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Archives

ON THE CONNECTION BETWEEN THE ZEROS OF THE AMPLITUDE NEAR $t = 0$

AND THE SLOPE OF THE DIFFRACTION PEAK

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A B S T R A C T

Using only axiomatic field theory, we show that the absorptive part of the amplitude $A_s(s, t)$ has no zeros inside the circle

$$|t| \leq t_1 = \left(\frac{2\pi}{15e}\right)^2 \mu^2 \left(\log \frac{s}{s_0}\right)^{-2}$$

for $s > s_0$. We then show that the quantity

$$\frac{1}{2} \alpha'_I \equiv \frac{d}{dt} \log A_s(s, t) \Big|_{t=0}$$

has the property

$$0 \leq \frac{1}{2} \alpha'_I \leq e \left(\frac{15}{\pi \mu}\right)^2 \left(\log \frac{s}{s_0}\right)^2$$

These bounds are the best one can obtain using only analyticity in one complex variable, because we show that improving t_1 , for instance, improves the Froissart bound.

In particular we show that if the amplitude does not become purely real for large physical energies the slope of the diffraction peak

$$\frac{d}{dt} \log \frac{d\sigma}{d\Omega} \Big|_{t=0}$$

increases at most as $(\log s)^2$, the Regge hypothesis giving $\log s$.

I.-

INTRODUCTION

One can show that the total cross-section $\sigma_{\text{tot}}(s)$ cannot vanish for any value of s , otherwise the amplitude would be identically zero ^{*}). The absorptive part $A_s(s, t)$ is then different from zero for $t = 0$, and holomorphic in t near $t = 0$. There must then exist a circle of radius $R(s)$ depending on s in which the amplitude does not vanish. In the first part of this paper, we obtain a value of $R(s)$ which decreases as $(\log s)^{-2}$ when $s \rightarrow +\infty$. This value is closely connected to the Froissart bound ^{1), 2)} $\sigma_{\text{tot}} < C(\log s)^2$, as shown in the conclusion, and, any improvement of the behaviour of $R(s)$, will improve the Froissart bound. But we know ³⁾ that using only one complex variable techniques, there is an example which saturates this bound. So, once again, we are obliged to go to two complex variable techniques, if we want to improve any of the preceding bounds.

In the second part we show that the knowledge of a circle around $t = 0$ in which the absorptive part does not vanish implies a bound for the quantity

$$\frac{1}{2} \alpha_I'(s) = \frac{d}{dt} \log A_s(s, t) \Big|_{t=0}$$

We show that $\alpha_I'(s)$ increases at most as $(\log s)^2$. In the special case of an amplitude, which is purely imaginary at high energies $\alpha_I'(s)$ is precisely the slope of the diffraction peak, and we see that this quantity behaves, at most, as $(\log s)^2$. This must be compared to the Regge behaviour which gives $\log s$.

^{*}) A. Martin - private communication.

2.

In the general case, we can say that if the amplitude does not become purely real at high energies [this would imply that $\sigma_{\text{tot}}(s) \rightarrow 0$] then the slope of the diffraction peak cannot increase faster than $(\log s)^2$. This result is still true if we suppose that there exists an ξ , as small as one likes, but independent of s , such that for $-\xi \leq t \leq 0$

$$\frac{|F(s, t)|}{|F(s, 0)|} < C$$

C independent of s .

Kinoshita ⁴⁾ and independently Kowalski ⁵⁾ have obtained for $|\alpha_I|$ the same bound as us. However, the aim of this paper is to show the close connection between the zeros of the amplitude near $t=0$, the slope of the diffraction peak and the Froissart bound, and the fact that improving our knowledge of the zeros, will improve the other bounds.

II.- THE ZEROS OF $A_s(s, t)$, $s > s_0$, $|t| < 4\mu^2$

II.1.- A theorem

If $f(z)$ is holomorphic inside a circle of radius R_1 ,
 $f(0) = 1$ and

$$M(R) = \sup_{|z| \leq R} |f(z)|$$

then $f(z)$ has no zeros inside the circle of radius :

$$r = \frac{R}{M(R)} \quad R < R_1 \quad (1)$$

Let us consider the auxiliary function :

$$\varphi_R(z) = M(R) \frac{f(z) - 1}{M^2(R) - f(z)} \quad (2)$$

We see easily that $\varphi_R(z)$ is holomorphic in $|z| \leq R$ and
 $\varphi_R(0) = 0$, and $|\varphi_R(z)| \leq 1$ for $|z| \leq R$.

If we apply the maximum modulus theorem to $\frac{R}{z} \varphi_R(z)$
 this last function being holomorphic in $|z| \leq R$ and bounded
 by 1 for $|z| = R$, its modulus being continuous on $|z| = R$,
 we get :

$$|\varphi_R(z)| \leq \frac{|z|}{R} \quad (3)$$

or

$$\left| \frac{f(z) - 1}{1 - \frac{f(z)}{M(R)}} \right| \leq |z| \frac{M(R)}{R} \quad (4)$$

If $|z| < R/\overline{M(R)}$, the right-hand side of equation (4) is smaller than 1 and

$$|f(z)-1| < \left| 1 - \frac{f(z)}{M^2(R)} \right| \quad (5)$$

and $f(z)$ cannot vanish.

II.2.- Some properties of the absorptive part

Let us consider the function

$$\Psi_s(t) = \frac{A_s(s,t)}{A_s(s,0)} \quad s > s_0 \quad (6)$$

This function is holomorphic in an ellipse ⁶⁾ of the t plane of focus 0 and vertex $4\mu^2$ (Fig. 1)

- Figure 1 -

On any ellipse Σ_{t_0} defined by its vertex t_0 , the function $A_s(s,t)$ is bounded by its value for $t = t_0$. This results from the fact that :

1) $|P_2(\cos \Theta)|$, when $\cos \Theta$ describes an ellipse of foci -1, +1 is maximum for Θ purely imaginary, because ⁷⁾

$$P_\ell(\cos \theta) = \sum_{k=0}^{\ell} \gamma_{\ell k} \cos k\theta \quad \gamma_{\ell k} \geq 0 \quad (7)$$

2)

$$\operatorname{Im} a_\ell(s) \geq 0$$

$$A_s(s, \cos \theta) = \sum_{\ell=0}^{\infty} (2\ell+1) \operatorname{Im} a_\ell(s) P_\ell(\cos \theta) \quad (8)$$

In particular, we deduce that :

$$- \Psi_s(t) \text{ is holomorphic inside a circle of radius } 4\mu^2. \quad (9)$$

$$- |\Psi_s(t)| \leq 1 \text{ for } -4\mu^2 \leq t \leq 0. \quad (10)$$

$$- |\Psi_s(t)| \leq \Psi_s(|t|). \quad (10)$$

We need now an upper bound for $|\Psi_s(t)|$ on the circle $|t| = 4\mu^2$.

We know that ⁸⁾

$$Cs^{-2} \leq |F(s, t)| \leq Cs^2 \quad (11)$$

This results essentially from the fact that $F(s, t)$ is an Herglotz function in the variable $z = (s - 2\mu^2 + \frac{t}{2})^2$.

But, on the other hand ⁸⁾ :

$$[\operatorname{Re} F(s, 0)]^2 \leq Cs(\log s)^2 A_s(s, 0) \quad s > s_0 \quad (12)$$

From (11) and (12), one deduces immediately :

$$A_s(s, 0) \geq Cs^{-5} (\log s)^{-2} \quad (13)$$

6.

And

$$|\Psi_s(t)| < C s^7 (\log s)^2 < \left[\frac{s}{s_0} \right]^{7(1+\varepsilon)} \equiv M \quad s > s_0. \quad (14)$$

II.3.- An upper bound for $\Psi_s(t)$ $|t| < 4\mu^2$

If we put

$$t = w^2 \quad (15)$$

we have

$$\Psi_s(t) = \Psi_s(w^2) = \varphi(w) \quad (16)$$

The function $\varphi(w)$ is holomorphic in (D), bounded by 1 on the segment $[-2\mu i, +2\mu i]$ and by M on (γ) (See Fig. 2)

- Figure 2 -

Let us consider the auxiliary function

$$h(w) = \varphi(w) \exp \left\{ -\frac{2i}{\pi} \log \mu [\log(w+i\mu) - \log(w-i\mu) - \pi] \right\} \quad (17)$$

This function is bounded by 1 on ∂D , holomorphic in (D) , and its modulus is continuous when W tends to a point on ∂D . In fact we must replace μ by $\mu(1 - \varepsilon_1)$ to fulfil this last condition. By the maximum modulus theorem $h(W)$ is bounded by 1 everywhere in $D + \partial D$.

So, on the arc of the circle γ_α , $|\varphi(W)| \leq M \frac{2\alpha}{\pi}$, this arc cuts the W real axis at $W_0 = 2\mu(1 - \varepsilon_1) \operatorname{tg} \frac{\alpha}{2}$. But

$$\frac{\alpha}{2} \leq \operatorname{tg} \frac{\alpha}{2}$$

so

$$\alpha \leq \frac{W_0}{\mu}$$

And

$$|\psi_3(W)| \leq S \frac{14}{\pi} \frac{\sqrt{14}}{\mu} \frac{1+\varepsilon}{1-\varepsilon_1} \quad (18)$$

and, by (10) :

$$|\psi_3(t)| \leq \left[\frac{S}{S_0} \right] \frac{14}{\pi} \frac{\sqrt{14}}{\mu} \frac{1+\varepsilon}{1-\varepsilon_1} \quad |t| < 4\mu^2 \quad (19)$$

II.4.- Application of theorem II.1

We can now apply the result of theorem II.1. and obtain that $A_s(s,t)$ will not vanish in the circle

$$|t| < |t_0| \left[\frac{s}{s_0} \right]^{-\frac{14}{\pi}} \frac{\sqrt{|t_0|}}{\mu} \frac{1+\varepsilon}{1-\varepsilon_1} \quad 0 \leq |t_0| < 4\mu^2 \quad (20)$$

The maximum of the right-hand side of (20) is at :

$$|t_0| = \mu^2 \left(\frac{2\pi}{14} \right)^2 \left(\frac{1-\varepsilon_1}{1+\varepsilon} \right)^2 \left(\log \frac{s}{s_0} \right)^{-2} \quad (21)$$

The absorptive part of the amplitude will not vanish in the circle :

$$|t| < \left(\frac{2\pi}{14\varepsilon} \right)^2 \mu^2 \left(\frac{1-\varepsilon_1}{1+\varepsilon} \right)^2 \left(\log \frac{s}{s_0} \right)^{-2} \quad (22)$$

We choose ε_1 and ε in such a way that the amplitude does not vanish inside and on the circle

$$|t| \leq \left(\frac{2\pi}{15\varepsilon} \right)^2 \mu^2 \left(\log \frac{s}{s_0} \right)^{-2} \quad (23)$$

III.- AN UPPER BOUND FOR THE SLOPE OF THE DIFFRACTION PEAK

III.1.- A general bound for α'

We have by definition

$$\alpha' = \frac{d}{dt} \log \frac{d\sigma}{d\Omega} \Big|_{t=0} \quad (24)$$

that is

$$\frac{1}{2} \alpha' = \frac{d}{dt} \log |f(s,t)| \Big|_{t=0} \quad (25)$$

or

$$\frac{1}{2} \alpha' = \operatorname{Re} \frac{d}{dt} \log \frac{f(s,t)}{f(s,0)} \Big|_{t=0} \quad (26)$$

Let us write

$$\varphi_s(t) = \log \frac{f(s,t)}{f(s,0)} \quad (27)$$

$f(s,t)/f(s,0)$ is holomorphic in $|t| < 4\mu^2$, and for $t = 0$ this ratio equals 1. It then exists a certain circle of radius $t_1(s)$ in which this ratio does not vanish. We see that $\varphi_s(t)$ is holomorphic in $|t| < t_1(s)$ and $\varphi_s(0) = 0$. The function

$$\rho_s(t) = \frac{\varphi_s(t)}{t} \quad (28)$$

is holomorphic in $|t| < t_1(s)$ and by (26) :

$$\frac{1}{2} \alpha' = \operatorname{Re} \rho_s(0) \quad (29)$$

We have

$$\frac{1}{2} |\alpha'| \leq |p_s(0)| \leq \frac{1}{|t_0|} \max_{|t| \leq |t_0|} |\varphi(s, t)| \quad |t_0| < t_1(s)$$

or

$$\frac{1}{2} |\alpha'| \leq \frac{1}{|t_0|} \max_{|t| \leq |t_0|} \left| \log \frac{f(s, t)}{f(s, 0)} \right| \quad |t_0| < t_1(s) \quad (30)$$

We recall Caratheodory's theorem ⁹⁾ :

If $f(z)$ is holomorphic for $|z| < R$, $f(0) = 0$
and

$$A = \max_{|z| \leq R} \operatorname{Re} f(z)$$

then

$$M(r) \leq \frac{2Ar}{R-r} \quad r < R. \quad (31)$$

where

$$M(r) = \max_{|z|=r} |f(z)|.$$

Applying this theorem to $\varphi_s(t)$ we get, taking

$$r = |t_0|, \quad R = t_1(s)$$

$$\frac{1}{2} |\alpha'| \leq \frac{2}{t_1(s) - |t_0|} \max_{|t| \leq t_1(s)} \log \left| \frac{f(s, t)}{f(s, 0)} \right| \quad 0 \leq |t_0| < t_1(s) \quad (32)$$

So finally, putting $|t_0| = 0$ in (32)

$$\left| \frac{1}{2} \alpha' \right| \leq \frac{2}{t_1(s)} \max_{|t| \leq t_1(s)} \log \left| \frac{f(s, t)}{f(s, 0)} \right| \quad (33)$$

We shall, now, use (33) in various different cases :

A - A bound for α_I

We can apply (33) for the case of a purely imaginary amplitude, or more precisely to the absorptive part of the amplitude, for which the preceding derivation applies. We then get that :

$$\frac{1}{2} \alpha_I'(s) = \left. \frac{d}{dt} \log A_s(s, t) \right|_{t=0} \quad (34)$$

has the property

$$0 \leq \frac{1}{2} \alpha_I'(s) \leq \frac{2}{t_1(s)} \max_{|t| \leq t_1(s)} \log \left| \frac{A_s(s, t)}{A_s(s, 0)} \right| \quad (35)$$

Here $t_1(s)$ can be chosen to be equal to :

$$t_1(s) = \left(\frac{2\pi}{15e} \right)^2 \mu^2 \left(\log \frac{s}{s_0} \right)^{-2} \quad (36)$$

by (23). And using (18), we see that

$$0 \leq \frac{1}{2} \alpha_I'(s) \leq \frac{2}{t_1(s)} \frac{15}{\pi} \frac{\sqrt{t_1(s)}}{\mu} \log \frac{s}{s_0} = e \left(\frac{15}{\pi \mu} \right)^2 \left(\log \frac{s}{s_0} \right)^2 \quad (37)$$

We see that the slope of the diffraction peak of the absorptive part of the amplitude does not increase faster than $(\log s)^2$, this is true also for an amplitude which becomes

purely imaginary for large physical energies. We shall see that in fact this is still true, if the amplitude does not become purely real for physical s and t .

B. - We make an extra-assumption to the axiomatic field theory

$$\left| \frac{f(s,t)}{f(s,0)} \right| < C \quad \text{for } -\varepsilon < t \leq 0 \quad (38)$$

with ε , as small as one likes, but independent of s and C independent of s .

By equation (11) we get :

$$\left| \frac{f(s,t)}{f(s,0)} \right| \leq Cs^4 \quad \text{for } |t| < 4\mu^2 \quad (39)$$

Applying the same method as in II-3, we deduce from (38), (39) and the holomorphy of $f(s,t)$ in t for $|t| < 4\mu^2$ that

$$\left| \frac{f(s,t)}{f(s,0)} \right| \leq C \left(\frac{s}{s_0} \right)^{\frac{8}{\pi}} \frac{\sqrt{|t|}}{\varepsilon} \quad (40)$$

It results from this bound that the ratio $f(s,t)/f(s,0)$ has no zeros inside a circle of radius $C(\varepsilon)(\log s/s_0)^{-2}$, and that

$$|\alpha'(s)| \leq C_1(\varepsilon) \left(\log \frac{s}{s_0} \right)^2 \quad (41)$$

C - The amplitude does not become purely real

If we suppose, now, that

$$\left| \frac{\operatorname{Im} f(s,t)}{\operatorname{Re} f(s,t)} \right| > \eta \quad \text{for} \quad -\varepsilon \leq t \leq 0 \quad s > s_0 \quad (42)$$

with η and ε as small as one likes, but independent of s , we see that (42) implies (38), and we find :

$$|\alpha'(s)| \leq C(\varepsilon, \eta) \left(\log \frac{s}{s_0} \right)^2 \quad (43)$$

IV.-

CONCLUSION

We would like to point out that no improvement of the bound (23) can be obtained in the framework of one complex variable. This results from the fact that in reference ¹⁰⁾ it has been shown that, if $\sigma_{\text{tot}} \rightarrow \infty$, then :

$$\sigma_{\text{tot}} \leq 36\pi \left. \frac{d}{dt} \log A_s(s,t) \right|_{t=0} \quad (44)$$

(44) is a consequence of unitarity alone.

In our case, this gives

$$\sigma_{\text{tot}} \leq 36\pi e \left(\frac{15}{\pi e} \right)^2 \left(\log \frac{s}{s_0} \right)^2 \quad (45)$$

i.e., is the Froissart bound. Any improvement of the bound (24) will of course improve the Froissart bound. But we know ³⁾ that this is not possible using only one complex variable.

One would like, at least, to get the Regge behaviour for $\alpha_I(s)$, that is $\alpha_I(s) \leq C \log s/s_0$, instead of $C(\log s/s_0)^2$. It may be possible to show this, using holomorphy combined with polynomial behaviour in s and t .

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