AN EXTENSION OF THE π - N AND π - Λ DISPERSION RELATIONS

J.D. Bessis and V. Glaser
CERN - Geneva

A B S T R A C T

It is shown, starting from the usual assumptions of "Axiomatic Field Theory", that for the π - N and π - Λ elastic scattering processes the dispersion relations hold for complex values of t inside the circle of radius $|t| = 4 \mu^2$, where μ is the pion mass.

I. INTRODUCTION

In his beautiful paper ¹⁾ on the analyticity of the S matrix, A. Martin has shown that, as a consequence of the positive definiteness of the Hilbert space metric, the fixed momentum transfer dispersion relations $(-t_0 \leq t \leq 0)$ of an elastic scattering amplitude extend also to complex values of the momentum transfer contained in a circle $|t| < R$. The essential ingredients of his proof are the existence of a finite number of subtractions for the dispersion relations in the cut s plane at $t = 0$ and the analyticity of the scattering amplitude in some complex neighbourhood of the real points $0 \leq t < R$ for some real s between the thresholds $[(M-\mu)^2 < s < (M+\mu)^2]$ in the case of $\pi-N$ or $\pi-\Lambda$ scattering, where μ = pion mass and M = nucleon or Λ mass. This neighbourhood has, of course, to be provided by the causal properties of the off-mass shell extrapolations of the S matrix. Subsequently Sommer ²⁾ has noticed that also a complex neighbourhood in t of $t = 0$ for fixed s above the threshold $[s > (M+\mu)^2]$ in the case we are going to treat in this paper will serve the same purposes. Such a neighbourhood is furnished by the small Lehmann ellipses, the maximal extension of which towards positive values of t turns out to be $R = 1.83 \mu^2$ in the case of $\pi-N$ and $\pi-\Lambda$ scattering.

In this paper we want to show that the small Lehmann ellipse is not the best domain of analyticity which the linear properties of the four-point function can yield. In fact, in order to apply Martin's original techniques, we will show that for all s in some interval $s_0 < s < (M+\mu)^2$ below threshold we have analyticity in t in a domain containing the semi-open interval $0 \leq t < 4\mu^2$. The use of positivity "à la Martin" yields then immediately the circle $|t| < 4\mu^2$, in which also the dispersion relations remain valid.

Such a result was to be expected in the light of the methods of analytic continuation developed in the paper II ³⁾ for the general proof of the "crossing analyticity". As a matter of fact this paper is nothing but a straightforward adaptation of the considerations described

2.

in II for the fixed $t < 0$ situation to the fixed $s > 0$ situation : there the primitive Steinmann domain of analyticity of the four-point function was restricted to the analytic manifold $p_1 + p_3 = (0, 0, 0, \sqrt{-t})$, $t < 0$ and partial analytic completions were performed in it using the Lehmann-Jost-Dyson formula and the "flattened" tube theorem. Here we simply consider the Steinmann sphere on the manifold $p_1 + p_2 = (\sqrt{s}, 0, 0, 0)$ and perform the analogous analytic completions in it.

Before plunging into technicalities of the proof let us make two general remarks. First, due to the elementary properties of the disk theorem, as explained in II, each point of analyticity gained on the manifold $\text{Im}(p_1 + p_2) = 0$, will be the centre of a full neighbourhood of analyticity in all the variables ; in particular every point in the t domain described in our conclusions as $t \in \mathcal{D}(s)$ for s real in $s_0 < s < (M + \mu)^2$ will be a point of analyticity in both t and s in an unspecified neighbourhood (which, by the way, could be explicitly estimated, a thing which is not worth while doing in view of Martin's much better conclusions). Second, our considerations are restricted in this paper to the Lorentz invariant scattering amplitudes and consequently the analytic completions are done in the invariants s, t, u . However, the same conclusions could be reached without assuming any covariance properties of the amplitudes, as it was done in II, but this would considerably complicate the proof. It is also not worth while doing it in view of the fact that Martin's consequent use of positivity is not based on purely geometric considerations, but relies heavily on the explicit form of the dispersion integrals, where a decomposition into conveniently chosen invariant amplitudes is essential ⁴⁾.

II. FORMULATION OF THE PROBLEM

We first remind the reader that the invariant parts of the elastic π -N or π - Λ scattering amplitudes $F(s, t, u)$ can be written in the following form

$$F(s, t, u) = \frac{\varphi^2(s)}{s - M^2} + \frac{\varphi^2(u)}{u - M^2} + \phi(s, t, u) \quad (1)$$

s, t, u being the usual invariants

$$\begin{aligned} s &= (p_1 + p_2)^2 \\ t &= (p_1 + p_3)^2 \\ u &= (p_2 + p_3)^2 \end{aligned} \quad s + t + u = 2M^2 + 2\mu^2 \quad (2)$$

explained by Fig. 1. The first two terms are the pole terms corresponding to the exchange of a single baryon. $\varphi(s)$ is the baryonic form factor, which is known to be analytic in the cut s plane with a single cut starting at $(M + \mu)^2$. It is also known that dispersion relations hold for the last term for fixed real t in $-t_0 < t \leq 0$ with

$$t_0 = \frac{32\mu^2}{3} \cdot \frac{2M_B + \mu}{2M_B - \mu} \quad B = N \text{ or } \Lambda \quad (3)$$

In this region ϕ has two cuts starting at $s = (M + \mu)^2$ respectively $u = (M + \mu)^2$.

According to LSZ the scattering amplitude (1) is the restriction to the mass shell $p_{1,3}^2 = M^2$, $p_{2,4}^2 = \mu^2$ of the "off-mass shell" amplitude $F(p_1, p_2, p_3, p_4)$ ($p_1 + p_2 + p_3 + p_4 = 0$) which is analytic in the "primitive domain of analyticity" consisting of 32 tubes best described by the Steinmann sphere of Fig. 2 and of the real points of analyticity given by the inequalities :

4.

$$p_1^2 < (M+\mu)^2, \quad p_2^2 < 9\mu^2, \quad p_3^2 < (M+\mu)^2, \quad p_4^2 < 9\mu^2 \quad (4)$$

$$\sigma^2 < (M+\mu)^2, \quad \tau^2 < 4\mu^2, \quad \nu^2 < (M+\mu)^2 \quad (5)$$

where for later convenience the vector variables

$$\sigma = p_1 + p_2, \quad \tau = p_1 + p_3, \quad \nu = p_2 + p_3 \quad (6)$$

have been introduced (compare, e.g., the papers ^{3),6)}, where more detailed references may be found). Since the variables (6) turn out to be more adapted to the problem of analytic continuation we note also the inverse formulae

$$\begin{aligned} p_1 &= \frac{1}{2} (\sigma + \tau - \nu) \\ p_2 &= \frac{1}{2} (\sigma - \tau + \nu) \\ p_3 &= \frac{1}{2} (-\sigma + \tau + \nu) \\ p_4 &= -\frac{1}{2} (\sigma + \tau + \nu) \end{aligned} \quad (7)$$

We note also that in the spectral conditions (4) and (5) the discrete one-baryon contributions have been omitted in accordance with the remark following formula (1).

As indicated in the Introduction we restrict now our attention to the manifold

$$\sigma = (\sqrt{s}, \vec{0}) \quad s > 0.$$

Here we are left with analyticity in a system of eight tubes in the remaining two four-vectors τ, ν as best explained by the Figs. 3 and 4. The domain of coincidence on real points is still given by the

Eqs. (4) and (5) but with σ now kept fixed. Since we are going to fix s below threshold, the condition $\sigma^2 < (M + \mu)^2$ will not be violated. Hence all the points of coincidence of the restricted problem will remain (by the local Edge of the Wedge Theorem) points of analyticity.

Our aim is to find a partial envelope of holomorphy of the restricted problem by first fixing ν real and essentially spacelike and solving the "degenerate" Edge of the Wedge problem along the "line" $\text{Im } \nu = 0$ in Fig. 4 in the four-vector τ with the help of the Lehmann-Jost-Dyson procedure and then doing once more the same with the roles of τ and ν reversed. This will provide us with two "degenerate" domains of analyticity, whose image in the invariants will be of the form u real $t \in \mathcal{O}(u)$ and t real $u \in \mathcal{O}(t)$. A new partial analytic completion will finally yield a domain in u and t with a non-empty intersection with the mass shell.

In this connection we would like to point out to the reader that in terms of our Fig. 4 the way leading to the small Lehmann ellipses consists in an Edge of the Wedge along $\text{Im } (\tau - \nu) = 0$, with p_1, p_2 fixed real on the mass shell and thus s fixed above the threshold (the Edge of the Wedge along $\text{Im } (\tau + \nu)$ with p_3, p_4 real amounts obviously to the same thing). It was, however, pointed out to us by Epstein ^{*)} that the manifold $\text{Im } (\tau - \nu) = 0$ has very degenerate properties of mapping into the invariants at the threshold and that this was the real "kinematical" reason - as a short calculation shows - for the shrinkage of the small Lehmann ellipses near threshold.

^{*)} We are very indebted to Dr. Epstein for this remark.

III. THE EDGE OF THE WEDGE CORRESPONDING TO $\text{Im } \mathcal{V} = 0$

If we cut the preceding system of tubes by $\text{Im } \mathcal{V} = 0$ with fixed real $\mathcal{V}$, we are left with an amplitude which depends only on τ , holomorphic in $\text{Im } \tau \in \mathcal{V}^{\pm}$ and a neighbourhood of the real points of coincidence given by (4) and (5).

We shall now specialize the vectors σ , τ and $\mathcal{V}$ as follows :

$$\begin{aligned} \sigma &= (\sqrt{s}, 0, 0, 0) \\ \mathcal{V} &= \left(-\frac{M^2 - \mu^2}{\sqrt{s}}, 0, \nu_1, 0\right) \\ \tau &= (\tau_0, \tau_1, 0, 0) \end{aligned} \quad \begin{aligned} s &> 0 \\ \nu_1 &\text{ real} \end{aligned} \quad (8)$$

The conditions (4) become

$$\begin{aligned} (H_1) &\equiv \left[\tau_0 + \left(\frac{M^2 - \mu^2}{\sqrt{s}} + \sqrt{s} \right) \right]^2 - \tau_1^2 < 4(M + \mu)^2 + \frac{(M^2 - \mu^2)^2}{s} - u \\ (H_2) &\equiv \left[\tau_0 - \left(\frac{M^2 - \mu^2}{\sqrt{s}} + \sqrt{s} \right) \right]^2 - \tau_1^2 < 4(M + \mu)^2 + \frac{(M^2 - \mu^2)^2}{s} - u \\ (H_3) &\equiv \left[\tau_0 - \left(\sqrt{s} - \frac{M^2 - \mu^2}{\sqrt{s}} \right) \right]^2 - \tau_1^2 < 36\mu^2 + \frac{(M^2 - \mu^2)^2}{s} - u \\ (H_4) &\equiv \left[\tau_0 + \left(\sqrt{s} - \frac{M^2 - \mu^2}{\sqrt{s}} \right) \right]^2 - \tau_1^2 < 36\mu^2 + \frac{(M^2 - \mu^2)^2}{s} - u \end{aligned} \quad (9)$$

ν_1 real implies that $u \leq (M^2 - \mu^2)^2/s$. From now on we shall restrict ourselves to

$$M^2 - \mu^2 \leq s \leq (M + \mu)^2 \quad (10)$$

In this case (5) is just

$$(H_0) \equiv \tau_0^2 - \tau_1^2 < 4\mu^2 \quad (11)$$

When

$$(M^2 - \mu^2) \leq s \leq (M + \mu)^2 \quad (12)$$

and

$$u \leq \frac{(M^2 - \mu^2)^2}{s} \quad (13)$$

one can show that the domain of coincidence has the shape shown in Fig. 5. We call C_0, C_1, C_2, C_3, C_4 the ordinates of the centres of the hyperbolae H_0, H_1, H_2, H_3, H_4 and $S_i^\pm$ the vertices of the upper or lower parts of the hyperbolae, and $H_i^\pm$ the upper or lower parts of the hyperbolae $[i = 0, 1, 2, 3, 4]$. The figure is symmetric with respect to the $\tau_0 = 0$ axis.

When (12) and (13) are fulfilled, it is easy to show that

$$H_2^+ \subset H_3^+ \subset H_4^+ \subset H_1^+ \quad (14)$$

so we are left with only H_0^+ and H_1^+ and the symmetric curves. In fact, (14) results from

$$\begin{aligned} C_2 &> C_3 > C_4 > C_1 \\ S_2^+ &> S_3^+ > S_4^+ > S_1^+ \end{aligned} \quad (15)$$

which is implied by the following expressions :

$$C_2 = \frac{M^2 - \mu^2}{\sqrt{s}} + \sqrt{s} ; \quad C_3 = \sqrt{s} - \frac{M^2 - \mu^2}{\sqrt{s}} ; \quad C_4 = -\left(\sqrt{s} - \frac{M^2 - \mu^2}{\sqrt{s}}\right) ; \quad C_1 = -\left(\sqrt{s} + \frac{M^2 - \mu^2}{\sqrt{s}}\right)$$

$$S_2^+ = \frac{M^2 - \mu^2}{\sqrt{s}} + \sqrt{s} + \sqrt{4(M + \mu)^2 + \frac{(M^2 - \mu^2)^2}{s} - u} ; \quad S_3^+ = \sqrt{s} - \frac{M^2 - \mu^2}{\sqrt{s}} + \sqrt{36\mu^2 + \frac{(M^2 - \mu^2)^2}{s} - u}$$

$$S_4^+ = -\left(\sqrt{s} - \frac{M^2 - \mu^2}{\sqrt{s}}\right) + \sqrt{36\mu^2 + \frac{(M^2 - \mu^2)^2}{s} - u} ; \quad S_1^+ = -\left(\sqrt{s} + \frac{M^2 - \mu^2}{\sqrt{s}}\right) + \sqrt{4(M + \mu)^2 + \frac{(M^2 - \mu^2)^2}{s} - u}$$

Finally one sees also that $S_1^+ > S_0^+ = 2\mu$. This is the situation represented in Fig. 5.

The set of singularities of the function $F(\tau)$ is the set of complex points of all hyperbolae of the form

$$\tau_0^2 - (\tau_1 - \xi)^2 = \eta^2(\xi) + \rho \quad \rho > 0 \quad (16)$$

ξ , and $\eta^2(\xi)$ are the parameters of the doubly inadmissible hyperbolae. Given ξ , the centre, we immediately determine the point of contact P with H_1^+ , by the fact that C_1 , ξ and P must lie on the same line.

So we get that when $\xi_0 < \xi < \xi_{\max}$ (see Fig. 5). We have

$$(\eta - \mu_1)^2 + \xi^2 = \xi_{\max}^2 \quad (17)$$

$$\xi_{\max} = \sqrt{s} + \frac{M^2 - \mu^2}{\sqrt{s}} \quad (18)$$

$$\mathcal{U}_1 = \sqrt{4(\eta + \mu)^2 + \frac{(\eta^2 - \mu^2)^2}{s} - u} \quad (19)$$

$$\xi_0 = \frac{\xi_{\max} \sqrt{[\mathcal{U}_1^2 - (\xi_{\max} + 2\mu)^2][\mathcal{U}_1^2 - (\xi_{\max} - 2\mu)^2]}}{\mathcal{U}_1^2 + \xi_{\max}^2 - 4\mu^2} \quad (20)$$

When $0 < \xi < \xi_0$, the doubly inadmissible hyperbolae are determined by the fact that they must all pass through the point of intersection I of H_1^+ and H_0^+ . So we get

$$(\xi - \omega)^2 + \eta^2 = \left\{ \frac{\mathcal{U}_1^2 - \xi_{\max}^2 - 4\mu^2}{2\xi_{\max}} \right\}^2 \quad (21)$$

with

$$\omega = \frac{\sqrt{[\mathcal{U}_1^2 - (\xi_{\max} + 2\mu)^2][\mathcal{U}_1^2 - (\xi_{\max} - 2\mu)^2]}}{2\xi_{\max}} \quad (22)$$

If we now restrict $\mathcal{L}$ to be of the form

$$\mathcal{L} = (0, \sqrt{-t}, 0, 0) \quad (23)$$

we see that the singularities are given by

$$\sqrt{-t} = \xi + i\sqrt{\eta^2 + \rho} \quad \rho > 0 \quad (24)$$

represented in Fig. 6. The circle (Γ_1) passes through the point $(0, 2\mu)$, has its centre on $\text{Im } \sqrt{-t} = 0$, and it is tangent to the circle Γ_2 of centre $(0, \mathcal{U}_1)$ and radius $\xi_{\max}$.

As a conclusion of this Section, we have proved that when

$$u \leq \frac{(M^2 - \mu^2)^2}{s} \quad (25)$$

the off-mass shell amplitude is holomorphic in $\sqrt{-t}$ in the domain of Fig. 6. With the choice (18) and (23) for σ, τ, ν , the mass-shell conditions are just given by

$$s + t + u = 2M^2 + 2\mu^2 \quad (26)$$

IV. THE EDGE OF THE WEDGE CORRESPONDING TO $\text{Im } \tau = 0$

If we cut now the system of tubes of Fig. 4 with $\text{Im } \tau = 0$, i.e., we fix τ real, we are left with an amplitude depending only on ν , holomorphic in $\text{Im } \nu \in \mathbb{V}^+$ and in a neighbourhood of the real domain of coincidence given by (4) and (5).

We again specialize the vectors σ , τ and ν by taking

$$\begin{aligned}\sigma &= (\sqrt{s}, 0, 0, 0) \\ \tau &= (0, \sqrt{-t}, 0, 0) \\ \nu &= (\nu_0, 0, \nu_1, 0)\end{aligned}\tag{27}$$

$$\begin{aligned}M^2 - \mu^2 &\leq s \leq (M + \mu)^2 \\ t &< 0.\end{aligned}\tag{28}$$

The real region of coincidence is given by

$$\begin{aligned}(H_1) &\equiv [\nu_0 - \sqrt{s}]^2 - \nu_1^2 < 4(M + \mu)^2 - t \\ (H_2) &\equiv [\nu_0 + \sqrt{s}]^2 - \nu_1^2 < 36\mu^2 - t \\ (H_0) &\equiv \nu_0^2 - \nu_1^2 < (M + \mu)^2\end{aligned}\tag{29}$$

By loosing some information, we shall try to extract from (29) a domain of coincidence which is t as well as s independent. This will simplify matters later on. Indeed, we find that the region of coincidence always contains the domain of Fig. 7. To see this it is necessary to verify that

12.

$$\begin{aligned}
 H_0^+ &\subset H_+^+ \quad \text{which is trivial} \\
 H_1^+ &\subset H_+^+ \quad \text{which is trivial} \\
 H_2^+ &\subset H_+^+ \quad \text{which results from } S_2^+ \geq -(M - S_R)
 \end{aligned} \tag{30}$$

and is true when (28) is fulfilled. In the same manner we see that

$$\begin{aligned}
 H_0^- &\subset H_-^+ \\
 H_1^- &\subset H_-^+ \quad \text{or } S_1^- < -(M + \mu) \\
 H_2^- &\subset H_-^+ \quad \text{or } S_2^- < -(M + \mu)
 \end{aligned} \tag{31}$$

Following the same reasoning as in the last Section we find that when ν is chosen of the form

$$\nu = \left(-\frac{M^2 - \mu^2}{\sqrt{3}}, 0, \nu_1, 0 \right) \tag{32}$$

the singularities in the ν_1 plane are given by

$$\left[M - 2\mu - \frac{M^2 - \mu^2}{\sqrt{3}} \right]^2 - [\nu_1 - \xi]^2 = \gamma^2(\xi) + p \quad p > 0 \tag{33}$$

$$- (2M - \mu) \leq \xi \leq 2M - \mu. \tag{34}$$

with

$$[\eta - 2(M + \mu)]^2 + \xi^2 = (2M - \mu)^2 \tag{35}$$

or

$$\nu_1 = \xi \pm \sqrt{\left[M - 2\mu - \frac{M^2 - \mu^2}{\sqrt{3}} \right]^2 - \gamma^2(\xi) - p} \tag{36}$$

but

$$\eta(\xi) = 2(M+\mu) - \sqrt{(2M-\mu)^2 - \xi^2} > \left| M-2\mu - \frac{M^2-\mu^2}{\sqrt{s}} \right| = \frac{M^2-\mu^2}{\sqrt{s}} - M + 2\mu. \quad (37)$$

and the $\sqrt{\quad}$ is always a pure imaginary number. The worse domain is obtained when $s = M^2 - \mu^2$ and the equation is

$$\operatorname{Re} v_1 = \xi \quad - (2M-\mu) \leq \xi \leq 2M-\mu \quad (38)$$

$$\operatorname{Im} v_1 = + \sqrt{h^2(\xi) - [\sqrt{M^2-\mu^2} - M + 2\mu]^2} + p \quad p > 0$$

But it is easily shown that

$$\sqrt{h^2(\xi) - [\sqrt{M^2-\mu^2} - M + 2\mu]^2} > 2M+\mu - \sqrt{(2M-\mu)^2 - \xi^2} \quad (39)$$

That is, our domain is always contained in the one of Fig. 8.

As a conclusion of this paragraph, we see that, when

$$t < 0 \quad (40)$$

and

$$M^2 - \mu^2 \leq s \leq (M+\mu)^2 \quad (41)$$

we have analyticity in the v_1 variable in the domain of Fig. 8 where u and v_1 are connected by

$$u = \frac{(M^2 - \mu^2)^2}{s} - v_1^2 \quad (42)$$

The mass-shell conditions are still given by only one equation

$$s + t + u = 2M^2 + 2\mu^2 \quad (43)$$

V. THE ENVELOPE OF HOLOMORPHY IN THE INVARIANTS

In the present case we could do exactly the envelope of holomorphy in the invariants, but when we cut at the end by the mass shell, we see easily that with the exact envelope we only give a larger domain in the imaginary of t , but nothing more in the real part of t , than the following approximation. Because we are essentially interested in a neighbourhood of t real, we shall do the following simplifications.

We have a function $F(s, \tau_1, \nu_1)$ with

$$t = -\tau_1^2 \quad (44)$$

$$u = \frac{(M^2 - \mu^2)^2}{s} - \nu_1^2 \quad (45)$$

holomorphic in the following domain

for ν_1 real, in the strip : $|\operatorname{Im} \tau_1| < 2\mu$ from Chapter 3 ;
for τ_1 real, in the strip : $|\operatorname{Im} \nu_1| < 2\mu$ from Chapter 4.

By the tube theorem, we see that $F(\tau_1, \nu_1)$ is holomorphic in the tube

$$0 \leq \operatorname{Im} \tau_1 \leq \alpha \quad \otimes \quad 0 \leq \operatorname{Im} \nu_1 \leq 2\mu - \alpha \quad \alpha \leq 2\mu. \quad (46)$$

Going back to the t and u variables (for which the mass-shell condition is simple : $s + t + u = 2M^2 + 2\mu^2$). We find that $F(s, t, u)$ is holomorphic in the topological product of Fig. 9. When we cut by the mass shell we get, that when $M^2 - \mu^2 \leq s \leq (M + \mu)^2$, the on-mass-shell amplitude is holomorphic in t in the domain of Fig. 10.

For a given value of s , φ is fixed and the union of the domains of holomorphy in t has its boundary which is simply the locus of the points of intersection M and M' of the parabolas P_t and P_t' . P_t is a parabola of focus origin and vertex α^2 ; P_t' is a parabola of focus :

$$\varphi = 2M^2 + 2\mu^2 - \left[s + \frac{(M^2 - \mu^2)^2}{s} \right] \quad (47)$$

and vertex $\varphi - (2\mu - \alpha)^2$. We see that $0 \leq \varphi \leq 4\mu^2$.

The point M (M' is the symmetric of M) has for equations

$$\begin{aligned} \operatorname{Re} t &= \alpha^2 - \frac{(\operatorname{Im} t)^2}{4\alpha^2} \\ \operatorname{Re} t &= \varphi - \left[(2\mu - \alpha)^2 - \frac{(\operatorname{Im} t)^2}{4(2\mu - \alpha)^2} \right] \end{aligned} \quad (48)$$

or

$$\begin{aligned} \operatorname{Re} t &= \frac{\varphi}{2} - \frac{1}{4(\mu - \alpha)} \left[1 - \frac{\varphi/2}{\alpha^2 + (2\mu - \alpha)^2} \right] \\ (\operatorname{Im} t)^2 &= 4\alpha^2 (2\mu - \alpha)^2 \left[1 - \frac{\varphi}{\alpha^2 + (2\mu - \alpha)^2} \right] \end{aligned} \quad (49)$$

on (49) we see that when we change $\alpha \rightarrow (2\mu - \alpha)$, $\operatorname{Im} t$ does not change and $\operatorname{Re} t - \frac{\varphi}{2}$ changes in its opposite. The curve is then symmetric with respect to the point $(\frac{\varphi}{2}, 0)$.

We have now to distinguish between two cases.

1) $0 \leq \varphi < 2\mu^2$ that is:

$$M^2 + \sqrt{2M^2 - \varphi} \leq S \leq (M + \mu)^2 \quad (50)$$

then $(\operatorname{Im} t)^2$ is always positive for $0 \leq \alpha \leq 2\mu$.

In this case one finds easily that the domain has the form indicated in Figs. 11, 12, 13 and 14.

2) If

$$2\mu^2 \leq \varphi < 4\mu^2 \quad (51)$$

then $(\operatorname{Im} t)^2$ is not always positive and the domain is made of two pieces as shown in Fig. 15.

VI. CONCLUSION

We see that for the π -N and π - Λ elastic scattering, the amplitude is for

$$M^2 + \sqrt{2M^2 - \mu^2} < s \leq (M + \mu)^2 \quad (52)$$

certainly holomorphic in a neighbourhood of the real $-3\mu^2 \leq t \leq 4\mu^2$ interval.

The existence of such a domain implies in fact that $F(s, t)$ is, for s in the range (52), holomorphic in the circle $|t| < 4\mu^2$. The reason for this is that

$$\phi(s_1, t) = \frac{F(s_1, t)}{s_1 - (M + \mu)^2 - t} \quad M^2 + \sqrt{2M^2 - \mu^2} < s_1 < (M + \mu)^2 \quad (53)$$

is holomorphic in a neighbourhood of $[-3\mu^2, +4\mu^2]$ of the t plane. But this function has all its derivatives with respect to t , positive at $t = 0$ ¹⁾. By Hadamard's theorem, we see that $\phi(s_1, t)$ is holomorphic in $|t| < 4\mu^2$. We can therefore apply Martin's argument and find that dispersion relations hold for $|t| < 4\mu^2$.

ACKNOWLEDGEMENTS

We thank Professors H. Epstein and A. Martin for fruitful discussions. One of us (D.B.) is grateful to Dr. G. Sommer for discussions.

R E F E R E N C E S

- 1) A. Martin, Nuovo Cimento 42A, 930 (1966).
- 2) G. Sommer, CERN preprint TH. 683 (1966), to be published in Nuovo Cimento.
- 3) J. Bros, H. Epstein and V. Glaser, Commun. Math. Phys. Vol. 1, N. 63 (1965), p. 240.
- 4) A. Martin, to be published.
- 5) "Dispersion Equations" in Annual International Conference on High Energy Physics at CERN (1958), p. 208.
- 6) J. Bros, H. Epstein and V. Glaser, Nuovo Cimento 31, 1265 (1964)

FIG.1 The elastic scattering amplitude $B + \pi \rightarrow B + \pi$ where B stands for the nucleon or the Λ .

FIG.2 The Steinmann sphere

FIG. 3 Section of Steinmann sphere by $\text{Im}(p_1+p_2) = 0$

FIG. 4 The eight tubes in the ν, τ variable

FIG.5 The domain of coincidence for τ .

FIG. 6 The domain of holomorphy in the $\sqrt{-t}$ plane.

FIG.7 The symmetrized domain of coincidence for v

FIG. 8 The domain of holomorphy in the v_1 plane

FIG.9 The domain of holomorphy of the off-mass shell amplitude in the invariants.

FIG. 10 The section of the domain of fig. 9 by the mass-shell.

case $\varphi = 0$
 $[s = (M + \mu)^2]$

FIG. 11

case $0 < \varphi < 4/3 \cdot \mu^2$

FIG. 12

case $4/3 \mu^2 < \varphi < 2\mu^2$

FIG. 13

case $\varphi = 2\mu^2$
 $[s = M^2 + \sqrt{2M^2 - \mu^2}]$

FIG. 14

case $2\mu^2 < \varphi < 4\mu^2$

FIG. 15

Domain of analyticity in t of the on-mass shell amplitude

$$\varphi = 2M^2 + 2\mu^2 - \left[s + \frac{(M^2 - \mu^2)^2}{s} \right]$$

$$M^2 - \mu^2 \leq s \leq (M + \mu)^2$$