CM-P00057361

A CRITERION FOR THE POWER BEHAVIOUR OF THE SCATTERING AMPLITUDE

AT HIGH ENERGY

D. Bessis and T. Kinoshita *)

CERN - Geneva

A B S T R A C T

Within the framework of axiomatic field theory, we have obtained a necessary and sufficient condition for the existence of the limit of the quantity

$$\alpha(s,t) = \frac{\ln f(s,t)}{\ln s}, \quad 0 \leq t < 4\mu^2$$

when $|s| \rightarrow \infty$, $\xi < \arg s < \pi - \xi$, $\xi > 0$, where $f(s,t)$ is a crossing symmetric elastic scattering amplitude. It is found to be related to the existence of a certain mean of the phase of $f(s,t)$. When this mean does not exist, $\alpha(s,t)$ oscillates indefinitely as $|s|$ increases. However, the variation of $\alpha(s,t)$ with s is very slow, the "period" of oscillation being an extremely rapidly increasing function of s . This oscillation may also be present in $f(s,t)$ for physical values of s superimposed on the oscillations of shorter range. It seems very difficult to detect the existence of such a long range effect experimentally.

*) Permanent address : Laboratory of Nuclear Studies, Cornell University, Ithaca, New York (U.S.A.).

I. - INTRODUCTION

In studying the asymptotic behaviour of the elastic scattering amplitude $f(s,t)$ for $s \rightarrow +\infty$, ¹⁾ it would be useful to introduce the function

$$\alpha(s,t) = \frac{\ln f(s,t)}{\ln s}, \quad s \geq 4\mu^2, \quad (1)$$

and see if the limit of $\alpha(s,t)$ for $s \rightarrow +\infty$ exists or not. When such a limit exists, it will correspond to the asymptotic behaviour of the form $\beta(s,t)s^{\alpha(t)}$, where $\alpha(t)$ is the limit of (1) and $\beta(s,t)$ is a function which varies less rapidly than any power of s . In most physical theories, such as the Regge pole theory, the existence of the limit $\alpha(t)$ is taken for granted. Even in more general considerations the existence of $\alpha(t)$ is assumed implicitly or explicitly. It would therefore be natural to ask to what extent such an assumption can be justified within the general framework of axiomatic field theory.

It is immediately obvious, however, that this question is unanswerable with our present knowledge of field theory : we have not found any reason within axiomatic field theory, or even in the Mandelstam representation for that matter, which prevents $\alpha(s,t)$ from oscillating indefinitely as $s \rightarrow +\infty$, $f(s,t)$ being the value of an analytic function on the boundary of its analyticity domain. Thus the only sensible question we can ask at present is the following : what are the conditions under which the limit of $\alpha(s,t)$ exists when we let s go to infinity in the complex direction $\varepsilon < \arg s < \pi - \varepsilon$, ε being small but positive ? It is this question that we want to examine in this paper. Clearly, even if such a limit exists, it does not necessarily mean that $f(s,t)$ has a power behaviour as $s \rightarrow +\infty$ along the real axis. However, it will provide us with the best available parametrization of the scattering amplitude $f(s,t)$ averaged over a certain energy range.

2.

We know already that, for very large $|s|$, $\alpha(s,0)$ cannot exceed 1 because of the Froissart bound ²⁾, and that $\alpha(s,t)$ is bounded by 2 for $0 < t < 4\mu^2$ ³⁾. These give upper bounds for $\alpha(s,t)$. On the other hand, when t is in the interval $0 \leq t < 4\mu^2$, $\alpha(s,t)$ is bounded from below by -2 ⁴⁾. Thus, for large values of $|s|$, we can assert that

$$-2 \leq \alpha(s,t) \leq 2, \quad 0 \leq t < 4\mu^2. \quad (2)$$

Of course, it may well be that $\alpha(s,t)$, although it satisfies the bound (2), has no limit when $|s| \rightarrow \infty$, and oscillates indefinitely between its least upper bound $\bar{\alpha}(t)$ and its greatest lower bound $\underline{\alpha}(t)$, where

$$-2 \leq \underline{\alpha}(t) \leq \bar{\alpha}(t) \leq 2. \quad (3)$$

However, in the special case where $\arg f(s,t)$ has a limit for $s \rightarrow +\infty$, it has been shown that $\underline{\alpha}(t) = \bar{\alpha}(t)$ ⁵⁾. That is, the limit does exist in this case.

What we would like to find is a general criterion for the existence of the limit $\lim_{|s| \rightarrow \infty} \alpha(s,t)$, $\xi < \arg s < \pi - \xi$. In this paper we have derived such a condition starting from the analyticity of $f(s,t)$ in s , which is obtained within the framework of axiomatic field theory. It should be emphasized, however, that we have used neither the analyticity in t nor the s - t crossing symmetry. If this neglected information were taken into account, it might be possible to obtain further conditions for the existence of the limit.

Within this limitation our result for $t = 0$ can be expressed as follows : $\alpha(s,0)$ for $|s| \rightarrow \infty$, $\varepsilon < \arg s < \pi - \varepsilon$, has a limit if and only if

$$\lim_{\lambda \rightarrow +\infty} \frac{1}{\ln \lambda} \int_c^\lambda \frac{\arg f(s,0)}{s} ds \quad (4)$$

exists. Obviously this limit exists if $\arg f(s,0)$ has a limit for $s \rightarrow +\infty$, in agreement with the previous result ⁵⁾. However, the condition (4) is more general since it allows us to define $\alpha(0)$ of the forward scattering amplitude $f(s,0)$ even when $\arg f(s,0)$ has no limit.

Unfortunately we have not been able to rule out within our scheme the possibility that the limit (4) may not exist. In fact, Martin ⁶⁾ has constructed an example which satisfies all requirements of analyticity and unitarity but fails to give a definite value to (4). Thus we have to concur with the frequently expressed suspicion that the axiomatic field theory (or even the Mandelstam representation) may not contain enough information to rule out the possible oscillation of the total cross-section, which, for instance, would be needed to prove the so-called Pomeranchuk theorem.

Our result reveals, however, one novel feature which has not previously been suspected. It shows that the circumstance in which (4) does not exist is such that $\alpha(s,0)$ undergoes an extremely slow change as $|s| \rightarrow \infty$ and furthermore the "period" of oscillation increases very rapidly as a function of $|s|$, being of order $|s|^{2e^\pi} \approx |s|^{46}$ in Martin's example (see Section III). Short range oscillations of $\arg f(s,0)$ by itself will not affect the existence of (4). Clearly such a long range behaviour would be very difficult to detect experimentally. Indeed, inasmuch as we have not been able

4.

to exclude such an oscillation on theoretical grounds, we cannot hope, strictly speaking, to determine the asymptotic behaviour of $f(s,0)$ from the experiment, even if $f(s,0)$ seems to have a well defined power behaviour over any finite energy range.

These results for $t = 0$ can be extended without difficulty to the case $0 < t < 4\mu^2$.

II. - FORMULATION OF THE PROBLEM

The case $0 < t < 4\mu^2$ can be treated in the same manner as the case $t = 0$. We shall therefore discuss only the case $t = 0$ explicitly.

Since $f(s, 0)$ is crossing symmetric by assumption, it is convenient to introduce the new variable

$$z = \left(\frac{s - 2\mu^2}{2\mu^2} \right)^2 \quad (5)$$

and write the scattering amplitude as $f(z)$. Then $f(z)$ satisfies the once subtracted dispersion relation

$$f(z) = f(0) + \frac{z}{\pi} \int_1^{\infty} \frac{\text{Im} f(x+i0) dx}{x(x-z)}, \quad (6)$$

where we have taken account of the fact that $f(z)$ satisfies the Froissart bound for large z ²⁾.

We are interested in the limit of

$$\frac{\ln f(z)}{\ln z} \quad (7)$$

Our problem is to establish a necessary and sufficient condition for the existence of the limit of (7) for $|z| \rightarrow \infty$ in complex direction $\xi < \arg z \leq \pi$.

In the representation (6), the discontinuity $\text{Im} f(x+i0)$ is positive and never vanishes in the interval $(1, +\infty)$ ⁷⁾. Thus, if $\text{Im} f(x+i0)$ has the normal threshold behaviour $[\propto (x-1)^{\frac{1}{2}}]$ at $x=1$ (no zero energy resonance), $f(z)$ is a Herglotz function ⁴⁾.

6.

Instead of the function (7) itself, we shall study a slightly different function

$$g(z) = \frac{\varphi(z)}{\ln z} \quad \text{with} \quad \varphi(z) = \ln \left(\frac{f(z) - f(0)}{z(f(1) - f(0))} \right), \quad (8)$$

which has essentially the same asymptotic behaviour as (7) but has some nice features such as being Herglotz. It is easy to show that $g(z)$ has the following properties :

- 1) - it is holomorphic in the z plane with two cuts $(-\infty, 0)$ and $(1, +\infty)$;
- 2) - it is real and positive in the interval $(0, 1)$;
- 3) - for $|z| \rightarrow \infty$ it is bounded by a constant in all directions in the cut z plane ;

- 4) - (9)

$$\operatorname{Im} g(x+i0) = \frac{\operatorname{Im} \varphi(x+i0)}{\ln x} > 0 \quad \text{for } 1 < x < +\infty;$$

- 5) - (10)

$$\operatorname{Im} g(x+i0) = \frac{-\pi \varphi(x)}{\pi^2 + (\ln|x|)^2} > 0 \quad \text{for } -\infty < x < 0.$$

Furthermore the phase of $g(x+i0)$ lies between 0 and π everywhere on the real axis. All these properties are sufficient to show that $g(z)$ is a Herglotz function⁸⁾. In particular, noting that $g(0) = 0$, we can write a once-subtracted dispersion relation for $g(z)$:

$$g(z) = \frac{z}{\pi} \int_{-\infty}^0 \frac{\operatorname{Im} g(x+i0) dx}{x(x-z)} + \frac{z}{\pi} \int_1^{\infty} \frac{\operatorname{Im} g(x+i0) dx}{x(x-z)}. \quad (11)$$

To study the behaviour of $g(z)$ in the complex direction it is sufficient to examine it for $z \rightarrow +i\infty$, because, if the limit exists in this direction, $g(z)$ must have the same limit in all complex directions according to Montel's theorem⁹⁾. Furthermore $g(z)$ has then the same limit for $z \rightarrow -\infty$ (i.e., $\arg z = \pi$) since there is no difficulty in moving the left-hand cut. If the limit does not exist for $z \rightarrow +i\infty$, it does not exist for any other direction either.

It is obvious from (8) that $\operatorname{Im} g(i\lambda) \rightarrow 0$ and $\operatorname{Re} g(i\lambda)$ is bounded when $\lambda \rightarrow +\infty$. Namely we have

$$\lim_{\lambda \rightarrow +\infty} \left(\lambda \int_{-\infty}^0 \frac{\operatorname{Im} g(x+i0) dx}{x^2 + \lambda^2} + \lambda \int_1^{\infty} \frac{\operatorname{Im} g(x+i0) dx}{x^2 + \lambda^2} \right) = 0 \quad (12)$$

and

$$\lim_{\lambda \rightarrow +\infty} \left| \lambda^2 \int_{-\infty}^0 \frac{\operatorname{Im} g(x+i0) dx}{x(x^2 + \lambda^2)} + \lambda^2 \int_1^{\infty} \frac{\operatorname{Im} g(x+i0) dx}{x(x^2 + \lambda^2)} \right| < \text{const.} \quad (13)$$

It is seen from (13) that the existence of $g(+i\infty)$ is equivalent to the existence of the limit

$$\lim_{\lambda \rightarrow +\infty} \lambda^2 \int_c^{\infty} \frac{[\operatorname{Im} g(x+i0) - \operatorname{Im} g(-x+i0)] dx}{x(x^2 + \lambda^2)}, \quad (14)$$

where c is a positive constant which we choose larger than e^2 for convenience [see (A7)]. If one takes (12) into account, one sees that the existence of (14) is equivalent to that of the limit

$$\lim_{\lambda \rightarrow +\infty} \int_c^{\lambda} \frac{dx}{x} [\operatorname{Im} g(x+i0) - \operatorname{Im} g(-x+i0)], \quad (15)$$

as is shown in the Appendix, A). Substituting (9) and (10) into (15), one finds [see the Appendix, B)] that the problem reduces finally to the existence of the limit

$$\lim_{\lambda \rightarrow +\infty} \frac{1}{\ln \lambda} \int_c^\lambda \frac{\arg f(x + i0) dx}{x}. \quad (16)$$

Namely, if (16) exists, $\ln f(z)/\ln z$ has a limit in any complex direction except possibly for $\arg z = 0$. Conversely, if (16) does not exist, $\ln f(z)/\ln z$ has no limit in any complex direction. This is what we wanted to find in this paper.

In general, the existence of (16) does not guarantee the existence of the limit of $\ln f(z)/\ln z$ for $\arg z = 0$ even if z goes to infinity through complex values, e.g., $z = x + iy$, $y > 0$, $x \rightarrow +\infty$. As is shown in the Appendix, C), however, $g(x+iy)$ for large x can be regarded as a sum of two terms, one having a limit if (16) exists and the other having a limit if and only if $\operatorname{Im} f(x)$ satisfies a certain condition on the short range variation on the positive real axis.

III. - DISCUSSION

The net result of our consideration is the following :

$$\frac{\ln f(z)}{\ln z} \rightarrow \alpha \text{ for } |z| \rightarrow \infty, \varepsilon < \arg z \leq \pi, \varepsilon > 0, (17)$$

if and only if

$$\xi(\lambda) \equiv \frac{1}{\ln \lambda} \int_c^\lambda \frac{\arg f(x+i0)}{x} dx \rightarrow \beta \text{ for } \lambda \rightarrow +\infty. (18)$$

This is a purely axiomatic result. If we want to extend this to $\varepsilon = 0$, we must impose a certain condition on the smoothness of $f(z)$ on the physical cut.

We would now like to discuss the implication of this result. First, we see that if $\arg f(x)$ has a limit for $x \rightarrow +\infty$, (18) has a limit, too. On the other hand, it may happen that (18) has a limit even if $\arg f(x)$, which lies between 0 and π , has no limit. This takes place when $\arg f(x)$ oscillates between two bounds but the centre of the short range oscillation itself does not drift as $x \rightarrow +\infty$ in the sense described below.

To understand this point, let us examine under what circumstances the quantity (18) has no limit. To simplify the matter let us assume that $\arg f(x)$ takes the discrete value a_i in the interval (x_i, x_{i+1}) , $i = 1, 2, \dots$, where $0 < a_i < \pi$.

In the interval (x_1, x_2) , $\xi(\lambda)$ has a value less than a_1 and approaches a_1 as closely as we wish if we choose x_2 large enough. As λ moves into the next interval (x_2, x_3) , however, $\xi(\lambda)$ deviates slowly from a_1 and approaches a_2 if we choose x_3

sufficiently large. In this manner, if we choose the sequence of points x_i in an appropriate fashion, we can force $\xi(\lambda)$ to approach any sequence of values $a_1, a_2, \dots$, successively. If the sequence $\{a_i\}$ has no limit, so has $\xi(\lambda)$ no limit either. We note that the length of the interval (x_i, x_{i+1}) increases very rapidly as x_i increases. This argument can be extended easily to the case where $\arg f(x)$ varies continuously, and also to the general case where $\arg f(x)$ has short range oscillations superimposed on the long range variation.

Martin ⁶⁾ has given an explicit example for $f(z)$ such that $\xi(\lambda)$ has no limit. It is of the form

$$f(z) = \exp \left\{ a \left[\ln(1 + \sqrt{1-z}) \right] \sin \left(\ln \ln(1 + \sqrt{1-z}) + b \ln(1-z) \right) \right\}_{(19)}$$

where a and b are appropriate constants chosen in such a way that $f(z)$ is a Herglotz function with a proper asymptotic behaviour. In this example one sees that $\alpha(s, 0)$ comes back to its value at $z = z_1$ when z reaches the point $z_2 \approx (z_1)^{e^\pi} \approx (z_1)^{23}$. Thus the "period" of oscillation increases very rapidly with z_1 . It would be very difficult to observe such an effect in any experiment.

ACKNOWLEDGEMENTS

We would like to thank Dr. A. Martin for useful discussions and for giving us an explicit example. We wish to thank Professor L. Van Hove and Professor J. Prentki for the warm hospitality at CERN.

- APPENDIX -

A. - We shall first show the equivalence of (14) and (15). For this purpose we note that

$$\begin{aligned} & \lambda^2 \int_c^\infty \frac{\Im g(x) dx}{x(x^2 + \lambda^2)} \\ &= \int_c^\lambda \frac{\Im g(x) dx}{x} - \int_c^\lambda \frac{x \Im g(x) dx}{x^2 + \lambda^2} + \lambda^2 \int_\lambda^\infty \frac{\Im g(x) dx}{x(x^2 + \lambda^2)}. \end{aligned} \quad (A1)$$

The second and third terms can be majorized as follows :

$$\int_c^\lambda \frac{x \Im g(x) dx}{x^2 + \lambda^2} \leq \lambda \int_c^\lambda \frac{\Im g(x) dx}{x^2 + \lambda^2}, \quad (A2)$$

$$\lambda^2 \int_\lambda^\infty \frac{\Im g(x) dx}{x(x^2 + \lambda^2)} \leq \lambda \int_\lambda^\infty \frac{\Im g(x) dx}{x^2 + \lambda^2}. \quad (A3)$$

The right-hand sides of (A2) and (A3) tend to zero as $\lambda \rightarrow \infty$ as is seen from (12). The equivalence of (14) and (15) follows immediately.

B. - Next we show that (15) and (16) are equivalent. We note that, taking (9) and (10) into account, we can express (15) in the form

$$\int_c^\lambda \frac{dx}{x} \left[\frac{\Im \varphi(x)}{\ln x} - \frac{1}{\pi^2 + (\ln x)^2} \left\{ x \int_1^\infty \frac{\Im \varphi(x') dx'}{x'(x' + x)} + \varphi(0) \right\} \right].$$

Apart from the contribution coming from $\varphi(0)$ which is convergent, this formula can be written as a sum of three integrals

$$I_1 = \int_c^\lambda \frac{dx}{x} \left[\frac{\operatorname{Im} \varphi(x)}{\ln x} - \frac{x}{\pi^2 + (\ln x)^2} \int_c^\lambda \frac{\operatorname{Im} \varphi(x') dx'}{x'(x'+x)} \right], \quad (A4)$$

$$I_2 = - \int_c^\lambda \frac{dx}{\pi^2 + (\ln x)^2} \int_1^c \frac{\operatorname{Im} \varphi(x') dx'}{x'(x'+x)}, \quad (A5)$$

$$I_3 = - \int_c^\lambda \frac{dx}{\pi^2 + (\ln x)^2} \int_\lambda^\infty \frac{\operatorname{Im} \varphi(x') dx'}{x'(x'+x)}. \quad (A6)$$

Noting that $0 \leq \operatorname{Im} \varphi(x) \leq \pi$, we obtain

$$\int_\lambda^\infty \frac{\operatorname{Im} \varphi(x') dx'}{x'(x'+x)} < \frac{\pi}{x} \ln \left(1 + \frac{x}{\lambda} \right).$$

Thus we find

$$\begin{aligned} |I_3| &< \pi \int_c^\lambda \frac{dx \ln \left(1 + \frac{x}{\lambda} \right)}{x (\ln x)^2} \\ &< \frac{\pi}{\lambda} \int_c^\lambda \frac{dx}{(\ln x)^2} \\ &< \frac{\pi}{1 - \frac{2}{\ln c}} \frac{1}{(\ln \lambda)^2}, \end{aligned} \quad (A7)$$

if we choose a large enough c satisfying $1 - \frac{2}{\ln c} > 0$. We have performed partial integration in the last step. It is clear from (A7) that I_3 tends to zero as $\lambda \rightarrow +\infty$.

As is easily seen, I_2 is bounded and monotonic for all λ . Thus I_2 must converge to a finite limit as $\lambda \rightarrow +\infty$.

In order to estimate I_1 , we first note that the order of x - and x' - integration can be interchanged for any finite λ . The resulting expression can be written as a sum of two terms

$$I_1' = \int_c^\lambda \frac{dx \operatorname{Im} \varphi(x)}{x} \left[\frac{1}{\ln x} - \int_c^\infty \frac{dx'}{(x'+x)(\pi^2 + (\ln x')^2)} \right], \quad (\text{A8})$$

$$I_1'' = \int_c^\lambda \frac{dx \operatorname{Im} \varphi(x)}{x} \int_\lambda^\infty \frac{dx'}{(x'+x)[\pi^2 + (\ln x')^2]}. \quad (\text{A9})$$

In order to evaluate I_1' , we note that for large x

$$\begin{aligned} & \frac{1}{\ln x} - \int_c^\infty \frac{dx'}{(x'+x)(\pi^2 + (\ln x')^2)} \\ & \approx \int_x^\infty \frac{dx'}{x'(\ln x')^2} - \int_c^\infty \frac{dx'}{(x'+x)(\ln x')^2} \\ & = \int_x^\infty \frac{dx'}{(\ln x')^2} \left(\frac{1}{x'} - \frac{1}{x+x'} \right) - \int_c^x \frac{dx'}{(x'+x)(\ln x')^2}. \end{aligned} \quad (\text{A10})$$

The first term of (A10) is less than $\ln 2/(\ln x)^2$. The second term can be majorized by

$$\left(\int_c^x \frac{dx'}{(x'+x)^2} \right)^{\frac{1}{2}} \left(\int_c^x \frac{dx'}{(\ln x')^4} \right)^{\frac{1}{2}}, \quad (\text{A11})$$

which is less than $c/(\ln x)^2$. It is now easy to see that I_1' converges to a finite limit as $\lambda \rightarrow +\infty$.

Finally, noting that

$$\frac{1}{\ln(\lambda+x)} - \frac{\pi^2}{3(\ln \lambda)^3} < \int_{\lambda}^{\infty} \frac{dx'}{(x'+x)[\pi^2 + (\ln x')^2]} < \frac{1}{\ln \lambda},$$

we can bound I_1'' from above and below as

$$\int_c^{\lambda} \frac{dx \operatorname{Im} \varphi(x)}{x} \left[\frac{1}{\ln(\lambda+x)} - \frac{\pi^2}{3(\ln \lambda)^3} \right] < I_1'' < \int_c^{\lambda} \frac{dx \operatorname{Im} \varphi(x)}{x \ln \lambda}. \quad (\text{A12})$$

Since $0 < \operatorname{Im} \varphi(x) < \pi$, these bounds are both finite and their difference approaches zero as $\lambda \rightarrow +\infty$. However, this does not mean that I_1'' has a limit since it is not monotonic in λ as is obvious from (A9). Thus the existence or non-existence of the limit of (15) is equivalent to the existence or non-existence of the integral I_1'' , which in turn is equivalent to (16) as is seen from (A12).

C. - We shall now discuss the asymptotic behaviour of $g(x+iy)$ when $x \rightarrow +\infty$ with fixed $y > 0$. We write

$$g(x+iy) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(xx' - x^2 - y^2) \operatorname{Im} g(x') dx'}{x'((x' - x)^2 + y^2)} + \frac{iy}{\pi} \int_{-\infty}^{\infty} \frac{\operatorname{Im} g(x') dx'}{(x' - x)^2 + y^2}. \quad (\text{A13})$$

Noting that $\operatorname{Im} g(x') \leq [C/\ln(|x'|)]$ as $x' \rightarrow \pm\infty$, it is easy to show that

$$\lim_{x \rightarrow +\infty} \operatorname{Im} g(x+iy) = 0.$$

Thus we have only to discuss the real part. Apart from a finite term coming from the integration over the interval $(-c, c)$, $\operatorname{Re} g(x+iy)$ can be written in the form

$$\int_c^{\infty} \frac{dx'}{x'} \left[\frac{xx' - x^2 - y^2}{(x' - x)^2 + y^2} \operatorname{Im} g(x') + \frac{xx' + x^2 + y^2}{(x' + x)^2 + y^2} \operatorname{Im} g(-x') \right] \quad (\text{A14})$$

which may be split into two parts

$$J_1 = \int_c^{\infty} \frac{dx'}{x'} \left\{ \operatorname{Im} g(-x') - \operatorname{Im} g(x') \right\} \frac{xx' + x^2 + y^2}{(x' + x)^2 + y^2} \quad (\text{A15})$$

and

$$J_2 = \int_c^{\infty} \frac{\operatorname{Im} g(x') dx'}{x'} \left\{ \frac{xx' - x^2 - y^2}{(x' - x)^2 + y^2} + \frac{xx' + x^2 + y^2}{(x' + x)^2 + y^2} \right\}. \quad (A16)$$

We shall first show that J_1 has a limit for $x \rightarrow +\infty$ if and only if (15) has a limit. Since the contribution to J_1 from the interval $(x, +\infty)$ tends to zero as $x \rightarrow +\infty$, we can ignore this part in our consideration. Thus we have only to show the equivalence of

$$\lim_{x \rightarrow +\infty} \int_c^x \frac{dx'}{x'} \left\{ \operatorname{Im} g(-x') - \operatorname{Im} g(x') \right\} \frac{xx' + x^2 + y^2}{(x' + x)^2 + y^2} \quad (A17)$$

and

$$\lim_{x \rightarrow +\infty} \int_c^x \frac{dx'}{x'} \left\{ \operatorname{Im} g(-x') - \operatorname{Im} g(x') \right\}. \quad (A18)$$

In this discussion y in (A17) is not important. We shall therefore consider

$$\lim_{x \rightarrow +\infty} \int_c^x \frac{dx'}{x'} \left\{ \operatorname{Im} g(-x') - \operatorname{Im} g(x') \right\} \frac{x}{x' + x}. \quad (A19)$$

This can be rewritten as

$$\begin{aligned} & \lim_{x \rightarrow +\infty} \int_c^x \frac{dx'}{x'} \left\{ \operatorname{Im} g(-x') - \operatorname{Im} g(x') \right\} \\ & - \lim_{x \rightarrow +\infty} \int_c^x \frac{dx'}{x' + x} \left\{ \operatorname{Im} g(-x') - \operatorname{Im} g(x') \right\}. \end{aligned}$$

But the second limit vanishes because $|\operatorname{Im} g(-x') - \operatorname{Im} g(x')| < \frac{C}{\ln x'}$.
This proves our assertion for J_1 .

Finally we consider J_2 . Clearly it is bounded for $x \rightarrow +\infty$. However, this does not mean that it has a limit since the integrand is not positive definite. In fact, for $y > 0$ and very large x , J_2 can be regarded as a certain average of $\operatorname{Im} g(x)$, whose main contribution arises from the interval $[(1-\varepsilon)x, (1+\varepsilon)x]$, $0 < \varepsilon \ll 1$. Thus, unlike J_1 , J_2 is sensitive to the short range behaviour of $\operatorname{Im} g(x)$ on the positive real axis. If $\operatorname{Im} g(x)$ can be regarded as a constant in this interval, it is easily seen that J_2 tends to zero like $1/\ln x$ as $x \rightarrow +\infty$. In general there is of course no reason why $\operatorname{Im} g(x)$ should be so smooth and thus it is not possible to claim the existence of the limit for J_2 .

-- R E F E R E N C E S --

- 1) To avoid unnecessary complication, we shall consider the elastic scattering of two neutral scalar particles of equal mass μ . Furthermore we shall treat explicitly only the case where $f(s,t)$ satisfies the s-u crossing symmetry. In the more general case our method applies to the s-u symmetrized amplitude. s, u, t are the familiar Mandelstam variables.
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