Extension of sine-Gordon field theory from generalized Clifford algebras.

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Abstract
Linearization of homogeneous polynomials of degree $n$ and $k$ variables leads to generalized Clifford algebras. Multicomplex numbers are then introduced in analogy to complex numbers with respect to usual Clifford algebra. In turn multicomplex extensions of trigonometric functions are constructed in terms of ‘compact’ and ‘non-compact’ variables. It gives rise to the natural extension of the $d$–dimensional sine-Gordon field theory in the $n$–dimensional multicomplex space. In dimension 2, the cases $n = 1, 2, 3, 4$ are identified as the quantum integrable Liouville, sine-Gordon and known deformed Toda models. The general case is discussed.

In most domain of research activities apparently disparate facts and data are first accumulated and later on classified and interpreted in terms of an underlying hidden order. There are numerous examples in physics ranging from Maxwell’s early unification of electricity and magnetism to the actual understanding of particle physics and elementary interactions. The hidden structure takes the form of underlying algebras. A recent celebrated example relates to the interpretation of the universality of critical phenomena in two dimensions in term of Virasoro algebra. Among the possible algebras those induced by bilinear relations have a special status, probably due to the bilinear aspects of fundamental objects such as quadratic metric, commutators, anticommutators, etc. However, algebras going beyond the quadratics ones have been constructed in the 70’s from underlying polynomials of degree higher than two. There are dubbed by mathematicians as Clifford algebras of polynomials [1, 2]. The matrix representations of such algebras [3] lead to natural algebraic extensions of the Clifford and Grassmann algebras [4, 5]. These families endow extension of complex numbers [6, 7] leading to the corresponding extension of trigonometric functions, dubbed multisine functions (see also [8]).

The potential usefulness of the multisine functions has already been pointed out and explored only briefly so far. In the context of field theory it is natural to focus on one of the ‘simplest’ (apparently) famous form, the sine-Gordon (SG) model, known for its integrability and fermionization properties related to the massive Thirring model for a specific relation between the couplings [9, 10, 11].

By analogy, Toda and affine Toda field theories (TFT and ATFT) based on simply or non-simply laced algebras are the natural simple Lie group extension of the SU(2) and \( \hat{SU}(2) \) Toda and affine Toda field theories, e.g. Liouville and SG models. The interest in such deformed Toda theories comes from their integrability and duality properties [12, 13] which can be used to pave the way to the understanding of electric-magnetic duality in four-dimensional gauge theories, conjectured in [14] and developed in [15]. Thereby non-perturbative analysis of the spectrum and of phase structure in supersymmetric Yang-Mills theory becomes possible.

Although the filiation of these models is recognized in the way already mentioned there is at present no common framework to link their studies. It is the purpose of this letter to point out such a possibility as a consequence of the properties of underlying generalized Clifford algebras.

The mathematical formalism is first introduced. A representation of the multisine functions is constructed on the basis of ‘compact’ and ‘non-compact’ directions. Two sets of variables appear: one set, namely \( \phi_p \), represents the \( E(n/2) \) ‘compact’ directions and the other, namely \( \varphi_p \), represents the \( n - 1 - E(n/2) \) ‘non-compact’ directions (where \( E(x) \) stands for the integer part of \( x \)). Clearly for \( n = 2 \) only one compact direction exists corresponding to the standard trigonometric angle. Then, the explicit expressions of the multisine are obtained for all \( n \).

We construct next the natural extensions of the sine-Gordon model with the substitution of the 2-dimensional complex space by the \( n \)-dimensional multicomplex space. The multicomplex extension of order \( n \) of the cosine function appears naturally hence the name of multisine-Gordon model (MSG).

In two dimensions and for specific values of the coupling constants, the first cases
$n = 1, 2, 3, 4$ are clearly identified as integrable quantum field theories (IQFT), $n = 1$ corresponding to $sl(2)$-TFT (Liouville), $n = 2$ to $sl(2)$-ATFT (SG), while $n = 3$ and $n = 4$ to $sl(2)$-ATFT deformed respectively by one and two fermionic interactions of the $\bar{\psi}\psi$ type. These two cases are related respectively to integrable deformations of the non-linear sigma model in Witten’s black hole metric and of the $N = 2$ supersymmetric SG.

Finally we envisage the cases $n = 5$ and 6. The integrability of the two models is discussed. Some conclusions and perspectives are finally drawn.

The set of multicomplex (MC) numbers is generated by one element $e$ which satisfies $e^n = -1$ [6] ($\mathbb{MC}_n = \left\{ x = \sum_{i=0}^{n-1} x_i e^i, e^n = -1, x_i \in \mathbb{R} \right\}$), and constitute a $n$–dimensional commutative algebra. It is worth stressing that most of the results of usual complex numbers analysis remains true for MC-numbers, were it be for algebraic [6] or analytic commutative algebra. It is worth stressing that most of the results of usual complex matrices, and, hence among the $n$ compact directions and $n$–basis (for $n$ even) $\sum_{i=1}^{n-1} \Theta_i e^i = \sum_{a=0}^{n/2-1} \left[ \phi_a p_a + \varphi_a (-p_a^2) \right]$, with $p_a$ ($a = 0, \cdots, n/2 - 1$) given by $p_a = 2 \sum_{j=0}^{n-1} \sin \left( (2a+1) j \frac{n}{2} \right) e_j$, we get

$$\text{mus}_k (\phi_a, \varphi_a) = \frac{2}{n} \sum_{a=0}^{n/2-1} \cos \left[ \frac{\phi_a - (2a+1)k\pi}{n} \right] \exp (\varphi_a), \quad \text{for} \ k = 0, \cdots, n-1,$$

with $\varphi_0 + \varphi_1 + \cdots + \varphi_{n/2-1} = 0$.

This last equation is necessary to ensure unimodular MC-numbers and to match the number of $\Theta$'s with the number of $\phi$ and $\varphi$'s. This constraint of unimodularity translates into a $n$–th order relation among the mus-functions [6].

In the odd case, the situation is more intricate, and the appropriate change of basis is $e^i, i = 1, \cdots, n - 1 \rightarrow p_a, -p_a^2 - 2d, a = 0, \cdots, (n - 1)/2 - 1$ with $p_a$ given as before and
\[
\sum_{i=1}^{(n-1)/2} (-1)^i e^i, \text{ leading to}
\]
\[
\text{mus}_k(\phi_a, \varphi_a) = \frac{2}{n} \sum_{a=0}^{(n-1)/2} \cos \left[ \phi_a - \frac{(2a + 1)k\pi}{n} \right] \exp(\varphi_a)
\]
\[
+ \frac{(-1)^k}{n} \exp(\tilde{\varphi}), \text{ for } k = 0, \ldots n - 1,
\]
with \( \tilde{\varphi} = -2 \sum_{a=0}^{(n-1)/2-1} \varphi_a. \)

As for \( n \) even the constraint among the non-compact variables translates into a \( n \)-th order constraint among the mus-functions.

Now, let us study the embedding of known models in multicomplex extensions. To construct some new models of field theory as extensions of the sine-Gordon model, the first obvious step is to consider potential terms as multisine functions. Although conceptually simple it is surprisingly fruitful in relating apparently different field theoretic models. We introduce a field theory associated to the \( n \)-dimensional multicomplex space by:

\[
\mathcal{A}_k^{(n)} = \int d^4 x \left[ \frac{1}{2} (\partial_\mu \Phi)^2 + J_k \text{mus}_k(\alpha_a \phi_a, \beta_b \varphi_b) \right],
\]
where \( J_k \) is real and \((\alpha_a, \beta_b)\) are real coupling constant.

- \( \Phi = (\phi_0, \ldots, \phi_{\frac{n}{2}-1}, \varphi_1, \ldots, \varphi_{\frac{n}{2}-1}) \) if \( n \) is even,
- \( \Phi = (\phi_0, \ldots, \phi_{\frac{n-1}{2}-1}, \varphi_0, \ldots, \varphi_{\frac{n-1}{2}-1}) \) if \( n \) is odd.

One may restrict to the case \( k = 0 \) since appropriate translations in the compact directions generate multisine functions of higher orders from \( \text{mus}_0 \) [6]. Then, in the sequel, we only consider (3) with \( k = 0 \) (\( J_0 = -\mu \)). Furthermore, we choose \( d = 2 \). At classical level these Lagrangians are unbounded from below except for the cases \( n = 1 \) and \( n = 2 \). However we shall consider the quantum versions which becomes properly defined due to the presence, upon fermionization, of appropriate counterterms.

In both cases (\( n \) even or odd), the QFT (3) can be cut in two parts. One part can be viewed as \( \frac{n}{2} - 1 \) (resp. \( \frac{n-1}{2} \)) conformal field theories (CFTs) \( a = 1, \ldots, \frac{n}{2} - 1 \) (resp. \( a = 0, \ldots, \frac{n-1}{2} - 1 \)):

\[
\mathcal{A}_k^{CFT} = \int d^2 x \left[ \frac{1}{2} (\partial_\mu \phi_a)^2 + \frac{1}{2} (\partial_\mu \varphi_a)^2 - \frac{2}{n} \mu e^{\beta x \varphi_a} \cos(\alpha_a \phi_a) \right],
\]
with an holomorphic stress-energy tensor (\( z = x_1 + ix_2, \bar{z} = x_1 - ix_2 \) are the complex coordinates) :

\[
T_a(z) = -2\pi \left[ (\partial_z \varphi_a)^2 + (\partial_{\bar{z}} \phi_a)^2 - 4Q_{\beta_a} \partial_{\bar{z}}^2 \varphi_a \right]
\]
with $Q_{\beta_a} = \frac{1}{4\alpha_a}$, $\alpha_a^2 - \beta_a^2 = 4\pi$ and central charge $c_a = 2 + \frac{3(8\pi)}{2\beta_a^2}$. For $n$ even, we add one other CFT (for $a = 0$) which corresponds to a free field theory (with central charge $c_0 = 1$). The other part (the last term of eqs. (1) and (2)) corresponds to a perturbation which destroys conformal invariance and generates masses.

By comparison for ATFT the addition of the extra root term destroys in the same manner the TFT conformal invariance. Of course, we could write these QFT in a way similar to ATFT (defining a root system, labels, etc...) but it does not seem to give more insight.

Except for the cases $n = 1$ and $n = 2$ (Liouville and sine-Gordon models) we do not know any Lax pair with a vanishing commutator for the classical equations of motion which could proof the classical integrability of these theories. Moreover some analog to the Cartan-Weyl basis, useful for Toda models, is not obvious here.

On the other hand, if a theory is quantum integrable in (1+1) dimensions, the conservation of currents can be obtained from massless perturbation theory in which the whole exponentials terms are treated as interaction terms. This is essentially equivalent to OPE techniques. As for massive two dimensional field theories, the ultraviolet divergent diagrams are tadpoles, removed by normal-ordering\textsuperscript{1}. We work in Euclidian space, with $(z, \bar{z})$ coordinates. The massless bosonic propagators are identical and such that $\langle \phi(z, \bar{z})\phi(0,0) \rangle = \langle \varphi(z, \bar{z})\varphi(0,0) \rangle = -\frac{1}{4\pi} \ln(2z\bar{z})$. Local holomorphic currents (with spin $s$) of the form ($n$ even):

$$T^{(s)} = \sum_{r_1, r'_1} X_{p_1 - p_2}^{r_0 - r_2 - 1} \cdots r'_2 \cdots (\partial^{p_0} \phi^{(r_0}(\partial^{r_2} - 1) \phi^{(r_2)\cdots(7)}$$

\[\times (\partial^{r_1} \varphi^{(r'_1)}(\partial^{r'_2} - 1) \varphi^{(r'_2)\cdots(6)} = \sum p_i r_i + p'_i r'_i = s \text{ and } X_{p_1 - p_2}^{r_1 - r_2} \cdots r'_2 \cdots (\partial^{p_0} \phi^{(r_0}(\partial^{r_2} - 1) \phi^{(r_2)\cdots(7)} \text{ determined - up to the shift } T^{(s)} \rightarrow T^{(s)} + \partial \Lambda^{(s)}$$

- by the conservation laws:

$$\partial \langle T^{(s)} \rangle \phi, \varphi \rangle = +\partial \langle T^{(s)} \rangle \phi, \varphi \rangle = 0,$$

where the expectation value is taken on the Fock space vacuum of the massless quantum field theory and $\mathcal{V}(\phi, \varphi)$ stands for the different vertex operators which appears in the multisine potential. The coefficients $X_{p_0 - p_2}^{r_0 - r_2 - 1} \cdots r'_2 \cdots$ possess the same symmetry properties as the Lagrangian, resulting in a set of strong constraints among them. Generalization to the case $n$ odd is straightforward, with a change $\frac{n}{2} - 1$ to $\frac{n-1}{2}$. The $X$’s are then determined in a way such as to cancel contributions coming from potential anomalies - e.g. local contributions which cannot be written as the $\partial$-derivatives of some suitable expression.\textsuperscript{2}

\textsuperscript{1}The normal-ordered version of vertex operator is: $\exp(\beta\phi) := \exp(\beta\phi)/a^2$ where $a$ is a cut-off. However, this multiplicative constant leads to no modification regarding OPE calculations.

\textsuperscript{2}Anti-holomorphic currents can be defined as in [17].
• Case $n = 1$: the Liouville model.

In this case, the multicomplex space is just the real space. Strictly speaking there is no exponential representation in the sense given before. However, if instead of unimodular numbers, positive numbers are used, one may write $x = \exp(\phi)$ leading to the potential $\exp(-2\beta\phi)$. This field theory is then integrable as it corresponds to a CFT, the well-known Liouville model:

$$A^{(1)}_0 = \int d^2 x \left[ \frac{1}{2} (\partial_\mu \phi)^2 - \mu \exp(-2\beta\phi) \right]. \quad (8)$$

• Case $n = 2$: the sine-Gordon model.

For $n = 2$ we immediately obtain the well-known sine-Gordon model [9, 11] which is integrable [10] for $d = 2$:

$$A^{(2)}_0 = \int d^2 x \left[ \frac{1}{2} (\partial_\mu \phi)^2 - \mu \cos(\alpha\phi) \right]. \quad (9)$$

The 2-d fermion-boson correspondence [9] gives:

$$A^{(2)}_0 = \int d^2 x \left( i \bar{\psi} \gamma_\mu \partial_\mu \psi - \frac{g}{2} (\bar{\psi} \gamma_\mu \psi)^2 - \mu \bar{\psi} \psi \right), \quad (10)$$

with:

$$g/\pi = \frac{4\pi}{\alpha^2} - 1. \quad (11)$$

• Case $n = 3$.

Starting from the expression (3) for $n = 3$ the action writes:

$$A^{(3)}_0 = \int d^2 x \left[ \frac{1}{2} (\partial_\mu \Phi)^2 - \frac{\mu}{3} \left( 2 \exp(\beta_0 \phi_0) \cos(\alpha_0 \phi_0) + \exp(-2\beta_0 \phi_0) \right) \right]. \quad (12)$$

This model is integrable [16] for:

$$\alpha_0^2 - \beta_0^2 = 4\pi, \quad (13)$$

and the integrals $Q_2$ of spin 2 are generated by the conserved currents $T_3$ of spin 3 which have the form:

$$T_3 = A(\partial \phi_0)^3 + B(\partial \phi_0)(\partial \phi_0)^2 + C(\partial^2 \phi_0)(\partial \phi_0), \quad (14)$$

with: $A = 2b^2 + a^2$, $B = 3b^2$, $C = -3b(1 + 4b^2)$ and $a = \frac{\alpha_0}{\sqrt{8\pi}}$, $b = \frac{\beta_0}{\sqrt{8\pi}}$.

For $\mu = \frac{3}{2}(\frac{\alpha_0}{\beta_0})^{\frac{1}{2}}$ and with the shift $\phi_0 \rightarrow \phi_0 - \frac{1}{3\beta_0} \ln(\sqrt{\beta_0})$ this QFT possesses [16] a dual representation corresponding to the complex sinh-Gordon (CSG), which is a sigma
model with Witten’s black hole metric. For small $\beta_0$, one can use the 2-d fermion-boson correspondence which gives:

$$A_0^{(3)} = \int d^2x \left[ \frac{1}{2} (\partial_\mu \varphi_0)^2 + i \overline{\psi}_0 \gamma_\mu \partial_\mu \psi - \frac{g}{2} (\overline{\psi}_0 \gamma_\mu \psi)^2 - M \exp(\beta_0 \varphi_0) \overline{\psi}_0 - \frac{M^2}{2 \beta_0^2} \exp(-2 \beta_0 \varphi_0) \right. + \left. CT^{(3)}(\varphi_0, \beta_0) \right],$$

(15)

with $g$ defined by (11) and $CT^{(3)}(\varphi_0, \beta_0) = -\frac{M^2}{2 \beta_0^2} \exp(2 \beta_0 \varphi_0)$ a usual counterterm added to cancel the fermion loop divergence. Furthermore, it has the $U(1)$ symmetry generated by the charge $Q = \int d^2x \overline{\psi}_0 \gamma_0 \psi$. The mass spectrum\(^3\) is made of fermions ($\psi$, $\psi^\dagger$) with mass $M$ [16].

- Case $n = 4$.

Starting from the expression (3) for $n = 4$, the following action results

$$A_0^{(4)} = \int d^2x \left[ \frac{1}{2} (\partial_\mu \varphi_1)^2 + \frac{1}{2} (\partial_\mu \phi_0)^2 + \frac{1}{2} (\partial_\mu \phi_1)^2 - \frac{\mu}{2} \left( \exp(-\beta_1 \varphi_1) \cos(\alpha_0 \phi_0) + \exp(\beta_1 \varphi_1) \cos(\alpha_1 \phi_1) \right) \right].$$

Restricting to $\alpha_0 = \alpha_1 = \alpha$ and $\beta_1 = \beta$, the model is integrable if (13) is verified [16](with the substitution $\alpha_0 \rightarrow \alpha$ and $\beta_0 \rightarrow \beta$).

The first non-trivial spin 4 current $T_4$ generating the conserved charge $Q_3$ is:

$$T_4 = A \left[ (\partial \phi_0)^4 + (\partial \phi_1)^4 \right] + B \left( \partial \varphi_1 \right)^4 + C \left[ (\partial \phi_0)^2 (\partial \phi_1)^2 + (\partial \phi_0)^2 (\partial \varphi_1)^2 + (\partial \phi_1)^2 (\partial \varphi_1)^2 \right] + D \left[ (\partial^2 \varphi_1) (\partial \phi_0)^2 - (\partial^2 \varphi_1) (\partial \phi_1)^2 \right] + E \left[ (\partial^2 \phi_0)^2 + (\partial^2 \phi_1)^2 \right] + F (\partial^2 \varphi_1)^2,$$

(17)

with $A = a^2 (1 + 3b^2)$, $B = \frac{b^2}{2} (3a^2 - 1)$, $C = 6a^2 b^2$, $D = 6a^2 b$, $E = 2(1 - 6a^2 + 10a^4)$, $F = 1 - 11a^2 + 7a^4$ and $a = \frac{2}{\sqrt{8\pi}}$; $b = \frac{\beta}{\sqrt{8\pi}}$.

Moreover, the QFT (17) has $U(1) \times U(1)$ symmetry generated by the charges $Q_a = \int d^2x \overline{\psi}_a \gamma_0 \psi_a$. For small $\beta$, the 2-d fermion-boson correspondence gives:

$$A_0^{(4)} = \int d^2x \left[ \frac{1}{2} (\partial_\mu \varphi_1)^2 + \sum_{a=0}^{a=1} \left( i \overline{\psi}_a \gamma_\mu \partial_\mu \psi_a - \frac{g}{2} (\overline{\psi}_a \gamma_\mu \psi_a)^2 \right) - \frac{\mu}{2} \sum_{a=0}^{a=1} \left( \exp(\beta_a \varphi_a) \overline{\psi}_a \psi_a \right) + CT^{(4)}(\varphi_1, \beta) \right],$$

(18)

with $g$ defined by:

$$\frac{g}{\pi} = \frac{4\pi}{\alpha^2} - 1 = \frac{\beta^2}{\alpha^2} = -\frac{\beta^2}{4\pi} \frac{1}{1 + \frac{\beta^2}{4\pi}},$$

(19)

\(^3\)The boson $\varphi_0$ is unstable and does not appear in the spectrum.
\( \beta_0 = 1 \) and \( CT^{(4)}(\varphi_1, \beta) \sim \cosh(2\beta \varphi_1) \).

For \( \mu = 2M \) and \( CT^{(4)}(\varphi_1, \beta) = -\frac{M^2}{2\beta} \cosh(2\beta \varphi_1) \) this QFT was studied in [16] and is related to \( N = 2 \) supersymmetric SG theory, anisotropic principal chiral field, the O(4) non-linear sigma model, etc. The factorized scattering theory gives the S matrix as a product of two SG S-matrices.

- Other cases.

The cases \( n = 5,4 \) and \( n = 6 \) initiate an ensemble (\( n \) even, \( n \) odd) of models possessing similar vertex operator structure. In keeping with the earlier cases, more generally for \( \beta_a \) small and for \( k = 0 \), the \( 2 - d \) fermion-boson correspondence [9, 11] can be used to rewrite the action (3) in a form more suitable for perturbative analysis, using for \( d = 2 \) the definitions (3). We can define also \( CT^{(n)}(\varphi, \beta) \), the appropriate counterterms to cancel the divergences coming from the fermions loops. For example the counterterms of \( n = 3 \) and \( n = 4 \) correspond respectively to the Liouville and sinh-Gordon potential (\( SU(2) \) and \( SU(2) \) Toda potential). These two series of QFT possess \( U(1) \times \ldots \times U(1) \) symmetry generated by the charges \( Q_a = \int dx \bar{\psi}_a \gamma_0 \psi_a \). The quantum integrability of these higher order QFT is an open question. There are some arguments pointing to the integrability of the case \( n = 5 \) using algebra embeddings. Indeed, the mus-fuctions we have considered, as well as there \( n \)-order associated constraint, are basically related to the matrix realization of the MC-algebra [6]. However, it is also possible to find a matrix representation of \( IMC_n \) of dimension higher than \( n \). This is essentially related to the embeddings \( IMC_n \subset IMC_m \) when \( n < m \) [6] (in fact to be more precise we have to distinguish odd and even \( n \), but this subtlety is irrelevant here). With such an embedding we can obtain other mus-fuctions of order \( n \). The expression of the mus are similar to those given in (1-2) but an other constraint is obtained in the non-compact direction. This translates into a relation of degree \( m \) on the mus instead of a relation of degree \( n < m \). Detailed work is in progress [18] and will be reported elsewhere. So far our investigations indicate that the cases \( n > 5 \) seem more problematic.

Finally, some remarks are necessary about the parameter \( \mu \) introduced which is very important as it defines the mass spectrum obtained for bosonic fields \( \varphi_a \). Then, the study of dynamics is more problematic as Yang-Baxter equations becomes more complicated and non-diagonal scattering may appear. However, integrability still holds.

- Concluding remarks.

The concept of unifying algebras has long proved fruitful in the understanding of apparently disparate physical phenomena. Major advances and developments in critical phenomena in two dimensions occurred from properties of local conformal invariance with the underlying Virasoro algebra. Quantum field theory in two dimensions has accumulated a variety of integrable models ranging from Liouville, sine-Gordon to Toda and affine Toda theories, etc...In TFT the coupling constant is chosen real or imaginary and a similar transformation leads to the equivalent extension of sinh-Gordon theory, opening

\(^{4}\)It can be checked that these cases correspond to \( C_2^{(1)} \) and \( D_2^{(2)} \)-ATFT deformed by two fermionic interactions [18]
of a new domain of investigation however beyond the scope of this exploratory letter. In the search for a common framework we have shown that generalized Clifford algebras and their associated multicomplex structure and trigonometry provide a strikingly unifying scheme for a whole set of QFT models of higher and higher complexities. The main characteristics of the primary models is their known integrability. It is believed that this new framework should provide some insight on the properties of QFT’s of higher order in the hierarchy as well as to lead to some genuine developments in models of statistical physics.

Finally to conclude, we would like to mention that the specific algebraic structure of the MC-numbers should be relevant for a general analysis. One can compare for instance the property of the model $n = 4$ studied where the matrix $S$ can be written as a product of two SG $S$--matrix with the property that $\mathbb{MC}_4$ contains the set of complex number [6]. Anyhow, if we use higher dimensional representations for the MC-number, we have, in principle, other models than those described in (3) with (1-2), having other constraints upon the non-compact variables.

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References


