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BOOTSTRAP SOLUTIONS IN A VECTOR MESON MODEL

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ABSTRACT

In a model with a left-hand discontinuity defined by the exchange of a vector meson with a sharp cut-off elastic-unitary bootstrap solutions are derived for a number of values of the vector meson mass, by solving the Fredholm integral equation obtained from the usual N/D decomposition. The corresponding self-consistent pseudoscalar-vector couplings are found to be larger than the experimental ones. In all cases the iteration expansion (in powers of the coupling) is shown to converge. It is then proved that the denominator functions are Herglotz; and that all the self-consistent solutions obtained correspond to ghost-free amplitudes. A sum-rule derived on grounds of the asymptotic and threshold properties is found to be fulfilled. Finally, Levinson's theorem is shown to be satisfied.

1. INTRODUCTION

Since the publication of the pioneering papers of Chew and Mandelstam¹⁾ and of Zachariasen and Zemach²⁾, the vector meson bootstrap has occupied a prominent position in the literature of elementary particles. The simplest approach proceeds through the usual N/D decomposition of the partial wave amplitude and, in the spirit of dominance of nearby singularities, defines the left-hand cut discontinuity by the exchange of a vector-meson and makes use of elastic unitarity on the right-hand cut.

In most applications of this approach the system of coupled integral N/D equations is solved approximately; questions of obtaining exact solutions and, in particular, of existence of exact solutions are not considered at all. The argument usually invoked is that, from the physical point of view, such an approach neglects important contributions like those of inelastic states or of multi-particle exchange; and, from the mathematical point of view, the resulting amplitudes are anyway going to violate crossing symmetry. Thus, it is said, mathematical rigour is physically useless.

Although this may well be true, in the present work we shall adopt a different attitude. First, we shall be interested in solutions leading to numerical results comparable with experiment. In the absence of a complete dynamical theory of strong interactions such a task necessitates the violation of at least one of the general principles. We shall be willing to abandon crossing-symmetry (or rather, to satisfy it approximately) and thus we shall study the properties of our amplitudes only in the direct channel. But we shall require that our results correspond to exact solutions of the system of N/D equations; and that our amplitudes, apart from crossing, do not violate any of the basic requirements imposed by partial-wave analyticity and threshold behaviour.

In our model the left-hand discontinuity is defined by the exchange of a vector meson, with a sharp cut-off Λ suppressing the distant part. From the physical point of view, such a cut-off procedure may be justified by the alleged existence of some damping mechanism which operates at high energy; e.g. Reggeization is known to provide such a mechanism³⁾. From the mathematical point of view, a cut-off Λ implies the introduction of an arbitrary parameter in our model. Then the bootstrap programme (i.e. self-consistent determination of the mass m and coupling λ of the vector meson) cannot be fully accomplished; in general, only one of the two parameters m , λ , can be bootstrapped. However, even in such a moderate approach, it is not clear that physically significant self-consistent solutions, which satisfy the above-mentioned mathematical conditions, do exist.

Section 2 presents the basic equations of our model. Section 3 contains the essential steps of our calculational procedure. Section 4 gives the numerical results obtained as well as certain checks of the procedure, and also discusses some immediate conclusions. Section 5 studies the mathematical consistency of the bootstrap solutions obtained and in particular the existence of unwanted ghost-poles. Section 6 discusses the asymptotic properties of the amplitude and the fulfilment of a sum-rule imposed by the threshold properties. Finally Section 7 summarizes the basic conclusions.

Our numerical solutions are derived with the additional purpose of applying them in the study of the problem of octet enhancement for the vector meson octet (V) in its interaction with the pseudoscalar octet (P)⁴⁾. Since $m_V^2/m_\rho^2 \simeq 4.3$ ⁴⁾ (in contrast to $m_\rho^2/m_\pi^2 \simeq 30$), this motivation has, to some extent, influenced our analysis towards small values of m .

2. GENERAL FORMALISM

Consider the elastic scattering of two pseudoscalar mesons (P) of mass μ . The partial wave amplitude $A_\ell(\nu)$ can be decomposed as follows:

$$A_\ell(\nu) = N(\nu) / D(\nu) \quad (2.1)$$

The variable ν (= square of centre-of-mass momentum) is related to the square of the invariant energy s by

$$s = 4(\nu + \mu^2) \quad (2.2)$$

As usual, in the decomposition (2.1), $N(\nu)$ has the left-hand cut $-\infty < \nu \leq -\nu_L$ and $D(\nu)$ the right-hand cut $0 \leq \nu < \infty$.

The present work will be concerned with the partial P-wave amplitude, when the threshold condition implies

$$N(\nu) \sim \nu \quad \nu \rightarrow 0$$

In order that the solutions satisfy automatically this condition, once subtracted representations for both N and D will be used. With the subtraction point for the N function at $\nu = 0$ and the normalization $D(\nu) = 1$ at $\nu = \nu_0$ one obtains

$$N(\nu) = \frac{\nu}{\pi} \int_{-\infty}^{-\nu_L} \frac{d\nu' \Delta A_\ell(\nu') D(\nu')}{\nu'(\nu' - \nu)} \quad (2.3)$$

$$D(\nu) = 1 - \frac{\nu - \nu_0}{\pi} \int_0^{+\infty} \frac{d\nu' \rho(\nu') N(\nu')}{(\nu' - \nu_0)(\nu' - \nu)} \quad (2.4)$$

$\Delta A_1(\nu)$ is the discontinuity of $A_1(\nu)$ along the left-hand cut and $\rho(\nu)$ the phase-space factor, which here is taken as

$$\rho(\nu) = \frac{1}{2} \left(\frac{\nu}{\nu + \mu^2} \right)^{1/2} \quad (2.5)$$

Substitution of (2.4) into (2.3) gives an integral equation for N . With the definition

$$f(\nu) = \left[\frac{\rho(\nu)}{\nu(\nu - \nu_0)} \right]^{1/2} N(\nu) \quad (2.6)$$

this integral equation can be written in the standard form:

$$f(\nu) = \lambda \beta(\nu) + \lambda \int_0^\infty d\nu' G(\nu, \nu') f(\nu') \quad (2.7)$$

Here

$$G(\nu, \nu') = \frac{1}{\pi} \left[\frac{\nu \rho(\nu)}{\nu - \nu_0} \frac{\nu' \rho(\nu')}{\nu' - \nu_0} \right]^{1/2} \left(\frac{\nu - \nu_0}{\nu} B(\nu) - \frac{\nu' - \nu_0}{\nu'} B(\nu') \right) \frac{1}{\nu - \nu'} \quad (2.8)$$

and

$$\beta(\nu) = \left[\frac{\rho(\nu)}{\nu(\nu - \nu_0)} \right]^{1/2} B(\nu) \quad (2.9)$$

The function $B(\nu)$, which is proportional to the Born term, is given by

$$B(\nu) = \frac{\nu}{\lambda \pi} \int_{-\infty}^{-\nu_L} \frac{d\nu' \Delta A_1(\nu')}{\nu'(\nu' - \nu)} \quad (2.10)$$

The parameter λ is specified below.

The discontinuity $\Delta A_1(\nu)$ defines the "force" of the interaction. In the spirit of dominance of nearby singularities this force will be assumed to be given predominantly by the exchange of a vector meson (V) of mass m . Furthermore, it will be assumed that, due to the presence of some damping mechanism which operates at high energy, for $\nu < -\Lambda$ the left-hand discontinuity is negligible. Then direct calculation gives

$$\Delta A_1(\nu) = \frac{3\pi g^2}{8} \frac{8\nu + m^2 + 4\mu^2}{2\nu} \left(1 + \frac{m^2}{2\nu}\right) \Theta(\nu + \Lambda) \Theta\left(-\nu - \frac{m^2}{4}\right) \quad (2.11)$$

where g is the PPV coupling constant⁵⁾ and $\Theta(x)$ the unit step function. In (2.11) the contribution of the one-vector-meson exchange graph as well as of its crossing-symmetric has been taken into account. The function $\Delta A_1(\nu)$ is plotted in Fig. 1. Now, if the parameter λ of (2.7) is taken:

$$\lambda = \frac{3}{8} g^2 \quad (2.12)$$

then introduction of (2.11) into (2.10) gives

$$B(\nu) = \left(4 + \frac{m^2 + 4\mu^2}{2\nu}\right) \left[-2 + \left(1 + \frac{m^2}{2\nu}\right) \log \frac{1 + \frac{4\nu}{m^2}}{1 + \frac{\nu}{\Lambda}}\right] + \\ + \frac{1}{2\Lambda} \left[5m^2 + 4\mu^2 + \frac{m^2(m^2 + 4\mu^2)}{2\nu}\right] - \frac{m^2(m^2 + 4\mu^2)}{8\Lambda^2}$$

For a finite value of the cut-off Λ the kernel $G(\nu, \nu')$ of (2.8) is easily seen to be Fredholm's. Since the inhomogeneous term $\beta(\nu)$ is square integrable, (2.7) is a Fredholm integral equation; and standard methods can be used to determine its solution⁶⁾.

Now, the existence of a self-consistent solution requires that at $\nu = \nu_R = \frac{\pi^2}{4} - \mu^2$ (> 0 , for all cases of interest to us) there is a resonance of $A_1(\nu)$; this leads to the first bootstrap condition:

$$\text{Re } D(\nu_R) = 0 \quad (2.13a)$$

The relation of the width of the produced resonance to the coupling constant of the exchanged particle leads to the second bootstrap condition:

$$\lambda = -\frac{3}{2} \left[\frac{\frac{1}{\nu} N(\nu)}{\frac{d}{d\nu} \text{Re } D(\nu)} \right]_{\nu \equiv \nu_R} \quad (2.13b)$$

Eqs. (2.13a, b) determine the self-consistent solutions of our problem.

3. CALCULATIONAL PROCEDURE

The kernel of the Fredholm Eq. (2.7) is non-separable.. To determine the solution this kernel can be approximated by a separable one. For this perhaps the simplest way is to replace the left-hand discontinuity by a finite sum of multiples of δ functions⁷⁾. Thus, the following substitution is made:

$$\Delta \tilde{A}_1(\nu) = -\lambda \pi \sum_{i=1}^n R_i \delta(\nu - \nu_i) \quad (3.1)$$

where ν_i are appropriately fixed points in the interval $-\Lambda \leq \nu < -m^2/4$ (see Sect. 4). With the substitution (3.1) the Born term becomes

$$\tilde{B}(\nu) = \nu \sum_{i=1}^n \frac{R_i}{\nu_i (\nu - \nu_i)} \quad (3.2)$$

The R_i 's will be determined by requiring that

$$\int_0^{+\infty} |B(\nu) - \tilde{B}(\nu)|^2 d\nu = \text{minimum} \quad (3.3)$$

It will be shown that in this way very accurate solutions can be obtained with small n ($n = 5$).

This approach allows the possibility of a slight generalization of the model by adding a term $-\lambda \pi R_0 \delta(\nu - \nu_0)$ to the right-hand side of (3.1), i.e. by taking the left-hand discontinuity to be

$$\Delta \hat{A}_1 = \Delta \tilde{A}_1(\nu) - \lambda \pi R_0 \delta(\nu - \nu_0) \quad (3.4)$$

The term $-\lambda \pi R_0 \delta(\nu - \nu_0)$ may be considered to represent the effect of the neglected distant part of the left-hand discontinuity (i.e. the effect of the "short-range part of the force") in a specific parametrization. For

convenience ν_0 in (3.4) coincides with the normalization point of the D function (2.4). In the numerical calculations ν_0 is taken arbitrarily

$$\nu_0 = -3\Lambda/2 \quad (3.5)$$

In Sect. 5 it is proved that the amplitude determined from the exact solution of (2.3) and (2.4) is independent of the specific choice of ν_0 .

Upon defining

$$N_i = \int_0^{+\infty} d\nu \frac{\rho(\nu) N(\nu)}{(\nu - \nu_0)(\nu - \nu_i)} \quad (i = 1, 2, \dots, n) \quad (3.6)$$

and using (3.4) the integral Eq.(2.7) is reduced into the following algebraic system of linear equations

$$\sum_{i=1}^n \left(\alpha_{ki} + \frac{\delta_{ki}}{\lambda} \right) N_i = \beta_k + \frac{R_0}{\nu_0} I(\nu_k, \nu_0) \quad (3.7)$$

(i, k = 1, 2, ..., n), where

$$\alpha_{ki} = \frac{1}{\pi} \frac{R_i (\nu_i - \nu_0)}{\nu_i} I(\nu_k, \nu_i) \quad \beta_k = \sum_{i=1}^n \frac{R_i}{\nu_i} I(\nu_k, \nu_i) \quad (3.8)$$

and

$$I(\nu_k, \nu_i) = \int_0^{+\infty} d\nu \frac{\rho(\nu) \nu}{(\nu - \nu_0)(\nu - \nu_k)(\nu - \nu_i)} \quad (3.9)$$

The determination of $N(\nu)$ and of $D(\nu)$ in terms of N_i is straightforward.

Now, the self-consistency conditions (2.13 a,b) can be written in the form:

$$\frac{R_0}{\nu_0} = \left(\sum_{j=0}^n \gamma_j \mathcal{F}_j(\nu_k) \right)^{-1} \left(\frac{\pi}{\lambda(\nu_k - \nu_0)} - \sum_{i=1}^n \delta_i \mathcal{F}_i(\nu_k) \right) \quad (3.10a)$$

$$\text{and} \quad \frac{3}{2} \frac{R_0}{\gamma_0} \sum_{j=0}^n \frac{\gamma_j}{\gamma_R - \gamma_j} + \frac{3}{2} \sum_{i=1}^m \frac{\delta_i}{\gamma_R - \gamma_i} = \frac{1}{\gamma_R - \gamma_0} +$$

$$+ \frac{\lambda}{\pi} (\gamma_R - \gamma_0) \left\{ \frac{R_0}{\gamma_0} \sum_{j=0}^n \gamma_j \mathcal{F}'_j(\gamma_0) + \sum_{i=1}^m \delta_i \mathcal{F}'_i(\gamma_R) \right\} \quad (3.10b)$$

In (3.10a,b):

$$\gamma_0 = 1 \quad \gamma_i = \frac{\gamma_0 - \gamma_i}{\pi} \frac{R_i}{\gamma_i} \sum_{j=1}^m (\bar{A})_{ij} I(\gamma_j, \gamma_0)$$

and

$$\delta_i = \frac{R_i}{\gamma_i} \left(1 + \frac{\gamma_0 - \gamma_i}{\pi} \sum_{k=1}^m (\bar{A}')_{ik} \beta_k \right) \quad (3.11)$$

where $\bar{A}^{-1}$ is the inverse of the matrix $A = [\alpha_{ki} + \frac{1}{\lambda} \delta_{ki}]$; furthermore

$$\mathcal{F}_i(\gamma_R) = P \int_0^{+\infty} \frac{d\nu R(\nu) \nu}{(\nu - \gamma_0)(\gamma - \gamma_R)(\nu - \gamma_L)} \quad (3.12)$$

$$\text{and } \mathcal{F}'_i(\gamma_R) = \frac{d}{d\gamma_R} \mathcal{F}_i(\gamma_R).$$

For a given value of the cut-off Λ and of the ratio R_0/ν_0 the system of Eqs. (3.10 a,b) gives a self-consistent set of values (n, λ) . Large values of R_0/ν_0 correspond to self-consistent solutions for which the effect of the "short-range force" is relatively important. Thus, within the limitations of the adopted parametrization, one can study the effect of the "short-range force" on the self-consistent values of the quantities (n, λ) .

However, from the bootstrap point of view, of particular interest are those self-consistent solutions which correspond to $R_0/\nu_0 = 0$; in this case the self-consistent values of n and λ will depend on only one free parameter, the cut-off Λ .

4. NUMERICAL RESULTS AND DISCUSSION

Following the procedure outlined in the previous section we have replaced the left-hand discontinuity by the sum of $n = 5$ multiples of δ function. The points ν_i have been chosen as in Fig. 1 and the R_i 's have been determined by minimizing the integral of (3.3).

The numerical solution of the system of Eqs. (3.10 a,b), as obtained by the CERN CDC 3400 computer, is presented in Fig. 2. For a number of fixed values of mass ($m^2 = 4.1, 5, 10$ and 30 , with the pseudo-scalar mass taken as unity) λ is plotted versus Λ with R_0/ν_0 as free parameter.

The first conclusion that arises immediately from Fig. 2 is that, at least for the values of m^2 considered (which cover the range of physical interest), the self-consistent value of λ is very insensitive to variations of the ratio R_0/ν_0 . In view of this and of the special interest in solutions depending on only one parameter (Λ), the subsequent discussion will be restricted to $R_0/\nu_0 = 0$.

The self-consistent values of Λ and λ which correspond to $m^2 = 4.1, 4.3, 5, 10$ and 30 and to $R_0/\nu_0 = 0$ are listed in Table I. The values of λ obtained have to be compared with the following. For the ρ -meson bootstrap in π - π scattering with only isospin ($SU(2)$) invariance, $m^2 \approx 30$, and $\Gamma_{\rho \rightarrow \pi\pi} \approx 120$ MeV yields $\lambda \approx 0.6$. For the V octet bootstrap in P-P scattering with $SU(3)$ invariance, as mentioned in the Introduction ($m^2 \approx 4.3$ ⁴⁾); now $\Gamma_{\rho \rightarrow \pi\pi} \approx 120$ MeV gives $\lambda \approx 0.89$ and $\Gamma_{K^* \rightarrow K\pi} \approx 50$ MeV gives $\lambda \approx 1.28$.

Next, certain tests of the accuracy of the solutions obtained will be considered. Through the procedure of Sections 3 and 4, the original integral equation (2.7) has been replaced by another Fredholm equation of the form

$$\tilde{f}(\nu) = \lambda \tilde{\beta}(\nu) + \lambda \int_0^{+\infty} d\nu' \tilde{G}(\nu, \nu') \tilde{f}(\nu') \quad (4.1)$$

where

$$\tilde{\beta}(\nu) = \left[\frac{\rho(\nu)}{\nu(\nu - \nu_0)} \right]^{1/2} \tilde{B}(\nu) \quad (4.2)$$

and $\tilde{G}(\nu, \nu')$ is a separable kernel:

$$\tilde{G}(\nu, \nu') = \frac{1}{\pi} \left[\frac{\nu \rho(\nu)}{\nu - \nu_0} \frac{\nu' \rho(\nu')}{\nu' - \nu_0} \right]^{1/2} \left(\frac{\nu - \nu_0}{\nu} \tilde{B}(\nu) - \frac{\nu' - \nu_0}{\nu'} \tilde{B}(\nu') \right) \frac{1}{\nu - \nu'} \quad (4.3)$$

$\tilde{B}(\nu)$ is given by (3.2). With the condition (3.3) being satisfied the error in replacing the inhomogeneous term $\beta(\nu)$ by $\tilde{\beta}(\nu)$ is very small. (For $\frac{n^2}{\mu^2} = 4.3$, the relative error on $\tilde{B}(\nu)$ is less than 10^{-4} throughout $\nu > 0$.)

The solution $\tilde{f}(\nu)$ will be a good approximation of the exact $f(\nu)$ (for $\nu \geq 0$) provided that the norm $\|G - \tilde{G}\|$ is sufficiently small⁸⁾.

This has been tested for the bootstrap solution with $n^2 = 4.3$ (which is of particular importance in the octet enhancement questions⁴⁾); it was found

$$\frac{\|G - \tilde{G}\|}{\|G\|} \simeq 10^{-3}$$

In view of the fact that $\|G\| = 0.171$ (Table 1) it is expected that the solution is highly accurate throughout $\nu \geq 0$.

A direct test of the accuracy of the function $N(\nu)$ was also made. For this, $\tilde{N}(\nu)$ has been calculated by means of the relation

$$\tilde{N}(\nu) = \lambda \nu \sum_{i=1}^n \frac{\delta_i}{\nu - \nu_i} \quad (4.4)$$

where the δ_i 's are given by (3.11) (remember $R_0/\nu_0 = 0$). Then $N(\nu)$ has been determined by directly solving (2.7) with the usual method of matrix inversion, and the quantity

$$\epsilon(\nu) = \left| \frac{N(\nu) - \tilde{N}(\nu)}{N(\nu)} \right| \quad (4.5)$$

has been calculated. For the bootstrap solution corresponding to $n^2 = 4.1$ it is found: $\epsilon(\nu) \lesssim 10^{-4}$ for all $0 < \nu < \infty$; and for $n^2 = 30$: $\epsilon(\nu) \simeq 10^{-2}$ for $0 < \nu < 5$, $\epsilon(\nu) \lesssim 10^{-4}$ for $5 < \nu < \infty$.

Finally, the existence of iteration solutions of the integral equation (2.7) is discussed. It is known⁸⁾ that the Neumann expansion (in powers of λ) converges for

$$|\lambda| < \|G\|^{-1} \quad (4.6)$$

In the fourth column of Table 1 the values of the norm $\|G\|$ corresponding to the bootstrap solutions for $n^2 = 4.1, 4.3, 5, 10$ and 30 are listed. Comparison to the relevant values of λ shows that in all cases an iteration solution does exist. However, the amplitude $A_1(\nu)$ obtained by a finite number of iterations will be a function of the subtraction point ν_0 . In contrast, the amplitude obtained from the exact solution will be shown to be independent of ν_0 (Sect. 5).

5. ABSENCE OF UNWANTED ZEROS OF THE D-FUNCTION

The solutions obtained in Sect. 4 satisfy elastic unitarity by construction of the basic equation for the D function [(2.4)]. Also, they satisfy the threshold condition by construction of the equation for the N function [(2.3)]. However, the corresponding D functions might have unwanted zeros lying on the physical sheet of the complex ν plane (ghost-zeros). It is the purpose of the present section to show that the D functions of our bootstrap solutions contain no such zeros.

Consider first, the case $\nu_0 = 0$. From (2.4) the corresponding D function is given by

$$D(\nu) = 1 - \frac{\nu}{\pi} \int_0^{+\infty} \frac{\nu' \rho(\nu') N(\nu')}{\nu'(\nu' - \nu)} d\nu' \quad (5.1)$$

Since $\rho(\nu) \geq 0$ for $\nu \geq 0$, a sufficient condition that $D(\nu)$ has no zeros on the first sheet of ν is:

$$N(\nu) \geq 0 \quad (5.2)$$

To see this, note first that for ν real and negative the condition (5.2) and Eq. (5.1) lead immediately to $D(\nu) \neq 0$; hence $D(\nu)$ has no real zeros. Next, for $\nu = x + iy$, $y \neq 0$, Eq. (5.1) gives directly

$$\text{Im } D(x + iy) = -\frac{y}{\pi} \int_0^{+\infty} \frac{\nu' \rho(\nu') N(\nu')}{(\nu' - x)^2 + y^2} d\nu' \quad (5.3)$$

Hence, if (5.2) is satisfied, $\text{Im } D(x + iy) \neq 0$ and $D(\nu)$ has no complex zeros either (the function $-D(\nu)$ is a Herglotz function).

Next, in order to derive sufficient conditions leading to (5.2), consider the integral equation for the N function. With $\nu_0 = 0$ Eqs. (2.7) to (2.9) lead to

$$N(\nu) = \lambda B(\nu) + \frac{\lambda \nu}{\pi} \int_0^\infty \frac{B(\nu) - B(\nu')}{\nu - \nu'} \frac{N(\nu')}{\nu'} d\nu' \quad (5.4)$$

In Sect. 4 it was found that an iteration solution exists for all our self-consistent values of λ (corresponding to $4.1 \leq m^2 \leq 30$). From (5.4) it is easily seen that in an iteration series solution for $N(\nu)$, all the terms will be positive provided that

$$(i) \quad \lambda B(\nu) \geq 0 \quad (ii) \quad \lambda \frac{dB(\nu)}{d\nu} \geq 0 \quad (\nu \geq 0) \quad (5.5)$$

In particular (ii) implies that $\lambda(B(\nu) - B(\nu'))/(\nu - \nu') \geq 0$ for any positive ν and ν' . The conditions (5.5) are sufficient (but not necessary) for (5.2) to be satisfied and thus for $D(\nu)$ to be free of unwanted zeros.

Figure 3 gives the Born terms $B(\nu)$ for physical values of ν and for the bootstrap solutions corresponding to $m^2 = 4.1, 5, 10$ and 30 . Apart from the case $m^2 = 30$ the conditions (5.5) are strictly satisfied. The absence of unwanted zeros of $D(\nu)$ for $m^2 = 30$ as well will be shown below.

It remains to be shown that the conclusions are not altered in the case $\nu_0 \neq 0$. For this, let D_{ν_0} be the denominator function corresponding to $\nu_0 \neq 0$ (i.e. for the N/D decomposition of Eqs. (2.3), (2.4)). Next define

$$D_0(\nu) = D_{\nu_0}(\nu) / D_{\nu_0}(0) \quad (5.6)$$

It will be shown that $D_0(\nu)$ is the denominator function corresponding to $\nu_0 = 0$. Indeed, from (2.3), (2.4) it follows that $D_{\nu_0}(\nu)$ satisfies the integral equation

$$D_{\nu_0}(\nu) = 1 + (\nu - \nu_0) \int_{-\infty}^{-\nu_L} d\nu' K(\nu, \nu'; \nu_0) D_{\nu_0}(\nu') \quad (5.7)$$

where

$$K(\nu, \nu'; \nu_0) = \frac{1}{\pi^2} \frac{\Delta A_1(\nu')}{\nu'} \int_0^{\infty} \frac{d\xi \cdot \xi \cdot \rho(\xi)}{(\xi - \nu)(\xi - \nu')(\xi - \nu_0)} \quad (5.8)$$

Hence

$$D_{\nu_0}(0) = 1 - \nu_0 \int_{-\infty}^{-\nu_L} d\nu' K(0, \nu'; \nu_0) D_{\nu_0}(\nu') \quad (5.9)$$

Dividing (5.7) by $D_{\nu_0}(0)$ (assumed $\neq 0$) and using the definition (5.6) one has

$$D_0(\nu) = [D_{\nu_0}(0)]^{-1} + (\nu - \nu_0) \int_{-\infty}^{-\nu_L} d\nu' K(\nu, \nu'; \nu_0) D_0(\nu') \quad (5.10)$$

However, $[D_{\nu_0}(0)]^{-1}$ can be calculated from (5.9) in terms of an integral over $D_0(\nu)$. Simple cancellation of terms in (5.10) gives

$$D_0(\nu) = 1 + \nu \int_{-\infty}^{-\nu_L} d\nu' K(\nu, \nu'; 0) D_0(\nu') \quad (5.11)$$

This proves the assertion. Hence, if $D_{\nu_0}(0) \neq 0$, absence of unwanted zeros of $D_0(\nu)$ implies the same for $D_{\nu_0}(\nu)$.

If $N_{\nu_0}(\nu)$ denotes the numerator function corresponding to $D_{\nu_0}(\nu)$ and $N_0(\nu)$ that corresponding to $D_0(\nu)$, use of (2.3) and of (5.6) leads to

$$N_0(\nu) = N_{\nu_0}(\nu) / D_{\nu_0}(\nu) \quad (5.12)$$

Hence

$$N_0(\nu) / D_0(\nu) = N_{\nu_0}(\nu) / D_{\nu_0}(\nu)$$

which means that the partial wave amplitude is independent of the subtraction point ν_0 .

To complete the study of zeros of $D(\nu)$ for our bootstrap solutions consider the case $n^2 = 30$, in which the sufficient condition (5.5) was not strictly applicable. Direct calculation by the CERN CDC 3400 computer gives $N_{\nu_0}(\nu) > 0$ throughout $0 < \nu < 10^3$ (for large ν , $N_{\nu_0}(\nu) \rightarrow \text{const.}$, as discussed in the next Section). Equation (5.12) implies that $N_0(\nu)$ has constant sign for all $\nu > 0$. Then, from Eqs. (5.1), (5.3) it follows that the denominator function has no zeros on the physical sheet of complex ν for the self-consistent solution $n^2 = 30$ as well.

6. ASYMPTOTIC BEHAVIOUR AND SUM-RULE

The form of the left-hand discontinuity that was chosen and elastic unitarity specify completely the asymptotic behaviour of the partial wave amplitude $A_1(\nu)$. This, combined with the threshold condition $A_1(0) = 0$, leads to a simple sum-rule. The explicit verification of this sum-rule will be an important test of the consistency of our bootstrap solutions.

Due to the presence of the cut-off Λ in (2.11), the equation (2.3) for the N function implies that for $\nu \rightarrow +\infty$

$$N(\nu) \rightarrow \text{const.} \quad (6.1)$$

With this, it can be shown from (2.4) that

$$D(\nu) \sim \log \nu \quad (6.2)$$

and therefore

$$A_1(\nu) = \frac{N(\nu)}{D(\nu)} \sim (\log \nu)^{-1} \quad (6.3)$$

Then, elastic unitarity gives

$$\text{Im } A_1(\nu) \sim (\log \nu)^{-2} \quad (6.4)$$

However, due to the assumed left-hand discontinuity,

$$\text{Im } A_1(\nu) = 0 \quad \nu < -\Lambda \quad (6.5)$$

In view of (6.4), (6.5) and of elastic unitarity, $A_1(\nu)$ satisfies the unsubtracted representation:

$$A_1(\nu) = \frac{1}{\pi} \int_{-\Lambda}^{-\nu_L} \frac{d\nu' \Delta A_1(\nu')}{\nu' - \nu} + \frac{1}{\pi} \int_0^{\infty} \frac{d\nu' \rho(\nu') |A_1(\nu')|^2}{\nu' - \nu} \quad (6.6)$$

But then, the threshold condition $A_1(0) = 0$ gives the following sum-rule⁹⁾

$$\int_{-\Lambda}^{-\nu_L} \frac{d\nu' \Delta A_1(\nu')}{\nu'} + \int_0^{\infty} \frac{d\nu' \rho(\nu') |A_1(\nu')|^2}{\nu'} = 0 \quad (6.7)$$

With (2.11) the first integral can be calculated explicitly; the second integral can be calculated in terms of $N(\nu)$ given by (4.4) and of the corresponding $D(\nu)$ given by (2.4).

The fulfilment of the sum rule (6.7) by our bootstrap solutions was examined by calculating the numerical value of $A_1(0)$ as given by (6.6). The results are presented in Table 2. For all solutions the ratio

$$A_1(0) / \frac{1}{\pi} \left| \int_{-\Lambda}^{-\nu_L} d\nu \frac{\Delta A_1(\nu)}{\nu} \right|$$

is of the order of 10^{-3} to 10^{-4} . These values give also a measure of the numerical accuracy of the results.

Thus, the sum-rule (6.7) is satisfied in practice by all bootstrap solutions. In view of the absence of zeros of $D(\nu)$ on the first sheet of complex ν , which was shown in Sect. 5, this conclusion is hardly surprising.

However, in this way, the value of the parameter Λ (cut-off) remains completely undetermined. At least within the present simple bootstrap model, to specify Λ an extra condition is necessary. In principle,

such a condition can be formulated if some information is given about the alleged damping mechanism, which operates at asymptotic energies.

Finally, the fulfilment of Levinson's theorem will be considered. The P-wave phase-shift $\delta_1(\nu)$ is defined by

$$A_1(\nu) = \bar{e}(\nu) \cdot e^{i\delta_1(\nu)} \sin \delta_1(\nu) \quad (6.8)$$

In the absence of CDD poles, which has been assumed throughout this work, Levinson's theorem gives

$$\delta_1(0) - \delta_1(\infty) = \pi n_B \quad (6.9)$$

where n_B is the number of bound states in the P-wave. For pseudoscalar-pseudoscalar scattering: $n_B = 0$; thus, with the definition $\delta_1(0) = 0$, (6.9) requires

$$\delta_1(\infty) = 0 \quad (6.10)$$

To show that this is true notice first that, as proved in Sect.5, $N(\nu) > 0$ for $\nu > 0$. Hence, $A_1(\nu)$ has no zeros in the physical region and, in view of (6.8),

$$0 < \delta_1(\nu) < \pi$$

Furthermore, it can be shown that, for large ν , $\cos \delta_1(\nu)$ is positive.

Indeed,

$$\operatorname{Re} A_1(\nu) = N(\nu) \frac{\operatorname{Re} D(\nu)}{|D(\nu)|^2}$$

for $\nu \rightarrow \infty$, Eq. (2.4) and $N(\nu) > 0$ imply that $\text{Re } D(\nu)$ increases (logarithmically) towards positive values; and therefore that $\text{Re } A_1(\nu) > 0$, which through (6.8) leads to $\cos \delta_1(\nu) > 0$. This and Eq. (6.3) lead immediately to (6.10). In particular, (6.3) implies that for $\nu \rightarrow \infty$:

$$\delta_1(\nu) \sim (\log \nu)^{-1} \quad (6.11)$$

To illustrate the conclusions of this work concerning the behaviour of $N(\nu)$ and $D(\nu)$, Fig. 4 gives the functions $N(\nu)$ (for $\nu > 0$) and $\text{Re } D(\nu)$ corresponding to the self-consistent solution $n^2 = 4.3$. As expected, at large ν , $N(\nu)$ tends to a constant and $\text{Re } D(\nu)$ increases logarithmically. Note that only the first zero of $\text{Re } D(\nu)$, at $\nu = \nu_R$, corresponds to a resonance; for, $(d \text{Re } D(\nu)/d\nu)_{\nu_R} < 0$, which leads to a positive resonance width.

7. CONCLUSIONS

The basic conclusions of the present work could be summarized as follows.

In a model, with a left-hand discontinuity defined by the exchange of a vector meson of mass m (between two pseudoscalars of mass μ) with a sharp cut-off Λ , bootstrap solutions exist for masses varying from just above threshold ($m^2/\mu^2 = 4.1$) up to $m^2/\mu^2 = 30$ (and, by all indications, for higher m^2/μ^2 too). The corresponding values of the cut-off vary from $\Lambda \simeq 11 \mu^2$ to $\Lambda \simeq 50 \mu^2$. Notice that these are rather small cut-offs; thus, the principle of dominance of nearby singularities, which forms the basis of the whole approach, is justified a posteriori. Furthermore, in a specific (one-pole) parametrization of the distant left-hand singularities (i.e. of the "short-range force") the corresponding self-consistent pseudoscalar-vector couplings are found to be practically insensitive to variations of the relevant parameter.

For $4.1 \leq m^2/\mu^2 \leq 30$ these coupling constants vary from ~ 2.5 to 8 times the experimental values (mainly depending on whether the comparison is made on the basis of an $SU(3)$ - or an $SU(2)$ - symmetric model).

In all these cases an iteration solution in powers of the coupling constant does exist. This and the independence of our exact solutions of the subtraction point are used to prove that the D functions obtained have no (unwanted) zeros on the physical sheet of the complex ν plane (ghost-free solutions). A simple sum-rule derived in terms of the asymptotic properties and the threshold behaviour of the amplitude is found to be satisfied. Finally, it is shown that the corresponding phase-shift satisfies Levinson's theorem and, for large ν , behaves like $(\log \nu)^{-1}$.

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- 5) The coupling g is defined in such a way that in the P-wave of the direct channel the residue of the function

$$\nu^{-1}A_1(\nu) = \nu^{-1}\rho^{-1}(\nu)e^{i\delta_1(\nu)} \sin \delta_1(\nu)$$

[cf. formula (6.8)] at the V meson pole is $-g^2$.

- 6) An alternative, which also leads to a Fredholm kernel, is to take $\Lambda = \infty$ in (2.11), but replace the range $0 \leq \nu \leq \infty$ of integration in (2.7) by the finite range $0 \leq \nu \leq \Lambda'$; this has been considered by Bander and Shaw (Ref.3) and by J.P. Fulco, G. Shaw and D.T. Wong, Phys.Rev. 137, B 1242 (1965).
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- 9) To our knowledge, consistency checks of the type of Eq.(6.7) have been first proposed by G. Frye and R.L. Warnock, Phys.Rev. 130, 478 (1963); and by A.P. Balachandran and F. von Hippel, Ann.Phys. (N.Y.) 30, 446 (1964) who have also considered higher-order restrictions. See also A. Martin, Nuovo Cimento 38, 1326 (1965). In this work we shall be content with the investigation of (6.7).

Table 1

m^2/μ^2	Λ/μ^2	λ	$\ G \ $
4.1	11.0	2.75	0.195
4.3	10.85	2.88	0.171
5.0	10.85	3.33	0.116
10.0	17.0	4.43	0.042
30.0	48.5	4.83	0.045

Table 2

m^2/μ^2	$\frac{1}{\pi} \int_{-\Lambda}^{-\nu_L} \frac{d\nu \Delta A_1(\nu)}{\nu}$	$\frac{1}{\pi} \int_0^{\infty} \frac{d\nu \rho(\nu) A_1(\nu) ^2}{\nu}$	$A_1(0)$
4.1	-7.0733	7.0749	0.0016
4.3	-6.8746	6.8763	0.0017
5.0	-6.4431	6.4452	0.0021
10.0	-5.2947	5.2978	0.0031
30.0	-4.8013	4.8053	0.0040

Figure Captions

- Fig. 1 The form of the left-hand discontinuity $\Delta A_1(\nu)$ and the distribution of the points ν_i of (3.1) and (3.4) used in the numerical calculations.
- Fig. 2 Plots of the coupling λ versus cut-off Λ for various values of m^2 and with R_0/ν_0 as free parameter. The points * correspond to $R_0/\nu_0 = 0$; $\square$ to $R_0/\nu_0 = 1$; and $\odot$ to $R_0/\nu_0 = -1$.
- Fig. 3 The Born terms $B(\nu)$ corresponding to the considered self-consistent solutions.
- Fig. 4 The functions $N(\nu)$ and $\text{Re } D(\nu)$ corresponding to the self-consistent solution $m^2 = 4.3$.

FIG. 1

FIG. 2

FIG. 3

FIG.4