simplicity $\hat{p}_T = 0$; then the distribution in φ_{12} for $n > 2$ is obtained by changing variables to $|\hat{p}_{T1}|$, $|\hat{p}_{T2}|$, φ_{12} , $\cos \varphi_1$, integrating over redundant variables and normalizing

$$P(\varphi_{12}) = \frac{2\pi \int_0^\infty d\rho_1 d\rho_2 \rho_1 \rho_2 \exp \left[-\alpha \frac{(n-1)}{(n-2)} (\rho_1^2 + \rho_2^2) - \frac{2\alpha}{n-2} \rho_1 \rho_2 \cos \varphi_{12} \right]}{\int d^2 p_{T1} d^2 p_{T2} P(\hat{p}_{T1}, \hat{p}_{T2})} \quad (31)$$

The integrals can be evaluated exactly, and the result is

$$P(\varphi_{12}) = \frac{1 - \xi^2}{\pi} \left[\left(1 - \frac{\xi^2 \cos^2 \varphi_{12}}{4} \right)^{-1} - \frac{\pi}{4} \xi \cos \varphi_{12} \left(1 - \frac{\xi^2 \cos^2 \varphi_{12}}{4} \right)^{-\frac{3}{2}} - \left\{ 1 - \frac{2}{\pi} \arctan \left(\frac{\frac{\xi}{2} \cos \varphi_{12}}{\left(1 - \frac{\xi^2 \cos^2 \varphi_{12}}{4} \right)^{\frac{1}{2}}} \right) \right\} \right] \quad (32)$$

$$\xi = \frac{2}{n-1}$$

It is important that this distribution does not depend on a . This results from the fact that a is the only dimensional parameter in our model and therefore it cannot enter into the expression for the dimensionless quantity $P(\varphi_{12})$. It is for this reason that we prefer to study the φ_{12} distribution and not the $\hat{p}_{T1} \cdot \hat{p}_{T2}$ distribution, which depends more on the distribution of $|\hat{p}_{T1}|$.

8. COMPARISON OF FORMULA (32) WITH THE DISTRIBUTION OF
 $\cos \varphi_{ik}$ OBTAINED FROM THE COVARIANT STATISTICAL MODEL

In order to confront our model with another approach we compare formula (32) for $n = 6$ with the predictions of the covariant statistical model. The statistical distribution was found ^{*)} for 6 pions at $E_{\text{cms}} = 3 \text{ GeV}$ and $E_{\text{cms}} = 9 \text{ GeV}$. Since the two sets of results are rather similar we discuss only the 9 GeV results.

100.000 events were obtained and in each the particles were labelled according to the algebraic values of the longitudinal components of the momentum. Thus within one event by definition $p_{\ell i} \geq p_{\ell k}$ if $i > k$. The histograms for all the 15 distributions φ_{ik} , $i = 1, \dots, 6$, $k = i, \dots, 6$ were plotted. It was found that they were significantly different, φ_{16} and φ_{34} , shown in Fig. 2 being the limiting cases. The statistical errors in the histograms are very small; a typical one is shown by the arrow in the first column of the histogram for φ_{16} . For 3 GeV the difference between $P(\varphi_{16})$ and $P(\varphi_{34})$ is slightly smaller, but there is no qualitative differences.

Our model neglects the differences between the $P(\varphi_{ik})$; but its predictions fall between the limiting statistical distributions, as seen from Fig. 2, or by comparing the moments of the distributions:

	$\cos^{\text{stat}} \varphi_{16}$	$\cos^{\text{Gauss}} \varphi$	$\cos^{\text{stat}} \varphi_{34}$
$\overline{\cos \varphi}$	- 0.224	- 0.158	- 0.085
$\overline{\cos^2 \varphi}$	0.529	0.509	0.504
$\overline{\cos 2 \varphi}$	0.057	0.019	0.008

*) We thank Dr. E. Quercigh for having made the Monte Carlo calculation.

9.

CONCLUSIONS

We have described a method, which makes it possible to compute the two and more-particle correlations, arising from the energy-momentum conservation, starting from the measured one-particle distributions. The method, when applied to the correlation between the transverse momenta of the secondaries produced in a high-energy reaction, gives, under reasonable assumptions about the one-particle transverse momentum distributions, a correlation similar to that one gets applying a Monte Carlo method to the statistical model. Any deviation of the actual experimental correlations from those given in Fig. 2 would mean a non-trivial correlation between the secondaries in a production process.

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APPENDIX : PROBLEM OF THE BOSE STATISTICS AND OF THE
SYMMETRIZATION OF THE AMPLITUDE

When the final state contains two or more identical particles (in general pions), some additional considerations are necessary, in order to take into account the identity of the particles. We shall discuss the case in which the identical particles are bosons.

The first problem is connected with the Bose statistics, obeyed by the identical particles. Let us assume that the particles are enclosed in a finite volume V . Then a sum over states ^{*)}, as the one represented as an integral in formula (2), is given by

$$\sum_{p_1, p_2, \dots, p_n} s^{(n)}(p - \sum_k p_k) \prod f_i(p_i) \quad (\text{A.1.})$$

Let us consider a set of momenta $p, p', p'', \dots, p^{(n)}$, and suppose, again for simplicity, that all the n particles are identical. Then if no two-momenta $p^{(i)}, p^{(k)}$ are equal, the set $p, p', p'', \dots, p^{(n)}$ is counted $n!$ times in (A.1.), while according to quantum statistics it should be counted only once. This factor $n!$ gives no trouble, since it may be included in the normalization constant. The sources of the difficulty are the momentum configurations where two or more momenta in the set $p, p', p'', \dots, p^{(n)}$, are equal. There the correction factor is somewhere between $1/n!$ and 1 . A proper counting of such states yields the quantum statistics for perfect Bose gases. In our case, however, this effect is unimportant, the reason being that the number of states is very large (of the order of $V^n/(2\pi)^{3n}$) in comparison with the number n of particles present in the final state of an actual experiment; therefore, the statistical weight of these states is vanishingly small and the combinatorial factor used for them is irrelevant. This corresponds to the well-known fact that at very low density the perfect gas is non-degenerate.

^{*)} We assume for simplicity that momentum is enough to define unambiguously a one-particle state.

A more important effect comes from the symmetrization of the one-particle amplitudes. For identical particles the amplitude must be a symmetric function of the particle variables. Let us now take the simple case of two identical Bose particles, and suppose that their amplitudes are ψ_1 and ψ_2 , then, taking into account the indistinguishability of the particles, we have for the distribution

$$\begin{aligned} & \left| \psi_1(p_1) \psi_2(p_2) + \psi_1(p_2) \psi_2(p_1) \right|^2 = \\ & = |\psi_1(p_1)|^2 |\psi_2(p_2)|^2 + |\psi_1(p_2)|^2 |\psi_2(p_1)|^2 + 2 \operatorname{Re} [\psi_1^*(p_1) \psi_2^*(p_2) \psi_1(p_2) \psi_2(p_1)] \end{aligned} \quad (\text{A.2.})$$

In the particular case which we have considered (see Section 2), we have $\psi_1 = \psi_2 = \psi$, $f_1 = f_2 = |\psi|^2$, and the symmetrization introduces only an irrelevant constant; however, for $\psi_1 \neq \psi_2$ one needs an additional assumption about the phases of ψ_1 and ψ_2 . The simplest possibility here is to take all the ψ 's to be real (or more generally with a phase which does not depend on momentum).

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FIGURE CAPTION

Figure 2 : Distribution of azimuthal angles in a system of 6 pions
at $E_{\text{cns}} = 9 \text{ GeV}$.

—————	Monte Carlo histogram for $\cos \varphi_{16}$
-----	Monte Carlo histogram for $\cos \varphi_{34}$
continuous curve	Distribution given by formula (32)

a)

b)

FIG.1

FIG. 2