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EXISTENCE OF SOLUTIONS OF PARTIAL WAVE DISPERSION RELATIONS
AND SINGULAR N/D EQUATIONS

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A B S T R A C T

It is shown that for left-hand discontinuities which asymptotically exceed certain limits imposed by unitarity, the partial wave dispersion relation admits no ghost-free solutions. However, for discontinuities which on the left tend to a limit λ , $0 < \lambda < 1$, the singular integral equation that arises by application of the usual N/D approach is shown to lead to a one-parameter manifold of ghost-free solutions; in particular, a specific solution which is analytic at $\lambda = 0$ belongs to this manifold. The variation of the zeros of the D function (for varying λ) is shown explicitly in a simple example.

1. INTRODUCTION

Recently, there has been some interest in the study of singular (non-Fredholm) integral equations which arise in the application of decomposition methods, notably the usual N/D , on scattering problems ¹⁾⁻⁴⁾.

One reason is that in some of the simplest examples treated in the strong interaction literature, e.g., pion-pion scattering by exchange of a ρ meson, a left-hand discontinuity defined by the exchange of an elementary particle of spin 1 (without a cut-off) tends asymptotically to a constant; and through an elastic unitary N/D approach it leads to a "marginally" singular integral equation for the N (or for the D) function ^{1),2),4)}. A second reason is that, under certain assumptions supported by the existing experimental data, diffraction scattering leads again to a marginally singular integral equation. A third reason is that, after a number of difficulties encountered in the peratization scheme, it could - in principle - be of interest to apply a unitary N/D approach to the problem of higher order corrections to weak interactions; then a left-hand discontinuity defined by the exchange of two (or more) elementary vector mesons increases asymptotically, and this leads to a strongly singular integral equation ³⁾.

In investigations of singular N/D equations, two problems are of particular interest. The first is the existence of solutions of the dispersion relation for the partial wave amplitude; for, in certain cases, the resulting D functions may introduce unwanted zeros on the first sheet of the complex plane of the energy variable (we shall call them ghosts). The second is the uniqueness of the ghost-free solutions; for, in general, the N/D equations have non-unique solutions ¹⁾⁻⁴⁾.

The first problem has already been considered ^{2),3)} on the basis of specific examples; in these, the exact integral equations are replaced by their asymptotic counterparts, which then lead to explicit solutions. However, so far, the relation between the manifold of solutions of the exact equations and of their asymptotic forms has not been rigorously established; thus, the existing considerations are not fully convincing.

In Section 2 of this work a proof is given that, for left-hand discontinuities which asymptotically exceed certain limits imposed by unitarity, no ghost-free solutions exist. In Section 3, it is shown that, provided that these unitarity limits are observed, ghost-free solutions do exist; in particular, a uniquely specified solution, which as a function of an appropriate eigenparameter λ is analytic at $\lambda = 0$ ²⁾, belongs to this class. However, it is also shown that, in general, a one-parameter infinity of ghost-free solutions exists too. Finally, in Section 4, the variation of the zeros of the D function (with varying λ) is studied in an explicit, simplified example.

2. A THEOREM ON THE EXISTENCE OF SOLUTIONS

Consider the ℓ th partial wave amplitude $A_\ell(\nu)$ for the elastic scattering of two pseudoscalar particles of equal mass μ ; ν is the square of the centre-of-mass momentum. It is assumed that the discontinuity $\Delta A_\ell(\nu)$ along the left-hand cut $-\infty < \nu < -\omega_L$ (< 0) is of the form

$$\Delta A_\ell(\nu) = \lambda(-\nu)^n, \quad (2.1)$$

where λ is a constant and n an integer ≥ 0 . The proof of this section can be easily extended to discontinuities which, after a finite number of oscillations, tend to asymptotic limits of the form (2.1). Then the cases of left-hand discontinuities defined by exchange of one or more elementary vector mesons will be included. In these cases λ is a function of the pseudoscalar vector coupling constant.

Consider now the partial wave dispersion relation for $A_\ell(\nu)$ with $n+1$ subtractions⁵⁾:

$$A_\ell(\nu) = \sum_{k=0}^n a_k \nu^k + \frac{\nu}{\pi} \int_{-\infty}^{-\omega_L} \frac{d\nu' \Delta A_\ell(\nu')}{\nu'^{n+1}(\nu'-\nu)} + \frac{\nu}{\pi} \int_0^{\infty} \frac{d\nu' \text{Im} A_\ell(\nu')}{\nu'^{n+1}(\nu'-\nu)} \quad (2.2)$$

Unitarity imposes the restriction

$$0 \leq \rho(\nu) |A_\ell(\nu)|^2 \leq \lim_{\nu \rightarrow \infty} |A_\ell(\nu)|^2 \leq 1 \quad 0 \leq \nu < \infty \quad (2.3)$$

where $\rho(\nu)$ is the usual phase space factor : $\rho(\nu) = [\nu/(\nu + \mu^2)]^{\frac{1}{2}}$.
It will be shown that in the cases :

$$a) \quad n = 0 \quad \text{and} \quad \lambda < 0 \quad \text{or} \quad \lambda > 1$$

$$b) \quad n \geq 1$$

solutions of (2.2) satisfying (2.3) do not exist.

To prove this, consider first the limit of $A_\ell(\nu)$ for $\nu \rightarrow -\infty + i\varepsilon$ ($\varepsilon > 0$, small). In the contribution of the left-hand cut, let $\nu = -\omega$, $\nu' = -\omega'$ and define

$$I(\omega) = \omega \int_{\omega_L}^{\infty} \frac{d\omega' \Delta A_\ell(-\omega')}{\omega'^{n+1} (\omega - \omega')}$$

This integral can be split as follows :

$$\begin{aligned} I(\omega) &= \int_{\omega_L}^{\omega(1-\delta)} \frac{d\omega' \Delta A_\ell(-\omega')}{\omega'^n (\omega - \omega')} + \int_{\omega(1-\delta)}^{\omega(1+\delta)} \frac{d\omega' \Delta A_\ell(-\omega')}{\omega'^n (\omega - \omega')} + \int_{\omega_L}^{\omega(1+\delta)} \frac{d\omega' \Delta A_\ell(-\omega')}{\omega'^{n+1}} + \omega \int_{\omega(1+\delta)}^{\infty} \frac{d\omega' \Delta A_\ell(-\omega')}{\omega'^{n+1} (\omega - \omega')} \\ &\equiv I_1 + I_2 + I_3 + I_4 \end{aligned}$$

(for some δ , $0 < \delta < 1$). With (2.1) it can be shown easily that

$$I_1 = O(1), \quad I_2 = 0, \quad I_3 = \lambda \log \omega + O(1), \quad I_4 = O(1)$$

Therefore, for $\omega \rightarrow \infty$

$$I(\omega) = \lambda \log \omega + O(1).$$

4.

Since $\Delta A_\ell(\nu) = O(\omega^n)$ the asymptotic behaviour of the integral over the left-hand cut is

$$\mathcal{F}_L(-\omega) = \frac{(-\omega)^{n+1}}{\pi} \int_{-\infty}^{-\omega_L} \frac{d\nu \Delta A_\ell(\nu)}{\nu^{n+1}(\nu+\omega)} = \frac{\lambda}{\pi} \omega^n \log \omega + O(\omega^n) \quad (2.4)$$

Next, consider the contribution of the right-hand cut $\mathcal{F}_R(-\omega)$ to the amplitude $A_\ell(-\omega)$ and distinguish the two cases a), b). In the case a), $\text{Im } A_\ell(\nu) \geq 0$ implies

$$\mathcal{F}_R(-\omega) = -\frac{\omega}{\pi} \int_0^\infty \frac{d\nu \text{Im } A_\ell(\nu)}{\nu(\nu+\omega)} < 0 \quad (\omega > 0) \quad (2.5)$$

On the other hand, since $\text{Im } A_\ell(\nu) \leq 1$:

$$\mathcal{F}_R(-\omega) > -\frac{1}{\pi} \log \omega + O(1) \quad (\omega \rightarrow \infty) \quad (2.6)$$

In this case $A_\ell(-\omega) = a_0 + \mathcal{F}_L(-\omega) + \mathcal{F}_R(-\omega)$. Thus, combination of (2.4) and (2.6) implies:

$$\frac{\lambda-1}{\pi} \log \omega < A_\ell(-\omega) < \frac{\lambda}{\pi} \log \omega \quad (\omega \rightarrow \infty) \quad (2.7a)$$

In the case b), the integral of $\mathcal{F}_R(-\omega)$ converges uniformly in the limit $\omega \rightarrow \infty$. It follows easily that

$$A_\ell(-\omega) = \frac{\lambda}{\pi} \omega^n \log \omega + O(\omega^n) \quad (2.7b)$$

Thus, it is concluded that in both cases a), b), for $\nu \rightarrow -\infty + i\epsilon$

$$A_\ell(\nu) = c |\nu|^n \log |\nu| + O(|\nu|^n)$$

where c is a real constant $\neq 0$. On the other hand, for $\nu \rightarrow +\infty$, $\rho(\nu) \rightarrow 1$ and (2.3) gives

$$|A_\ell(\nu)| \leq 1$$

Direct application of a weak form of the Phragmén-Lindelöf theorem ⁶⁾ along the positive imaginary axis of ν , $\nu = iy \rightarrow i\infty$, implies :

$$|A_\ell(\nu)| \leq C |\nu|^{n/2} (\log |\nu|)^{1/2} \quad (2.8)$$

where C is a positive constant.

The contradiction can be shown by computing the limit of $A_\ell(\nu)$ for $\nu = iy \rightarrow i\infty$ directly from the partial wave dispersion relation (2.2). It is easily shown that

$$\mathcal{F}_\ell(iy) = (-i)^n \frac{1}{\pi} y^n \log y + O(y^n) \quad (2.9)$$

In the case a), and because of (2.3) :

$$0 \geq \mathcal{F}_\ell(iy) \geq -\frac{1}{\pi} \log y \quad (y \rightarrow \infty)$$

and therefore

$$\frac{1}{\pi} \log y \leq A_\ell(iy) \leq \frac{1}{\pi} \log y \quad (y \rightarrow \infty) \quad (2.10a)$$

In the case b), the integral of $\mathcal{F}_R(iy)$ converges uniformly in the limit $\nu = iy \rightarrow i\infty$; thus :

$$A_\ell(iy) = (-i)^n \frac{1}{\pi} y^n \log y + O(y^n) \quad (2.10b)$$

The conclusion is that in both cases

$$|A_\ell(iy)| = C_1 y^n \log y + O(y^n) \quad (2.11)$$

where $C_1 > 0$. This contradicts (2.8).

In general, the N/D method leads to explicit solutions in the cases a) and b), as well. In view of the preceding theorem, these solutions cannot satisfy (2.2). One possible way to break (2.2) is by the existence of zeros of $D(\nu)$ along $\nu < 0$, which introduce pole terms in (2.2) (ghosts). However, unitarity requires an infinity of such terms. This is shown to be the case in a specific example corresponding to $n=0$ and $\lambda < 0$ (Section 4).

Notice that in case b), since the discontinuity (2.1) does not oscillate at infinity and requires at least two subtractions, the conclusions of this Section can be considered as a consequence of Kinoshita's recent work ⁷⁾; nevertheless, an independent proof is, perhaps, worth while. Notice also that the present methods can be easily extended to the case $0 < n < 1$, which is not covered by Ref. 7).

3. GHOST-FREE SOLUTIONS

In the case $n=0$ and $0 < \lambda < 1$ the analysis of Section 2 does not lead to any contradiction [see Eq. (2.9a)]. The purpose of the present Section is to show that, in this case, the N/D approach does lead to ghost-free solutions: and that, in general, these are not unique.

Consider a left-hand discontinuity $\Delta A_\ell(\nu)$ which, after a finite number of oscillations, behaves asymptotically like (2.1). Suppose that the N/D decomposition is carried as follows :

$$A_\ell(\nu) = \nu^\lambda N(\nu)/D(\nu) \quad (3.1)$$

and that N and D satisfy once-subtracted representations :

$$N(\nu) = N(\nu_0) + \frac{\nu - \nu_0}{\pi} \int_{-\infty}^{-\omega_L} \frac{\Delta A_\ell(\nu') D(\nu') d\nu'}{\nu'^\lambda (\nu' - \nu_0)(\nu' - \nu)} \quad (3.2)$$

$$D(\nu) = 1 - \frac{\nu - \nu_0}{\pi} \int_0^{\infty} \frac{\rho(\nu') \nu'^n N(\nu') d\nu'}{(\nu' - \nu_0)(\nu' - \nu)} \quad (3.3)$$

Equation (3.3) holds for elastic unitarity. In this Section, it will be assumed that $l \geq n+1$ so that the usual threshold properties of $A_l(\nu)$ imply $N(0)=0$. With $\nu_0=0$ replacement of (3.3) in (3.2) gives :

$$N(\nu) = B(\nu) + \frac{\nu}{\pi} \int_0^{\infty} d\nu' \rho(\nu') \nu'^{n-1} \frac{B(\nu) - B(\nu')}{\nu - \nu'} N(\nu') \quad (3.4)$$

where

$$B(\nu) \equiv \frac{\nu}{\pi} \int_{-\infty}^{-\nu_L} \frac{\Delta A_l(\nu') d\nu'}{\nu'^{n+1}(\nu' - \nu)} \quad (3.5)$$

In all cases $n \geq 0$, (3.4) is a singular integral equation.

In view of Section 2, solutions of (3.4) satisfying also (2.2) may exist only in the case $n=0$. A sufficient condition that $D(\nu)$ has no zeros on the first sheet of complex ν is :

$$N(\nu) \geq 0 \quad \text{for} \quad \nu \geq 0 \quad (3.6)$$

For if (3.6) is satisfied, Eq. (3.3) implies

$$\text{Im} D(\nu) < 0 \quad \text{for} \quad \text{Im} \nu > 0,$$

i.e., that $-D(\nu)$ is a Herglotz function, and

$$D(\nu) \neq 0 \quad \text{for} \quad \nu < 0;$$

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thus, $D(\nu)$ has no ghost zeros. Now, sufficient conditions that (3.6) be satisfied are the following :

- i) convergence of the Neumann series expansion of (3.5)
(in powers of λ) and
- ii) $B(\nu) \geq 0 \quad (\nu \geq 0)$ (3.7)
- iii) $dB(\nu)/d\nu > 0 \quad (\nu \geq 0)$

For if i)-iii) are satisfied all the terms of the Neumann series of (3.4) will be positive.

For a left-hand discontinuity which asymptotically tends to a constant λ the general resolvent of (3.4) (corresponding to $l = 1$) was shown to have a branch point at $\lambda = 1$ ²⁾. However, a unique resolvent which is analytic at $\lambda = 0$ was also shown to exist. The corresponding solution of (3.4) will admit a power series expansion in λ for $|\lambda| \leq \lambda_0 \leq 1$ [condition i] ⁸⁾.

An example satisfying the conditions ii), iii) is given by a positive, asymptotically constant, left-hand discontinuity

$$\Delta A_l(\nu) \geq 0 \quad (-\infty < \nu < -\omega_l), \quad \Delta A_l(\nu) \xrightarrow{\nu \rightarrow -\infty} \lambda (< 1) \quad (3.8)$$

With $\nu' = -\omega$ (3.5) gives

$$B(\nu) = \frac{\nu}{\pi} \int_{\omega_l}^{\infty} \frac{d\omega \Delta A_l(-\omega)}{\omega(\omega + \nu)}$$

$$dB(\nu)/d\nu = \frac{1}{\pi} \int_{\omega_l}^{\infty} \frac{d\omega \Delta A_l(-\omega)}{(\omega + \nu)^2} \quad (3.9)$$

so that (3.7) are satisfied. Hence, the resolvent which is analytic at $\lambda = 0$ can be used to construct a ghost-free solution.

The structure of the integrals in (3.9) indicates that the condition (3.8) can be easily relaxed: it is sufficient that the positive contribution of $\Delta A_\ell(\nu)$ dominates. An important example is a Born term $B(\nu)$ defined by the exchange of an elementary vector-meson of mass m , when ²⁾

$$B(\nu) = \frac{\lambda}{4\pi} \frac{8\nu + 4\mu^2 + m^2}{\nu} Q_1\left(1 + \frac{m^2}{2\nu}\right); \quad (3.10)$$

λ is proportional to the square of the pseudoscalar-vector coupling constant. It was found that (3.7) are satisfied, at least for ratios m/μ in the interval $2 < m/\mu < 10$ ⁹⁾.

Now, the uniqueness of the ghost-free solutions will be discussed. For simplicity, the considerations are restricted to a constant left-hand discontinuity

$$\Delta A_\ell(\nu) = \lambda; \quad (3.11)$$

this is a special case of (3.8). With the simplification $\rho(\nu) = 1$ (which corresponds to $\mu = 0$) the integral equation (3.4) can be solved exactly ²⁾. The general solution is:

$$N(\nu) = B(\nu) + \lambda \int_0^\infty R(\nu, \nu'; \lambda) \frac{B(\nu')}{\nu'} d\nu' + A_\nu P_{-s_0} \left(1 + 2\frac{\nu}{\omega_l}\right) \quad (3.12a)$$

where $R(\nu, \nu'; \lambda)$ is the resolvent analytic at $\lambda = 0$, P_{-s_0} the Legendre function of the first kind, A an arbitrary constant and

$$s_0 = \frac{1}{\pi} \arcsin \sqrt{\lambda} \quad \left(0 < \arcsin < \frac{\pi}{2}\right). \quad (3.12b)$$

In (3.12a), the last term is the general solution of the associated homogeneous integral equation. Now, in view of the preceding discussion, the sum of the first two terms in the right-hand side of (3.12a) is positive.

On the other hand, for $0 < \lambda < 1$, $0 < s_0 < \frac{1}{2}$, and for $\nu > 0$ ¹⁰⁾ :

$$P_{-s_0} \left(1 + 2 \frac{\nu}{\omega_L} \right) > 0$$

Hence, if $A > 0$ the condition (3.6) is satisfied and the resulting $D(\nu)$ is free of unwanted zeros. Thus, in this case, the ghost-free solutions form a one-parameter manifold ($A > 0$).

These considerations can be generalized to conclude that asymptotically constant left-hand discontinuities lead to non-unique ghost-free solutions. Extra requirements (like, e.g., analyticity in λ at $\lambda = 0$) seem to be necessary for a unique specification.

This Section will be closed with a remark concerning asymptotically increasing left-hand discontinuities [(2.1) with $n \geq 1$]. Consider the case of a Born term $B(\nu)$ which satisfies the conditions (3.7). Next, suppose that (3.4) has a Neumann expansion which converges for some $|\lambda| < \lambda_0$. Then, the considerations of this Section imply that the corresponding $D(\nu)$ is free of ghost-zeros. But then the corresponding solution satisfies (2.2) and (2.3); and this contradicts Section 2. Thus, in this case, the Neumann expansion cannot converge for any finite λ ; the general solution of (3.4) must be singular at $\lambda = 0$. This was found to be the case in all the specific examples of Ref. 3).

4. AN EXPLICIT EXAMPLE

This Section is restricted to asymptotically constant left-hand discontinuities ($n=0$). It was shown in Section 2 that, e.g., for $\lambda < 0$ the solutions which can be obtained by the N/D approach must be defective. On the other hand, Section 3 concluded that as soon as λ becomes positive (but < 1) the defects disappear; and that this includes the solution which is analytic at $\lambda = 0$. To demonstrate a possible mechanism for this transition, the present Section considers a simplified example which leads to a simple solution for the D function. It is found that $D(\nu)$ has an infinity of zeros on the first sheet of ν , along $\nu < 0$; but that for $0 < \lambda < 1$ these zeros pass onto higher sheets.

The example is :

$$\Delta A_\ell(\nu) = \lambda \quad \rho(\nu) = 1 \quad \omega_L = 0 \quad (4.1)$$

This is rather unrealistic since the gap $-\omega_L < \nu < 0$ has been contracted to zero (which means $m=0$) and the masses of the scattered particles have been neglected. Nevertheless, it has been found very instructive, in particular as far as the asymptotic behaviour (for $|\nu| \rightarrow \infty$) of the solutions of the marginally singular equation is concerned ²⁾.

In this case it will be assumed $\nu_0 \neq 0$ and, to simplify the example, $N(\nu_0) = 0$. After the changes $\nu = -\omega$, $\nu' = -\omega'$, $\nu_0 = -\omega_0$ and with the definition

$$D(\nu)/(\nu - \nu_0) = f(\omega) \quad (4.2)$$

replacement of Eq. (3.2) in (3.3) gives the integral equation :

$$f(\omega) = \frac{1}{\omega - \omega_0} + \frac{\lambda}{\pi^2} \int_0^\infty d\omega' \frac{\log \frac{\omega'}{\omega}}{\omega' - \omega} f(\omega') \quad (4.3)$$

The norm of the kernel of (4.3) diverges logarithmically at large ω , ω' ; and in this respect (4.3) is similar to (3.4) (for $n=0$). However, (4.3) can be diagonalized by Mellin transform; a (unique) resolvent which is analytic at $\lambda = 0$ was found to be ²⁾ :

$$R(\omega, \omega'; \lambda) = \frac{1}{\pi [\lambda(1-\lambda)]^{1/2}} \frac{\sinh(s_0 \log \frac{\omega'}{\omega})}{\omega' - \omega}. \quad (4.4)$$

s_0 was introduced in (3.12b) so that for general λ :

$$\lambda = \sin^2(\pi s_0). \quad (4.5)$$

The corresponding solution is

$$f(\omega) = \frac{1}{\omega - \omega_0} + \lambda \int_0^{\infty} d\omega' R(\omega, \omega'; \lambda) \frac{1}{\omega' - \omega_0}$$

The reality of $D(\nu)$ for $\nu < 0$ implies that the last integral be considered as a principal value. Application of Mellin transforms ¹¹⁾ gives :

$$f(\omega) = \frac{\cosh(s_0 \log \frac{\omega}{\omega_0})}{\omega - \omega_0}$$

so that :

$$D(-\omega) = \cosh(s_0 \log \frac{\omega}{\omega_0}) \quad (4.6)$$

Notice that $D(-\omega_0) = 1$ in accord with (3.3). Thus, the zeros of the denominator function appear at

$$\omega = \omega_0 \exp \left[\left(k + \frac{1}{2} \right) \frac{\pi}{i s_0} \right] \quad k = 0, \pm 1, \pm 2, \dots \quad (4.7)$$

Consider first the case $\lambda < 0$. With $s_0 = \xi / i \pi$ this corresponds to

$$\lambda = -\sinh^2 \xi \quad -\infty < \xi < \infty \quad (4.8)$$

For any $\xi \neq 0$, Eq. (4.7) gives two infinite sets of zeros lying on the positive real ω axis; e.g., for $\xi > 0$ the first set (corresponding to $k=0, 1, 2, \dots$) accumulates at $\omega = +\infty$ and the second ($k=-1, -2, \dots$) accumulates at $\omega = 0$. In accord with Eq. (4.6) the physical sheet of ν (defined by $0 < \arg \nu < 2\pi$) corresponds to $-\pi < \arg \omega < \pi$. It is concluded that these zeros lie on the physical sheet of the ν plane, along $\nu < 0$ (ghost-zeros); and this is in accord with Section 2 ¹²⁾.

Now, consider the case $0 < \lambda < 1$ which, according to (3.12b), corresponds to $0 < s_0 < \frac{1}{2}$. From (4.7) it is easily seen that for all k :

$$|\arg w| > \pi$$

Here none of the zeros defined by (4.7) lies on the physical sheet of v ; and this leads to a ghost-free solution, in accord with the conclusions of Section 3.

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