CM-P00057563

Ref. TH. 764

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DOUBLE REGGE ANALYSIS OF HIGH ENERGY EXPERIMENTS
PRODUCING THREE FINAL PARTICLES

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A B S T R A C T

A general systematic analysis based on a double Regge model is suggested for high-energy hadron collisions producing three particles in the final state. It is intended only for events in the central region of the Dalitz plot where all two particle invariant masses are high. The procedure is described in detail and applied to two existing experiments :

$$A) \quad \bar{\pi}^+ p \rightarrow \bar{\pi}^+ \pi^0 p \quad \text{at} \quad 8 \text{ GeV}/c$$

$$B) \quad \bar{\pi}^- p \rightarrow K_1^0 K_1^0 n \quad \text{at} \quad 12 \text{ GeV}/c.$$

The results are encouraging.

I. -

INTRODUCTION

In this paper a double Regge analysis recently suggested for three-body events (Fig. 1) in high energy hadron collisions ¹⁾ is applied to two experiments :

(A)

$$\begin{aligned} & \pi^+ p \text{ at } 8 \text{ GeV}/c \quad 2) \\ & \pi^+ p \rightarrow \pi^+ \pi^0 p \end{aligned} \quad (1)$$

(B) $\pi^- p$ at $12 \text{ GeV}/c$ ³⁾

$$\pi^- p \rightarrow K_1^0 K_1^0 n \quad (2)$$

The model as proposed is expected to be valid only for non-resonant events in the central region of the Dalitz plot where all two-body effective masses are large (region IV of Fig. 2). In this region it is assumed that the reaction of Fig. 1 is dominated by double Regge graphs of the form shown in Fig. 3, and by similar graphs with the three final particles permuted. The suggested analysis is based on the observation that the Regge vertices are sharply peaked at small values of the four momentum transfers, and may be reasonably well-approximated by exponentials, e.g., for Fig. 3, $\exp(\Omega_a t_{13})$ and $\exp(\Omega_b t_{25})$. In accordance with current Regge models for two-particle reactions we assume that

$$\Omega_a = a + \alpha'_a \ln s_{34} \quad (3)$$

$$\Omega_b = b + \alpha'_b \ln s_{45} \quad (4)$$

where a and b are constant parameters. This "diffractive" property of the vertices leads to several important kinematical consequences ¹⁾.

2.

For example, for the graph shown in Fig. 3 :

- (C1) - The longitudinal momenta $p_{\ell i}$ of the final particles, measured in the total c.m. frame in the direction of particle 1, tend to be ordered algebraically as follows : $p_{\ell 3} > p_{\ell 4} > p_{\ell 5}$;
- (C2) - The events populate only one corner of the Dalitz plot, namely that corresponding to small values of $\xi = s_{34}s_{45}/s$. (Fig. 2) ;
- (C3) - At a fixed point on the Dalitz plot, the kinematics is made very rigid. This in general leads to a peak of the distribution function in almost any variable. A particularly interesting case is the distribution in the angle θ_{14} between particles 1 and 4 in the total c.m. system. The peak in θ_{14} is approximately Gaussian [Eq.(A11)]. Its position and width both depend on the "diffractive" parameters a and b .

For similar graphs with the final particles permuted, which may also contribute to the same reaction, similar properties will apply, only with the indices interchanged.

In view of these properties, the following procedure for analysis is suggested ¹⁾ :

- (P1) - Events in region (IV) of the Dalitz plot (Fig. 2) are separated first into six categories, according to the six different orderings of the longitudinal momenta of the final particles. Each category then corresponds to only one double Regge graph.
- (P2) - Events of each category are plotted on a separate Dalitz plot. The model predicts that in each category only one corner will be populated.
- (P3) - In each category, where by definition $p_{\ell 3} \geq p_{\ell 4} \geq p_{\ell 5}$, the experimental value of θ_{14} for each event is computed. A best fit for θ_{14} over all events in the category is then made with the theoretical peak in order to determine the diffractive parameters a and b .

(P4) - These values of a and b are then used to predict other quantities, such as the distribution of events on the Dalitz plot.

The procedure can be applied systematically to all three-body processes including cases where some of the final particles are metastable resonances. The mathematical formulae and technical details relevant for the analysis are collected in the Appendix.

In the following Sections, the method is illustrated in detail for the two previously quoted experiments. Our interests differ somewhat in the two cases. The reaction of experiment (A) receives contributions from several double Regge graphs, and is thus ideal for testing our separation procedure (P1) and (P2). Serious applications of (P3) and (P4) however, are limited both by the incoming energy and the present low statistics ⁴⁾. In experiment (B), only one graph contributes ¹⁾, so that (P1) and (P2) are not so relevant, but the higher statistics and energy make an application of (P3) and (P4) more quantitatively significant. Wherever possible in the analysis, we make use of data from two particle reactions. All other parameters we try to determine from the present experiments within the framework of our model.

2. - π^+p EXPERIMENT (A) AT 8 GeV/c

The data consist of 754 events obtained by the Aachen-Berlin-CERN collaboration in a bubble chamber experiment done at CERN ^{2),5)}. The region (IV) on the Dalitz plot (Fig. 2) is defined by

$$s_{\pi^+\pi^0} > 2 \text{ GeV}^2, \quad s_{\pi^+p} > 3 \text{ GeV}^2, \quad s_{\pi^0p} > 3 \text{ GeV}^2. \quad (5)$$

The limits (5) have been chosen quite arbitrarily. They must not, however, be taken so low as to make the application of any Regge model absurd. In fact, limits somewhat higher than (5) would be preferred, but one is here restricted both by the incident energy and by the low statistics. Discarding events outside this region, one is then left with 78 events. The Dalitz plot for these is shown in Fig. 4(i).

(P1) - The double Regge graphs which can contribute to this reaction are shown in Fig. 5. In these only the leading Regge pole(s) is retained. The six graphs correspond to the six different orderings of the longitudinal momenta of the final particles. Among these graphs, (e) and (f) are expected to be negligible since they require double baryon exchange, and are not enhanced by the kinematical effect discussed in Section IIIb of Ref. ¹⁾.

Separating then the 78 events into six categories according to the ordering of $p_{\ell i}$, one obtains the following :

(a)	$p_{\ell\pi^+} > p_{\ell\pi^0} > p_{\ell p}$	38 events
(b)	$p_{\ell\pi^0} > p_{\ell\pi^+} > p_{\ell p}$	22 events
(c)	$p_{\ell\pi^+} > p_{\ell p} > p_{\ell\pi^0}$	13 events
(d)	$p_{\ell\pi^0} > p_{\ell p} > p_{\ell\pi^+}$	5 events
(e)	$p_{\ell p} > p_{\ell\pi^+} > p_{\ell\pi^0}$	0 event
(f)	$p_{\ell p} > p_{\ell\pi^0} > p_{\ell\pi^+}$	0 event.

- (P2) - Plotting the events of each category on a separate Dalitz plot, we obtain for categories (a)-(c) Fig. 6(i)a-c. The cornering effect in (a) and (b) is clearly discernable. In (c) the corner is displaced to one side by a kinematic effect due to the heavy mass of the proton being exchanged.
- (P3) - We next analyze the events in each category separately according to the procedure described in the Appendix. First we assume that we know from two-particle reactions the following ⁶⁾

$$\begin{aligned}
 \alpha_p(0) &= 1 \\
 \alpha_\rho(0) &\approx \alpha_{A_2}(0) \approx \alpha_\omega(0) \approx 0.5 \\
 \alpha_p(0) &\approx \alpha_n(0) \approx -0.3 \\
 \alpha'_p(0) &\approx 0 \\
 \alpha'_\rho(0) &\approx \alpha'_{A_2}(0) \approx \alpha'_\omega(0) \approx \alpha'_p(0) \approx \alpha'_n(0) \approx 1 \text{ GeV}^{-2}
 \end{aligned} \tag{6}$$

Assuming the approximate form (A1) for the amplitudes, one is then left, apart from the normalization factor B_0 , with only two parameters for each graph in Fig. 5, - namely the "diffractive" parameters a and b .

Because of the poor statistics, and the low limits (5) which imply sizeable contamination from, e.g., resonances, we have not been successful in applying the condition (A15) for the determination of the parameters a and b ⁷⁾. We shall therefore proceed instead as follows. For each graph we first take some arbitrary value for the parameter a . The condition (A13) then determines b uniquely. With these values of a and b , the distribution function on the Dalitz plot can be calculated using (A19). Generating then random events with the weight (A19), one can simulate a Dalitz plot for events in this category. The parameter a is adjusted until satisfactory agreement is obtained between this and the corresponding experimental plot. The results of this calculation for categories (a), (b) and (c) are collected

in Table I. This method of determining a and b is of course only qualitative. It is sensitive only to changes of a and b by about $0.5 \sim 1.0 \text{ GeV}^{-2}$.

The histogram for the distribution of the 38 events of category (a) in $(\theta_{\text{ex}}^i - \theta_0^o)$ is shown in Fig. 7. Histograms for categories (b) and (c) are similar; in each case, a peak exists. The experimental widths of these peaks are given in column 6 of Table I, and are to be compared with the theoretical widths (column 7) as calculated from the listed values of the "diffractive" parameters.

- (P4) - The values of a and b so determined may then be used to predict other distributions. The simulated Dalitz plots shown in Fig. 6(ii)a-c have been obtained by random generation, where the number of events has been normalized to the experimental number in each case. The sum of these is shown in Fig. 4(ii), which is to be compared with the original experimental Dalitz plot Fig. 4(i). Examples of other distributions for the events in category (a) calculated by the Monte Carlo method are shown in Figs. 8-12.

The separation procedure (P1) is of course not absolute; its validity depends on the sharpness of the diffractive vertices, i.e., on a and b . To check the consistency of our method, we have calculated from the listed values of a and b , the fraction of events in each category which have a "wrong" ordering of longitudinal momenta. The result is listed in column 8 of Table I.

3. - $\bar{u}p$ EXPERIMENT (B) AT 12 GeV/c

The data consist of 1425 events obtained by the CERN-ETH collaboration ³⁾ with a magnetic spark chamber. Because of the better statistics and higher incident energy of this experiment, we can afford somewhat better limits than (5) in the definition of region (IV), Fig.2. Restricting ourselves to events with

$$s_{KK} > 3.5 \text{ GeV}^2, \quad s_{Kn} > 3.5 \text{ GeV}^2 \quad (7)$$

we are then left with 224 events. The Dalitz plot for these is shown in Fig. 13, where every event has been plotted twice, since experimentally the two K mesons are not distinguished. One notes also that the detection efficiency of the apparatus varies considerably over the Dalitz plot ⁸⁾. In Fig. 13, of course, only the raw data have been plotted. In all following calculations, however, each event is given its proper weight as estimated from the geometry of the apparatus.

(P1) - The quantum numbers of this reaction are such that only one double Regge graph can contribute, namely that of Fig. 14 ¹⁾. This implies in our model that of the three possible categories :
 (a) $p_{\ell K_1^0} > p_{\ell K_1^0} > p_{\ell n}$, (b) $p_{\ell K_1^0} > p_{\ell n} > p_{\ell K_1^0}$,
 (c) $p_{\ell n} > p_{\ell K_1^0} > p_{\ell K_1^0}$, only (a) is appreciable. Experimentally, there is no event out of the 224 satisfying (7) which belongs to either (b) or (c).

(P2) - Moreover, the model implies that, within the limit of validity of this separation procedure, $p_{\ell K_0} > p_{\ell \bar{K}_0} > p_{\ell n}$, so that K_0 and $\bar{K}_0$ are distinguished by their longitudinal momenta. With this identification, the same events yield the Dalitz plot Fig. 15, where each event has been plotted only once. The cornering effect is clearly seen. We note again that in Fig. 15 the raw data have been plotted ; there is thus a concentration of events near the centre of the Dalitz plot which is mainly due only to a peak there in the detection efficiency of the apparatus.

(P3) - The higher statistics and lower background contamination in this case permit an attempt on the determination of unknown parameters via condition (A15). First from two-particle reaction data, we take ⁹⁾

$$\begin{aligned}\alpha_{\rho}(0) &\approx \alpha_{A_2}(0) \approx 0.5 \\ \alpha_{K^*}(0) &\approx \alpha_{K^{**}}(0) \approx 0.3 \\ \alpha'_{\rho}(0) &\approx \alpha'_{A_2}(0) \approx 1.0 \text{ GeV}^{-2}.\end{aligned}\tag{8}$$

Since there does not as yet exist direct evidence for α'_{K^*} and $\alpha'_{K^{**}}$ in two-particle reactions, we shall try to determine these also from the present data, assuming that $\alpha'_{K^*} \approx \alpha'_{K^{**}}$.

Condition (A15) then determines a , b and α'_{K^*} uniquely. The results of this calculation are collected in Table II. The slow dependence on the background control parameter μ (see Appendix), suggests that the sample is parity pure and contamination rather small. In order to check whether our parameters a , b and α'_{K^*} give approximately the correct dependence on energy, we have also repeated the calculation with different limits on s_{KK} and s_{Kn} . The parameters are found to be stable against moderate variations. Examples of these checks are quoted also in Table II.

One interesting physical point to note in the result is the small value consistently obtained for α'_{K^*} , which reflects a much slower energy dependence of the meson vertex a than the nucleon vertex b [$\alpha'_{\rho} = 1$]. At this stage, we do not wish to interpret α'_{K^*} as the slope of the K^* and K^{**} trajectories. We consider it only as a convenient parametrization of the vertex's dependence on energy.

In Fig. 16, we present the (weighted) histogram for the distribution of the 224 events satisfying (7) as a function of $(\theta_{\text{ex}}^i - \theta_o^i)$ for the parameters given in the first row of Table II. It compares favourably with the Gaussian peak predicted by the model.

(P4) - With all the parameters now determined, distributions in any other variables can be calculated. In Figs. 17-21 are presented examples of such distributions for events satisfying (7). They have been calculated by the Monte Carlo method from the parameters given in the first row of Table II, and are to be compared to the corresponding experimental distributions given in the same figures. In order to check the dependence on energy, we have calculated with the same parameters again the same distributions for events with $s_{\text{KK}} > 5 \text{ GeV}^2$ and $s_{\text{Kn}} > 5 \text{ GeV}^2$. In Figs. 22, 23, we present the two histograms which differ most from those with $s_{\text{KK}} > 3.5 \text{ GeV}^2$ and $s_{\text{Kn}} > 3.5 \text{ GeV}^2$. It is seen that some features dependent on the limits are faithfully reproduced by the model. In all the histograms, both experimental and theoretical, of Figs. 17-23, the particles have been labelled according to the convention $p_{\ell 3} > p_{\ell 4} > p_{\ell 5}$. The typical errors shown are only statistical.

No simulated Dalitz plot is shown since it cannot be compared with the raw data experimental plot of Fig. 15, because of the variation in detection efficiency.

The expected fractions of wrongly-ordered events have also been calculated and found to be 23% and 22%, respectively, for the limits at 3.5 GeV^2 and 5.0 GeV^2 . This unpleasantly high percentage of wrong-ordering, in this case almost entirely between the two mesons, is due to the unusual flatness of the meson vertex. It affects only the fit (P3), its effect on the histograms Figs. 17-23 having already been accounted for.

4. - REMARKS

Our aim in this paper is to construct a possible systematic analysis of the three-body final states in high energy collisions. For reasons of generality therefore, we have preferred to work always within the Regge framework. It is clear, however, that up to now, the full Regge model is not really necessary for our analysis. What is truly essential is just the fact that both the upper and lower vertices in the graph of Fig. 3 are peaked in t , a property which is shared by any double-peripheral model ¹⁰⁾ modified when necessary by form factors ¹¹⁾ including for example the Deck effect ¹²⁾.

In all our analyses reported in Sections 2 and 3, we have encountered only three factors which may properly be assigned to the Regge model :

- (i) - the existence of only one graph in reaction (2) ; this depends on the quantum numbers of existing Regge trajectories ;
- (ii) - the fact that the substitution of Regge pole intercepts $\alpha_i(0)$ obtained in two-particle reactions seems to give the correct distributions ; in this connection it should be noted that $\alpha_i(0)$ play no role in our determination of parameters via (A13) and (A15) ;
- (iii) - the fact that $\alpha_\rho^! = 1$ and $\alpha_{K^*}^! \approx 0$ in Section 3 ; this points definitely to a relatively different dependence on the energy of the two vertices, though the absolute energy dependence of either vertex is itself uncertain.

It cannot be claimed, however, that any of these have been well substantiated.

The proposed analysis is expected to apply to all high energy reactions with three-body final states, including cases where one or more of the final particles are metastable. It should be stressed, however, that only events in the centre of the Dalitz plot can be so analyzed. Application of the suggested procedure, which is specifically parametrized

for the "asymptotic" region, to events with small s_{ij} is meaningless. For example, by (3) and (4), the vertices will not be diffractive at all for s_{ij} small.

It may be questioned whether this restriction to events with large s_{ij} is ever compatible with the cornering effect ; in other words, whether a double Regge graph will ever be important in production processes. The answer, in general, is yes. The lower limits on s_{ij} required by the Regge model are constants independent on the incoming energy s , whereas the size of the corner populated by a double Regge graph is, according to Eq. (19) of Ref. ¹⁾ and (A1) of this paper, given by

$$s_{34} s_{45} \lesssim s \left[\left\{ (2 \Omega_a)^{-1/2} + (2 \Omega_b)^{-1/2} \right\}^2 + m_4^2 \right] \quad (9)$$

and grows in proportion to the incoming energy and the size of the Dalitz plot. Double Regge graphs are thus expected to increase in importance for increasing incident energy.

The parameters we have chosen are supposed to be constants in the Regge model, so that once determined for a reaction at one incident energy, they allow us in principle to predict all distributions as well as the absolute cross-section for the same reaction at all other energies.

ACKNOWLEDGEMENTS

We are indebted to Drs. L. Bertocchi, H. Högaasen, J. Ranft, L. Van Hove and K. Zalewski for numerous suggestions and discussions, and to the Aachen-Berlin-CERN and CERN-ETH collaborations for permission to use their unpublished data.

Two of us (K.K. and G.R.) wish to thank Professors L. Van Hove and J. Prentki for the kind hospitality extended to them at the Theoretical Study Division at CERN.

- APPENDIX -

The following approximate form of the amplitude has been suggested in Ref. ¹⁾ for the graph in Fig. 3

$$|A|^2 = B_0 \exp[2(\Omega_a t_{13} + \Omega_b t_{25})] s_{34}^{2\alpha_a} s_{45}^{2\alpha_b} \quad (A1)$$

with Ω_a and Ω_b as given in (3) and (4).

At a fixed point on the Dalitz plot, i.e., fixed s , s_{34} and s_{45} , the distribution in $\theta = \theta_{14}$ is as follows ¹⁾:

$$q(\theta) \sim e^{C \cos \theta} I_0(D \sin \theta) \quad (A2)$$

where

$$C = \sqrt{\frac{\lambda(s, m_1^2, m_2^2)}{s^2 \lambda(s, s_{35}, m_4^2)}} \left\{ \Omega_b \left[s(s_{45} - s_{34} + m_3^2 - m_5^2 - 2m_4^2) + (s_{34} - m_5^2)(s_{34} + s_{45} - m_3^2 - m_5^2) \right] - \Omega_a \left[s(s_{34} - s_{45} - m_3^2 + m_5^2 - 2m_4^2) + (s_{45} - m_3^2)(s_{34} + s_{45} - m_3^2 - m_5^2) \right] \right\} \quad (A3)$$

$$D = 2(\Omega_a + \Omega_b) \sqrt{\frac{-\lambda(s, m_1^2, m_2^2) g(s_{34}, s_{45}, s, m_4^2, m_3^2, m_5^2)}{s \lambda(s, s_{35}, m_4^2)}} \quad (A4)$$

with

$$\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx \quad (\text{A5})$$

$$\begin{aligned} U(x, y, z, u, v, w) = & x^2y + xy^2 + z^2u + v^2w + vw^2 + zu^2 \\ & + xzw + xuv + yzv + yuw - xy(z+u+v+w) \\ & - zu(x+y+v+w) - vw(x+y+z+u) . \end{aligned} \quad (\text{A6})$$

In practice, in the central region of the Dalitz plot in which we are interested, $D \sin \theta \gg 1$, so that the Bessel function $I_0(z)$ is well approximated by $e^z/\sqrt{z}$, thus

$$\rho(\theta) \sim \frac{e^{C \cos \theta + D \sin \theta}}{\sqrt{D \sin \theta}} . \quad (\text{A7})$$

The exponential in (A7) has a maximum at $\theta = \theta_0$, where

$$\tan \theta_0 = \frac{D}{C} . \quad (\text{A8})$$

It is convenient to rewrite (A7) as

$$\rho(\theta) \sim \frac{\exp[v \cos(\theta - \theta_0)]}{\sqrt{D \sin \theta}} \quad (\text{A9})$$

where

$$v = \sqrt{C^2 + D^2} \quad (\text{A10})$$

In all cases so far encountered, the peak in (A9) at θ_0 is sharp with $\sqrt{(\theta - \theta_0)^2} \sim 10^\circ = 0.17$ radian, so that it is sufficient to keep terms of leading order in $(\theta - \theta_0)^2$. We have then

$$\rho(\theta) \sim \exp\left[-\frac{\nu}{2}(\theta - \theta_0)^2\right], \quad (\text{A11})$$

a simple Gaussian distribution, with

$$\overline{(\theta - \theta_0)^2} = \frac{1}{\nu}. \quad (\text{A12})$$

Both the position of the peak θ_0 , and the "width" $\nu^{-\frac{1}{2}}$ (13) depend via (A3) and (A4) on s_{34} and s_{45} , i.e., on the position on the Dalitz plot. However, in all cases studied, the "width" $\nu^{-\frac{1}{2}}$ varies at most by only $\sim 1^\circ$ in the region of the Dalitz plot under consideration, and can for most purposes be taken as constant. We note also that at a fixed point on the Dalitz plot, the position of the peak θ_0 depends by (A8) only on Ω_a/Ω_b , i.e., only on the relative sharpness of the two "diffractive" vertices, and not on Ω_a and Ω_b individually. The reason for this is intuitively clear from the kinematics.

The existence or otherwise of this peak in $\theta = \theta_{14}$ is already a good check for the validity of the present model. Moreover, since θ_0 , as defined by (A8) can vary over a wide range, it can yield a quite sensitive means for determining unknown parameters of the model, e.g., the "diffractive" parameters a and b . For this we propose the following procedure. For each experimental event i , with two-particle invariant masses s_{34}^i and s_{45}^i , we denote by θ_{ex}^i the experimental value for $\theta = \theta_{14}$ in the total c.m. frame. On the other hand, the theoretical value θ_0^i depends, via (A8), on s_{34}^i and s_{45}^i , and on the diffractive parameters a and b . We require first then that a and b be such that

$$F_i(a, b) \equiv \sum_i [\theta_{\text{ex}}^i - \theta_0^i(a, b)]^2 = \min. \quad (\text{A13})$$

where the sum is meant to run over all events in the category.

The condition (A13) by itself is in practice insufficient to determine a and b uniquely, since θ_0 depends only on the ratio between Ω_a and Ω_b and not on their actual values (the slow dependence on s_{34} and s_{45} due to the log terms in (3) and (4) matters little in practice for the present experiments). We shall therefore supplement (A13) by imposing the condition that the distribution of θ_{ex}^i about θ_0^i be the same as (A11), e.g., by requiring

$$\sum_i (\theta_{ex}^i - \theta_0^i)^2 = \sum_i 1/v^i \quad (A14)$$

with v^i given by (A10). This we can do since v is known to be almost constant throughout the region of interest in the Dalitz plot. Combining conditions (A13) and (A14), we require then that a and b be such that

$$F(a,b) \equiv F_1(a,b) + \lambda F_2(a,b) = \min. \quad (A15)$$

where

$$F_2(a,b) \equiv \left[\sqrt{\sum_i (\theta_{ex}^i - \theta_0^i)^2} - \sqrt{\sum_i 1/v^i} \right]^2 \quad (A16)$$

and $\lambda > 0$ is a Lagrange parameter. The condition (A15) is in general sufficient to determine the parameters a and b uniquely. The result should not depend on the control parameter μ so long as it is sufficiently large. In actual practice, for angles measured in radians, we have found that values of λ , $10 < \lambda < 100$, are convenient.

The condition (A14), and hence (A15) also, are expected to be accurate so long as the sample of events is pure, i.e., containing mainly double Regge events and little else. If the sample is contaminated, say by resonances between two final particles, there will in general be

a background superimposed on the distribution (A11). The condition (A14) which stresses events far from the centre of the peak will thus yield too large a value for the width $(\nu)^{-\frac{1}{2}}$, i.e., too small values for the parameters a and b . Because of this we propose the following technique for background control. In place of (A14) we require

$$\begin{aligned} & \sum_i (\theta_{ex}^i - \theta_o^i)^2 \exp[-\mu (\theta_{ex}^i - \theta_o^i)^2] \\ &= \sum_i \overline{(\theta - \theta_o^i)^2 \exp[-\mu (\theta - \theta_o^i)^2]} = \sum_i \sqrt{\frac{\nu^i/2}{(\mu + \nu^i/2)^3}} \end{aligned} \quad (A17)$$

where a bar denotes an average over the distribution (A11). Equation (A17) reduces identically to (A14) when $\mu = 0$, and even for $\mu \neq 0$ would yield the same condition as (A14) on a and b if the sample is pure. However, for $\mu > 0$, background contribution to the first sum will be suppressed. In the condition (A15) we have then instead of (A16)

$$F_2(a, b) \equiv \left[\sqrt{\sum_i (\theta_{ex}^i - \theta_o^i)^2 \exp[-\mu (\theta_{ex}^i - \theta_o^i)^2]} - \sqrt{\sum_i \sqrt{\frac{\nu^i/2}{(\mu + \nu^i/2)^3}}} \right] \quad (A18)$$

If the sample is reasonably pure, the values of the parameters a and b as determined from (A15) and (A18) should not depend strongly on the parameter μ . This thus affords a control on the background and a check on the values of the parameters so determined. Note, however, that μ should not be taken $\mu > \nu^i/2$, otherwise (A15) may become unstable. The convenient range for μ is found to be $0 < \mu < 15$, for angles measured in radians.

We note also that the condition (A13) attempts to fit for each event a number, namely the angle θ_{14} , and is thus sensitive and likely to apply even for low statistics. Whereas, condition (A14) or (A17) attempt to fit the distribution of events about a peak, and is

expected to be meaningful only when statistics is good. For experiments with low statistics and large background, one can only determine the ratio Ω_a/Ω_b via (A13); the actual values of Ω_a and Ω_b may have to be determined by other means (see Section 2).

The parameters so determined can then be used to calculate distributions in any other variable(s). For example, the distribution function over the Dalitz plot can be obtained by integrating (A2) over θ . This integral can be performed in closed form, yielding ¹⁾,

$$\frac{\partial^2 \sigma}{\partial s_{34} \partial s_{45}} = \frac{B_0}{(4\pi)^3} \frac{s_{34}^{2\alpha_a} s_{45}^{2\alpha_b}}{\lambda(s, m_1^2, m_2^2)} \cdot$$

$$\exp\left\{\Omega_a\left[s_{45}-s+m_1^2+m_2^2+m_3^2-\frac{1}{s}(m_2^2-m_1^2)(s_{45}-m_3^2)\right]\right.$$

$$\left.+\Omega_b\left[s_{34}-s+m_1^2+m_2^2+m_5^2+\frac{1}{s}(m_2^2-m_1^2)(s_{34}-m_5^2)\right]\right\} \times$$

$$\sinh \left| \frac{\lambda(s, m_1^2, m_2^2)}{s^2} \left\{ \Omega_a^2 \lambda(s, s_{45}, m_3^2) + \Omega_b^2 \lambda(s, s_{34}, m_5^2) \right. \right.$$

$$\left. + 2\Omega_a \Omega_b \left[s(s-s_{34}-s_{45}-m_3^2-m_5^2+2m_4^2) - (s_{34}-m_5^2)(s_{45}-m_3^2) \right] \right\}$$

$$\times \frac{\left\{ \Omega_a^2 \lambda(s, s_{45}, m_3^2) + \Omega_b^2 \lambda(s, s_{34}, m_5^2) + 2\Omega_a \Omega_b \left[s(s-s_{34}-s_{45} \right. \right. \right.$$

$$\left. \left. - m_3^2 - m_5^2 + 2m_4^2) - (s_{34}-m_5^2)(s_{45}-m_3^2) \right] \right\}}{.}$$

(A19)

Other theoretical distributions are most conveniently obtained by Monte Carlo calculations. A particularly interesting case is the distribution in the angle φ as defined in Ref. ¹⁾

$$\cos \varphi \equiv \frac{\varepsilon_{\mu\nu\rho\sigma} p_1^\mu p_3^\nu p_4^\rho \varepsilon_{\alpha\beta\gamma\delta} p_2^\alpha p_5^\beta p_4^\gamma}{|\varepsilon_{\mu\nu\rho\sigma} p_1^\mu p_3^\nu p_4^\rho| |\varepsilon_{\alpha\beta\gamma\delta} p_2^\alpha p_5^\beta p_4^\gamma|}$$

(A20)

This angle occurs naturally in the coupling vertex between two Regge poles and a particle. In all our calculations so far, the dependence of such a vertex on φ is assumed to be weak and neglected. A careful check on this assumption is thus of much theoretical interest. The diffractive vertices imply in general a peak in the amplitude at $\varphi = \pi^{-1}$. This peak may, however, be much suppressed by phase space when the mass m_4 is large, as, e.g., in the baryon-exchange graphs (c) and (d) of Fig. 5.

- Table I -

Graph (Fig. 5)	Final state	No of events	a (GeV) ⁻²	b (GeV) ⁻²	$\sqrt{(\theta_{ex}^i - \theta_0^i)^2}$ (degrees)	$\sqrt{1/\gamma^i}$ (degrees)	Wrongly- ordered events.
(a)	$\pi^+ \pi^0 p$	38	2.5	0.8	12.9	8.3	12 %
(b)	$\pi^0 \pi^+ p$	22	0.5	1.2	13.1	8.6	9 %
(c)	$\pi^+ p \pi^0$	13	2.0	1.8	9.0	8.1	9 %

- Table II -

Limit on s_{KK} and s_{Kn} (GeV) ²	No of events	Background control parameter μ	a (GeV) ⁻²	b (GeV) ⁻²	$\alpha_{K^*}^{-2}$ (GeV) ⁻²	$\sqrt{(\theta_{ex}^i - \theta_0^i)^2}$ (degrees)	$\sqrt{1/\gamma^i}$ (degrees)
3.5	224	0	0.21	0.08	0.08	9.58	9.58
3.5	224	10	0.22	-0.03	0.06	9.57	9.88
4.5	98	0	0.36	0.13	-0.06	9.67	9.67

REFERENCES AND FOOTNOTES

- 1) - Chan Hong-Mo, K. Kajantie and G. Ranft - CERN Preprint TH. 719 (1966) ; to be published in Nuovo Cimento.
- 2) - Aachen-Berlin-CERN collaboration - Phys.Letters 18, 351 (1965).
- 3) - CERN-ETH collaboration - Berkeley Conference (1966).
- 4) - The statistics of this experiment is soon to be much improved.
- 5) - A parallel analysis of the $\bar{\pi}^+ \pi^+ n$ events from the same experiment has been attempted. The results are similar though quantitatively less satisfactory. We believe that there is here a large contamination due to, for example, resonance tails. The results are therefore not presented pending a more detailed analysis.
- 6) - See, e.g., R.J.N. Phillips and W. Rarita - Phys.Rev. 139, B1336 (1965).
 Little is known about the baryon trajectories. Our numbers for these are obtained simply by parallel extrapolation from the particles.
- 7) - The values of a and b determined from (A15) are too low and depend strongly on the control parameter μ , both of which facts are indicative of a sizeable background.
- 8) - The detection efficiency can vary by as much as a factor 5.
- 9) - The parameters for the ρ and A_2 trajectories are well known. See, e.g., Ref. ⁶⁾. The intercepts for K^* and K^{**} are obtained by parallel extrapolations.
- 10) - D. Amati, S. Fubini and A. Stanghellini - Nuovo Cimento 26, 896 (1962).

- 11) - J. Joseph and H. Pilkuhn - Nuovo Cimento 33, 1407 (1964).
- 12) - R.T. Deck - Phys.Rev.Letters 13, 169 (1964).
- 13) - We use here a definition of the "width" different from that given in the Appendix of Ref. ¹⁾. The present definition is more appropriate for a Gaussian peak.

- FIGURE CAPTIONS -

Figure 1 - Reaction producing three final particles.

Figure 2 - Dalitz plot for the reaction shown in Fig. 1.

Figure 3 - Double Regge graph for a reaction producing three final particles.

Figure 4 - (i) Dalitz plot of the 78 experimental events with $M_{\pi p}^2 > 3 \text{ GeV}^2$ and $M_{\pi\pi}^2 > 2 \text{ GeV}^2$;
 (ii) Simulated Dalitz plot from the model using parameters listed in Table I. Events of category (d) are missed out.

Figure 5 - Double Regge graphs contributing to the reaction $\pi^+ p \rightarrow \pi^+ \pi^0 p$.

Figure 6 - (i) Dalitz plots of experimental events belonging respectively to the categories (a), (b) and (c).
 (ii) Dalitz plots for respectively the categories (a), (b) and (c) simulated using the parameters listed in Table I. The events have been generated randomly, the number in each category being normalized to the respective experimental number.

Figure 7 - Distribution of the 38 events in category (a) as a function of $\theta_{\text{ex}}^i - \theta_o^i$.

Figure 8 - Distributions of the events in category (a) as functions of φ [see Appendix, Eq.(A20)]. The dotted line represents the experimental distribution ; the full line represents the distribution as calculated from the parameters given in Table I.

Figure 9 - Distributions of the events in category (a) as functions of $s_{34} = M_{\pi^+\pi^0}^2$. The dotted line represents the experimental distribution ; the full line represents the distribution as calculated from the parameters given in Table I.

Figure 10 - Distributions of the events in category (a) as functions of $s_{45} = M_{\pi^0 p}^2$. The dotted line represents the experimental distribution as calculated from the parameters given in Table I.

Figure 11 - Distributions of the events in category (a) as functions of $t_{13} = (p_{\pi^+}^f - p_{\pi^+}^i)^2$. The dotted line represents the experimental distribution ; the full line represents the distribution as calculated from the parameters given in Table I.

Figure 12 - Distributions of the events in category (a) as functions of $t_{25} = (p_p^f - p_p^i)^2$. The dotted line represents the experimental distribution ; the full line represents the distribution as calculated from the parameters given in Table I.

Figure 13 - Dalitz plot of the 224 experimental events with $M_{KK}^2 > 3.5 \text{ GeV}^2$ and $M_{Kn}^2 > 3.5 \text{ GeV}^2$. Every event is plotted twice.

Figure 14 - Double Regge graph for the reaction $\pi^- p \rightarrow K_0 \bar{K}_0 n$.

Figure 15 - Dalitz plot of the 224 experimental events with $M_{KK}^2 > 3.5 \text{ GeV}^2$, $M_{Kn}^2 > 3.5 \text{ GeV}^2$ and $M_{\bar{K}n}^2 > 3.5 \text{ GeV}^2$. Each event is plotted only once.

Figure 16 - Weighted histogram of the 224 events in $\theta_{ex}^i - \theta_o^i$. The Gaussian shape predicted by the model is shown in comparison.

- Figure 17 - Distributions in φ for events with $s_{ij} > 3.5 \text{ GeV}^2$:
experiment (dotted line) ; as calculated from parameters
given in the first row of Table II (full line).
- Figure 18 - Distributions in $s_{34} = M_{KK}^2$ for events with $s_{ij} > 3.5 \text{ GeV}^2$:
experiment (dotted line) ; as calculated from parameters
given in the first row of Table II (full line).
- Figure 19 - Distributions in $s_{45} = M_{Kn}^2$ for events with $s_{ij} > 3.5 \text{ GeV}^2$:
experiment (dotted line) ; as calculated from parameters
given in the first row of Table II (full line).
- Figure 20 - Distributions in $t_{13} = (p_K - p_\pi)^2$ for events with
 $s_{ij} > 3.5 \text{ GeV}^2$: experiment (dotted line) ; as calculated
from parameters given in the first row of Table II (full
line).
- Figure 21 - Distributions in $t_{25} = (p_n - p_o)^2$ for events with
 $s_{ij} > 3.5 \text{ GeV}^2$: experiment (dotted line) ; as calculated
from parameters given in the first row of Table II (full
line).
- Figure 22 - Distributions in $s_{34} = M_{KK}^2$ for events with $s_{ij} > 5.0 \text{ GeV}^2$:
experiment (dotted line) ; as calculated from parameters
given in the first row of Table II (full line).
- Figure 23 - Distributions in $t_{13} = (p_K - p_\pi)^2$ for events with
 $s_{ij} > 5.0 \text{ GeV}^2$: experiment (dotted line) ; as calculated
from parameters given in the first row of Table II (full
line).

FIG. 1

FIG. 2

FIG. 3

FIG. 4(i)

FIG. 4(ii)

FIG. 5

FIG. 6(i) a

FIG. 6(i) b

FIG. 6(1)C

FIG. 6(ii) c

FIG. 7

FIG. 8

FIG. 9

FIG. 10

$-t_{13} \text{ (GeV}^2\text{)}$ FIG. 11

38 EVTS

FIG. 12

FIG. 13

FIG. 14

FIG. 15

FIG. 16

FIG. 17

FIG. 18

FIG. 19

FIG. 20

FIG. 21

FIG. 22

FIG. 23