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IMPACT PARAMETER DESCRIPTION OF RELATIVISTIC SCATTERING

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A B S T R A C T

An impact parameter description of the scattering of relativistic particles is established which, at high energy, is completely analogous to the picture of high energy scattering in the Schrödinger theory. The investigation is based entirely on general properties like analyticity and unitarity of the relativistic scattering amplitude.

In this investigation, an impact parameter description of the scattering of relativistic particles will be established, which at high energy is completely analogous to the picture of high energy scattering in non-relativistic Schrödinger theory. The following considerations will be based entirely on general properties like analyticity and unitarity of the relativistic scattering amplitude.

To make this statement more explicit, the situation in high energy Schrödinger theory shall first briefly be reviewed.

In potential scattering, the amplitude $f(s, t)$ for scattering of two spinless particles at high energy may be approximated by the eikonal form ¹⁾

$$f(s, t) = 2q\sqrt{s} \int_0^{\infty} db b h(b, s) J_0(b\sqrt{-t}) \quad (1)$$

s and t are the usual Mandelstam variables, q is the magnitude of the centre-of-mass momentum; the normalization of $f(s, t)$ and further kinematical details are indicated in the next section. The quantity b has an interpretation as impact parameter, i.e., the distance of the straight line of propagation of the incoming wave packet from the scattering centre. Thus, (1) gives a description of the scattering directly in terms of a spacial parameter. J_0 is the zero order Bessel function. If we define

$$h(b, s) = \frac{1}{2i} \left(e^{i\chi(b, s)} - 1 \right) \quad (2)$$

the eikonal $\chi(b, s)$ is given by ($q \rightarrow \infty$)

2.

$$\chi(b,s) = -\frac{m}{q} \int_{-\infty}^{+\infty} dz V(\sqrt{z^2 + b^2}) \quad ; \quad (3)$$

m is the reduced mass of the colliding particles. In our normalization, $f(s,t)$ is given through (1) up to terms of order q^0/d , where d is a measure of the region over which the potential $V(|\vec{x}|)$ varies slowly. On the other hand, the main term, given in (1), is for instance of order q^1 (for fixed t) for an energy independent potential, as may be seen with the help of (3). The approximation is valid for values of t in a region

$$-\frac{q}{d} \ll t \leq 0 \quad , \quad (4')$$

and for energies such that

$$qd \gg 1 \quad , \quad \frac{V}{q^2} \ll 1 \quad (4'')$$

Formulae (2) and (3) may be compared with the quasi-classical approximation to the partial wave amplitudes, valid at high energy and for large ℓ ²⁾. With the expression

$$f_\ell(s) = \frac{1}{2i} \left(e^{2i\delta_\ell(s)} - 1 \right) \quad (5)$$

for the partial wave amplitudes $f_\ell(s)$, this approximation reads

$$\delta_\ell(s) = -\frac{m}{q} \int_0^\infty dz V\left(\sqrt{z^2 + \left(\frac{\ell+1/2}{q}\right)^2}\right) \quad (6)$$

If we define an interpolating function $\delta(\ell, s)$ of the phase shifts $\delta_\ell(s)$, i.e., $\delta(\ell, s) = \delta_\ell(s)$ if $\ell = 0, 1, \dots$, by allowing ℓ to take arbitrary real values in equation (6), we find with the definition

$$f(\ell, s) = \frac{1}{2i} \left(e^{2i\delta(\ell, s)} - 1 \right) \quad (7)$$

of the corresponding interpolation of the partial wave amplitudes

$$h(b, s) = f\left(\ell = bq^{-\frac{1}{2}}, s\right). \quad (8)$$

This relation is only valid for large values of ℓ . However, in the integral (1) only a small region near $b = 0$ violates this condition and the contribution from this interval to the integral is negligible compared to the residual term. In view of the considerations in the following, we may neglect for instance contributions from $0 \leq b \leq b_0$ $Bq^{-1+\epsilon_2}$, $B > 0$ and constant and $\epsilon_2 > 0$ (ϵ_2 may be arbitrarily small); for $b \gg b_0$, we can use (8), where now $\ell \gg \ell_0 = Bq^{\epsilon_2 - \frac{1}{2}}$. The neglected term in this case is at most of order $q^{-1+2\epsilon_2}$ for an energy independent potential. Relation (8) shows clearly the meaning of the quantity b as impact parameter. Furthermore, we see also that the expression (1) for the scattering amplitude at high energy in its range of applicability gives a substitute for the partial wave expansion, the latter not being easy to handle in the considered energy region. This, because the number of contributing terms grows proportionally to q (for interactions of finite range) and the Legendre polynomials P_ℓ show an oscillatory behaviour in ℓ . It should also be remarked that despite its restricted angular range of validity, the impact parametrization (1) describes the predominant part of high energy scattering which is of diffraction type.

Expressions like (1) together with relation (8) are currently applied either in the phenomenological analysis of the observed high energy diffraction scattering in terms of (interpolated) partial wave amplitudes or in model calculations of the weight function $h(b,s)$, e.g., Refs. 3),4),5),6).

Thus it is desirable to establish the validity of an impact parameter description also within the framework of a relativistic theory of the scattering amplitude. As already mentioned in the beginning, it is possible to arrive at such a result by basing the argument entirely on general analyticity and unitarity properties. It should be emphasized that both of these ingredients are necessary in order to arrive at a picture completely analogous to Schrödinger theory, at least within the framework proposed above.

The basic assumptions about the relativistic scattering amplitude will be stated in the next section. In Section 3, we introduce, essentially on the basis of the analyticity properties, a representation of the scattering amplitude which is already of impact parameter type. This representation is valid over the whole energy range, i.e., $q \geq 0$, and in a domain of values of t containing the entire angular range $-4q^2 \leq t \leq 0$. Corresponding to the fact that exchange interactions are present in the relativistic case, there are now two weight functions $h_1(b,s)$ and $h_2(b,s)$. After a consideration of some general aspects of this impact parametrization of the amplitude, we discuss briefly some proposals of other authors [Refs. 5),7),8),9),10),11)] to introduce an impact parameter description within a relativistic framework.

In Section 4, the properties of the weight functions $h_1(b,s)$, especially in the variable b , are examined for arbitrary energy². Also, their connection with the partial wave amplitudes is given. In the next section, the form of the functions $h_2(b,s)$ for high energy is investigated and relations analogous to equation (8) are found, connecting them with the Carlson interpolations¹²⁾ of the partial wave amplitudes. We give two versions of the argument; in the first, some

further information on the scattering amplitude is needed in addition to what we choose as basic properties. In the second, we stay within our starting assumptions. This result, combined with some specific properties of the Carlson interpolations established by Kinoshita, Loeffel and Martin ¹³⁾, will lead in Section 6 to a high energy form of the scattering amplitude analogous to equation (1) [with the relation (8)]. In these considerations unitarity, in addition to the analyticity ingredient, plays an essential role.

Finally, in Section 7, a few statements are added about combining the impact parameter description at high energy with more detailed information from unitarity in the sense of Van Hove ¹⁴⁾. Also, the question of finding an optical potential reproducing the observed scattering is briefly discussed.

For reasons of simplicity, the case of spinless particles with equal masses will be investigated. It may be remarked that the following considerations can be carried through in perfect analogy for the helicity amplitudes of particles with spin.

6.

2. BASIC ASSUMPTIONS ABOUT THE SCATTERING AMPLITUDE

We consider the scattering of two relativistic non-identical particles. For reasons of simplicity, we suppose that their spins are zero and their masses equal ($=\mu$). We use the Mandelstam variables s, t, u with the condition

$$s + t + u = 4\mu^2 \quad (9)$$

as well as the magnitude q of momentum and the scattering angle θ in the s channel centre-of-mass system. The following relations hold :

$$q^2(s) = \frac{1}{4}(s - 4\mu^2) \quad (10)$$

$$z = \cos \theta = 1 + \frac{t}{2q^2} = -1 - \frac{u}{2q^2} \quad (11)$$

$$t + u = -4q^2. \quad (12)$$

In the following, an important role will be played by

$$\sqrt{-t} = 2q \sin \frac{\theta}{2}$$

The differential cross-section is given by

$$\frac{d\sigma}{d\Omega} = \frac{1}{s} |f(s, t, u)|^2, \quad (13)$$

where $f(s, t, u)$ is the invariant scattering amplitude.

We now assume that the function $f(s, t, u)$ has the following properties for $s \gg 4\mu^2$:

a)

$$f(s, t, u) = f_1(s, t) + f_2(s, u) \quad (14)$$

b) $f_1(s, t)$ and $f_2(s, u)$ are holomorphic in domains D_1 and D_2 , where

$$\begin{aligned} D_1 &= \left\{ t; |t| < t_0 \text{ or } \pi - \tau < |\arg t| \leq \pi \right\}, \\ D_2 &= \left\{ u; |u| < t_0 \text{ or } \pi - \tau < |\arg u| \leq \pi \right\}, \end{aligned} \quad (15)$$

t_0 is a positive number independent of s and $0 < \tau \leq \pi$.
 τ is independent of s and may be chosen arbitrarily small.

D_1 and D_2 are shown in Figs. 1 and 2.

c) For $t \rightarrow \infty$ or $u \rightarrow \infty$ in the interior or on the boundaries of the domains D_1 and D_2 , the functions $f_1(s, t)$ and $f_2(s, u)$ are bounded by powers in t or u , whose exponents do not depend on s .

d) For $s \rightarrow \infty$, the functions $f_1(s, t)$ and $f_2(s, u)$ are bounded by a power in s , with t or u independent exponents, for all t and u in D_1 or D_2 or on their boundaries.

e) If we introduce with

$$f(s, t, u) = \frac{\sqrt{s}}{q} \sum_{\ell=0}^{\infty} (2\ell+1) f_{\ell}(s) P_{\ell} \left(1 + \frac{t}{2q^2} \right), \quad (16)$$

$$f_e(s) = \frac{1}{2} \frac{q}{\sqrt{s}} \int_{-4q^2}^0 \frac{dt}{2q^2} P_e \left(1 + \frac{t}{2q^2} \right) f(s, t, u) \quad (17)$$

the partial wave expansion of $f(s, t, u)$, the unitarity of the S matrix has as a consequence

$$|f_e(s)|^2 \leq \text{Im } f_e(s) \leq 1. \quad (18)$$

With regard to the preceding assumptions, we remark the following :

- a) is motivated by the existence of a dispersion relation in the t channel ($s_0 \leq s \leq 0$). It is assumed that there exists an analytic continuation of f_1 and f_2 in the variable s from the physical values in the t channel to the physical values in the s channel. Then we have the structure of $f(s, t, u)$ for $s \geq 4\mu^2$ given by equation (14).
- b) means that when we perform the analytic continuation just indicated, the singularities in t or u do not enter the domains D_1 or D_2 .

That $f(s, t, u)$ is holomorphic in the interior of circles with origins $t = 0$ and $t = -4q^2$ has recently been proved by Martin ¹⁵⁾ from basic principles of field theory and unitarity. The radii of these circles are independent of s and for instance equal to $4\mu^2$ in the case we consider.

The other parts of the domains are chosen in a minimal way, to carry through the following considerations on the basis of analyticity and unitarity properties only. This is motivated by the fact we have not yet a definite idea about the singularity

picture in the t variable on the basis of general principles [holomorphy of $f_1(s,t)$ and $f_2(s,u)$ in domains of this type is prove to every order in perturbation theory for complex s ¹⁶⁾]. $\mathcal{C} = \mathcal{C}$ means Mandelstam analyticity of $f(s,t,u)$. Without loss of generality the same $\mathcal{C}$ can be taken to characterize both D_1 and D_2 .

- c) and d) are standard assumptions about a scattering amplitude and e) is a rigorous consequence of unitarity.

Later we have to restrict further the consideration to those functions with a certain minimal growth for $q \rightarrow \infty$. This point is discussed in detail in the context of Eq. (59) in Section 5. Strictly speaking, this restriction should be included among our basic properties.

3. IMPACT PARAMETER DESCRIPTION OF THE SCATTERING AMPLITUDE

Because of the structure of $f(s, t, u)$, as given by Eq. (14), we limit the discussion first to $f_1(s, t)$. b) and c) allow to find a Cauchy type representation

$$f_1(s, t) = \sum_{k=0}^{N-1} a_k(s) t^k + \frac{1}{2\pi i} t^N \int_{C_1} dt' \frac{f_1(s, t')}{t'^N (t' - t)} \quad (19)$$

with a suitably chosen integer N . The integration along C_1 goes in the sense indicated in Fig. 1.

Now we use the relation

$$\frac{t^N}{t'^N (t' - t)} = \frac{1}{t' - t} - \sum_{k=0}^{N-1} \frac{1}{k!} t^k \left(\frac{d^k}{dt^k} \frac{1}{t' - t} \right)_{t=0} \quad (20)$$

and 17)

$$\frac{1}{t' - t} = \int_0^\infty db b K_0(b \sqrt{t'}) J_0(b \sqrt{-t}) , \quad (21)$$

$$\operatorname{Re} \sqrt{t'} > |\operatorname{Im} \sqrt{-t}| , \quad (22)$$

where J_0 is the Bessel function of order zero and K_0 is the modified Bessel function of the third kind of order zero. The root is defined by $\sqrt{x} > 0$, $x > 0$.

Since in (19) the values of t' lie on the boundary C_1 , we can use (21) for all values of t contained in a sufficiently small domain around the negative t axis, which contains certainly all physical t values. We call this region S_1 .

Now we introduce the representation (21) for $(t'-t)^{-1}$ into equation (20). In the second term, it is allowed to interchange the b integration and the differentiation. We get

$$\frac{t^N}{t'^N(t'-t)} = \int_0^\infty db b K_0(b\sqrt{t'}) \left\{ J_0(b\sqrt{-t}) - \sum_{k=0}^{N-1} \frac{1}{k!} t^k \left(\frac{d^k}{dt^k} J_0(b\sqrt{-t}) \right) \right\}_{t=0}, \quad (23)$$

$$t' \in C_1, \quad t \in S_1.$$

To verify that the interchange of differentiation and integration is allowed, one uses the fact that $K_0(b\sqrt{t'})$ ($t' \in C_1$) decreases exponentially for large b and the relation

$$\frac{d^k}{dt^k} J_0(b\sqrt{-t}) = \left(\frac{b}{2\sqrt{-t}} \right)^k J_k(b\sqrt{-t}), \quad (24)$$

where $J_k(b\sqrt{-t})$ is the Bessel function of order k , which behaves like $(b\sqrt{-t})^k$ for $t \rightarrow 0$.

The right-hand side of (24) may be evaluated explicitly for $t = 0$ with the help of the definition of $J_k(b\sqrt{-t})$ by a power series,

$$J_k(b\sqrt{-t}) = \left(\frac{1}{2} b \sqrt{-t} \right)^k \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(k+m)!} \left(\frac{1}{2} b \sqrt{-t} \right)^{2m}. \quad (25)$$

We get

$$\left[\left(\frac{b}{2\sqrt{-t}} \right)^k J_k(b\sqrt{-t}) \right]_{t=0} = \left(\frac{b}{2} \right)^{2k} \frac{1}{k!} \quad (26)$$

Now, the sum in (23) reads

$$\sum_{k=0}^{N-1} \frac{1}{(k!)^2} (-1)^k \left(\frac{1}{2} b \sqrt{-t} \right)^{2k} \quad (27)$$

and comparing with (25), we see that (27) are just the first N terms in the defining sum of $J_0(b \sqrt{-t})$ [this may also be seen directly, since (25) shows that $J_0(b \sqrt{-t})$ is an entire function of $\bar{t}$]. Thus, (23) becomes

$$\frac{t^N}{t'^N(t'-t)} = \int_0^\infty db \, b \, K_0(b \sqrt{t'}) J_0^N(b \sqrt{-t}) \quad (28)$$

$$J_0^N(b \sqrt{-t}) = \sum_{k=N}^{\infty} \frac{1}{(k!)^2} (-1)^k \left(\frac{1}{2} b \sqrt{-t} \right)^{2k}, \quad (29)$$

$t' \in C_1, t \in S_1.$

This representation for the function $t^N/t'^N(t'-t)$ will now be introduced into equation (19) and we are allowed to interchange the t' and the b integration. This gives the result

$$f_1(s, t) = \sum_{k=0}^{N-1} a_k(s) t^k + 2\sqrt{s} \int_0^\infty db \, b \, h_1(b, s) J_0^N(b \sqrt{-t}), \quad (30)$$

$t \in S_1,$

$$h_1(b, s) = \frac{1}{2\sqrt{s}} \frac{1}{2\pi i} \int_{C_1} dt \, K_0(b \sqrt{-t}) f_1(s, t). \quad (31)$$

That the interchange of integration is allowed, is a consequence of the regularity properties of the integrand in (19), with the relation (28) introduced, and the properties of the function $h_1(b,s)$ as defined by equation (31). To arrive at this conclusion, it is essential to have a minimal singularity free region around the negative t axis as given by our choice of the domain D_1 . To make the discussion more transparent, we shall investigate details about $h_1(b,s)$ in the next section. In short, it follows from our analyticity assumptions and those about the behaviour at infinity (in the variable t), that for $b \rightarrow 0$ $h_1(b,s) \rightarrow \infty$, but such that the integral in (30) exists. The existence of this integral requires to keep the modified form $J_0^N(b\sqrt{-t})$ of the Bessel function under the integral. For $b \rightarrow \infty$, more exactly for $b > \text{const.} \log q$, it will be shown that $h_1(b,s)$ becomes exponentially small.

The contribution $f_2(s,u)$ to $f(s,t,u)$ may be handled in the same way and we get

$$f_2(s,u) = \sum_{k=0}^{N-1} b_k(s) u^k + 2q\sqrt{s} \int_0^{\infty} db b h_2(b,s) J_0^N(b\sqrt{-u}) ,$$

$$u \in S_2 , \quad (30')$$

$$h_2(b,s) = \frac{1}{2q\sqrt{s}} \frac{1}{2\pi i} \int_{C_2} du K_0(b\sqrt{u}) f_2(s,u) .$$

$$(31')$$

Adding (30) and (30'), we get for $f(s, t, u)$ the form

$$f(s, t, u) = \sum_{k=0}^{N-1} d_k(s) t^k + \sum_{k=0}^{N-1} b_k(s) u^k + \\ + 2q \sqrt{s} \int_0^{\infty} db b \left\{ h_1(b, s) J_0^N(b\sqrt{-t}) + h_2(b, s) J_0^N(b\sqrt{-u}) \right\}, \\ t \in S_1, u \in S_2; s \geq 4\mu^2. \quad (32)$$

This form of the scattering amplitude reminds us already of an integral over an impact parameter-like quantity.

We observe that we have used until now only the analyticity properties in t and u of $f(s, t, u)$ as well as the temperedness condition c).

Now, what really may be called impact parameter is a quantity b , which for $q \rightarrow \infty$ obeys the relation

$$\ell = bq - \frac{1}{2} \quad (33)$$

between momentum q and angular momentum ℓ , in perfect analogy to the quasi-classical situation in potential scattering. Therefore, in order that the quantity b in (32) can be interpreted as impact parameter, in the limit of high energy it should become coupled to angular momentum according to equation (33). In this case, (32) may be called impact parameter description of the scattering amplitude. It should be noted that (32) is valid for all energies.

In the following, it shall be shown that

$$f_2(\ell = bq - \frac{1}{2}, s) \longrightarrow h_2(b, s) \quad (34)$$

for $q \rightarrow \infty$, if the scattering amplitude obeys our basic conditions (with the quantities introduced above an additional assumption is necessary ; however, after a suitable redefinition of the used quantities this difficulty can be eliminated). $f_2(\ell, s)$ is the Carlson interpolation¹²⁾ of the partial wave amplitudes of $f_1(s, t)$ or $f_2(s, u)$. Thus, the quantity b in (25) may indeed be interpreted as the impact parameter. Furthermore, (34) shows that at high energy $h_2(b, s)$ obey partial wave type unitarity relations (up to terms vanishing in the limit $q \rightarrow \infty$) and that there is a simple way to find an optical potential reproducing the observed scattering (details are given in Section 7).

In this context, it is important to realize that the spectral functions $h_2(b, s)$ depend only on the scattering amplitude and not on the power N of t chosen in order to derive the Cauchy representation. Thus, $h_2(b, s)$ are free from arbitrariness and may indeed be interpreted as amplitudes for the scattering with definite impact parameter.

Finally, it will be shown that for $q \rightarrow \infty$ (32) assumes a form analogous to the Schrödinger expression (1).

With the assumptions a)-e) about the scattering amplitude, we stay essentially within a framework of analyticity and unitarity. No further hypothesis have to be made, as it is done at one point or another in existing works in connection with the impact parameter description. It is important to realize that the exploration of unitarity and temperedness in the variable s , as they are stated in e) and d), is indispensable in order to arrive at the result. This will be seen in more detail in Sections 5 and 6.

We also add a remark with respect to our analyticity assumption b). It is in the sense minimal that our considerations may be carried through for any singularity distribution in the t or u plane which lies outside the domains D_1 or D_2 . As already mentioned, there

is on the other hand a certain analyticity region around the negative real t and u axes necessary in order to be able to find the result on the basis of analyticity and unitarity properties. In short, it is essential that the singularities in t or u stay to a minimal extent away from the negative values of these variables and as a consequence, it becomes possible to choose in the Cauchy formula an integration contour like C_1 or lying even outside the domain D_1 .

In view of the appearance of the impact parameter in the quasi-classical situation, it might be appealing to apply the theory of representing arbitrary functions as Fourier-Bessel integrals ¹⁸⁾. However, this is not possible in the present framework since this theory is only applicable to functions which fulfil $\int_0^\infty d\sqrt{-t} \sqrt[4]{-t} |f(t)| < \infty$. In this case the scattering amplitude should decrease faster than $t^{-3/4}$ for $t \rightarrow \infty$.

We add now a few remarks on some previous attempts to introduce the impact parameter into the description of relativistic particle scattering. We will come back on various points in more detail during the following sections.

A discussion of impact parametrization by Blankenbecler and Goldberger ⁷⁾ is mainly based on the use of the Fourier-Bessel theory together with the Mandelstam representation and has to be considered as heuristic from the point of view of general properties of the scattering amplitude. Also, they work with Regge asymptotic behaviour; this is not necessary for our purposes. One can, however, find a remark in their work pointing to the usefulness of the representation (21) for $(t'-t)^{-1}$.

More recently, Cottingham and Peierls ⁵⁾ introduced an impact-like parametrization of the scattering amplitude to discuss high energy π meson-proton and proton-proton scattering, which has also been applied to proton-antiproton scattering by Kokkedee ⁶⁾. Here again the Fourier-

Bessel theory is used, but now it is worked only with the physical values of the scattering amplitude, i.e., with a function which is equal to the scattering amplitude for the physical and zero for the non-physical negative t values. This is of course only a very pragmatic procedure. As it has become clear in the preceding discussion, a minimum of dynamical properties is necessary in order to arrive at the impact parameter concept. In our context, one arrives in an approximation at this approach if it is in addition assumed that the scattering amplitude is exponentially small in some domain outside the diffraction region. An attempt by Predazzi⁸⁾ to introduce an impact parametrization is easily seen to be identical to the procedure of Cottingham and Peierls. Here, the relation¹⁹⁾

$$P_e\left(1 + \frac{t}{2q^2}\right) = 2q \int_0^{\infty} db J_{2e+1}(2qb) J_0(b\sqrt{-t}),$$

$$-4q^2 < t < 0, \quad (35)$$

is introduced into the partial wave expansion, integration and summation become interchanged and one formally arrives at an integral over an impact parameter-like quantity. But the right-hand side of (35) is also defined for $-\infty \leq t < -4q^2$ and for these values it vanishes identically¹⁹⁾. As a consequence, this procedure leads to the situation studied in⁵⁾ with its lack of dynamical basis. A further study by Adachi and Kotani⁹⁾ is in its content identical to these two proposals.

A third line of approach has been indicated by Ida¹⁰⁾ and Omnès¹¹⁾. Here, the integral over an impact parameter-like quantity converges only in a part of the physical region of the t values, contrary to the relation (32). Furthermore, no typical high energy behaviour, as discussed in the context of Eq. (33), is deduced.

Neither the cited proposals accounts for the necessary dynamical input of unitarity, analyticity and temperedness properties.

4. PROPERTIES OF THE FUNCTIONS $h_1(b,s)$

In this section we study the behaviour of the function $h_1(b,s)$ in the variable b . Also the relationship between $h_1(b,s)$ and the partial wave amplitudes will be discussed. s has any fixed value $s \geq 4\mu^2$. Without loss of generality we can restrict ourselves to $h_1(b,s)$ and we shall drop in the following the index for convenience.

a) Analyticity properties in b

To analyze the function $h(b,s)$, we use the defining equation (31). From the integral representation (20)

$$K_0(w) = e^{-w} k(w), \quad (36)$$

$$k(w) = \int_0^{\infty} dv e^{-v} \frac{1}{\sqrt{v(v+2w)}}, \quad (37)$$

for the Bessel function K_0 , valid for w in a plane cut from 0 to $-\infty$, follows

$$K_0(w) = A w^{-\frac{1}{2}} e^{-w}, \quad |w| \rightarrow \infty \quad (38)$$

and

$$K_0(w) = B \log w \cdot e^{-w}, \quad |w| \rightarrow 0, \quad (39)$$

where A and B are independent of w .

This shows that $h(b,s)$ has a singularity for $b = 0$ and that it is holomorphic in b in the domain $|\arg b| < \frac{\tau}{2}$, where τ is the parameter introduced to characterize the holomorphy domain D_1 of $f(s,t)$.

b) Behaviour for $b \rightarrow 0$

From (31) follows

$$|h(b,s)| \leq \frac{1}{2q\sqrt{s}} \frac{1}{2\pi} \int_{C_1} |dt| |K_0(b\sqrt{t})| |f(s,t)|. \quad (40)$$

As a consequence of temperedness in t , we may choose $N_0 \geq 0$ as the smallest integer such that

$$\frac{|f(s,t)|}{|t|^{N_0}} < C(s) |t|^{-\varepsilon}, \quad \varepsilon > 0, \quad (41)$$

for $t \rightarrow \infty$, where $C(s)$ is independent of t . By choosing C sufficiently large, this bound can be used along the whole path C_1 of integration. In the integral (40) $f(s,t)$ may thus be bounded by

$$|f(s,t)| < C(s) |t|^{N_0 - \varepsilon} \quad (42)$$

With the help of (38), (39) and (40) we find now that the integral over the circular part of C_1 can be bounded by $\log b$ and the rest by $b^{-2(N_0+1-\varepsilon)}$ for $b \rightarrow 0$. Thus, for the behaviour of $h(b,s)$ for $b \rightarrow 0$ we find

$$|h(b,s)| < C_1(s) b^{-2(N_0+1-\varepsilon)}, \quad b \rightarrow 0. \quad (43)$$

This result also shows that the integral in (30) exists for $N \geq N_0$ because $J_0^N \sim b^{2N}$, $b \rightarrow 0$, as may be seen from the definition (29) of J_0^N .

c) Behaviour for $b \rightarrow \infty$

Using (38) and (40) we derive now that

$$|h(b, s)| < C_2(s) e^{-\sin \frac{\pi}{2} \cdot \sqrt{t_0} b}, \quad b > 0, \quad (44)$$

where $C_2(s)$ is independent of b and t_0 and π are the parameters characterizing the domain C_1 (See Fig. 1). This result shows already the impact character of the parameter b since, if we choose

$$\sqrt{t_0} = 2\mu$$

this is the smallest mass which can be exchanged through the interaction. This mass in turn is related to the range b_1 of the interaction by

$$b_1 \sim (2\mu)^{-1}$$

If we write (44) as

$$|h(b, s)| < e^{\log C_2(s) - \sin \frac{\pi}{2} \cdot 2\mu b} \quad (45)$$

we may define b_1 by

$$b_1 = \frac{\log C_2(s)}{\sin \pi/2} (2\mu)^{-1}$$

Because of the temperedness of $f(s, t)$ in s , we can put $C_2(s) < \tilde{C} \cdot q^{N'}$ with $\tilde{C}$ independent of s . Then we have

$$b_1 = \frac{\log \tilde{C} + N' \log q}{\sin \tau/2} (2\mu)^{-1} \quad (46)$$

This gives the well-known range, which expands logarithmically with energy, of an interaction corresponding to our amplitude assumptions.

Equations (43) and (44) show that the spectral function $h(b, s)$ for $b \rightarrow 0$ is sensitive to the behaviour of the scattering amplitude for large momentum transfer while the behaviour for $b \rightarrow \infty$ is determined by the singularities nearest to the small momentum transfer region.

d) Connection with the partial wave amplitudes

Introducing the form (32) of $f(s, t, u)$ with $N = N_0 + 1$, where N_0 has been introduced in (41), into equation (17) for the partial wave amplitudes, we define

$$f_\ell(s) = f_\ell^1(s) + (-1)^\ell f_\ell^2(s), \quad (48)$$

where $f_\ell^1(s)$ [$f_\ell^2(s)$] contains only $t(u)$ contributions. We first consider $f_\ell^1(s)$. For $\ell > N_0$, the polynomial in t does not contribute to $f_\ell^1(s)$. Since with this choice of N the total integrand of the double integral is continuous for $b \rightarrow 0$ because $J_0^{N_0+1}(b\sqrt{-t}) \rightarrow 0$ sufficiently rapidly, we are allowed to interchange the two integrations. Since $J_0^{N_0+1}(b\sqrt{-t})$ is a difference of $J_0(b\sqrt{-t})$ and a polynomial of degree N_0 in t , again for $\ell > N_0$ this polynomial does not contribute. Thus, for $\ell > N_0$ we can evaluate the double integral with help of the inversion of (35) ²¹⁾

$$\frac{1}{2} \int_{-4q^2}^0 \frac{dt}{2q^2} P_e \left(1 + \frac{t}{2q^2} \right) J_0(b\sqrt{-t}) = \frac{1}{qb} J_{2e+1}(2qb),$$

$$b \geq 0.$$

Finally we get

$$f_e^1(s) = 2q \int_0^\infty db h_1(b, s) J_{2e+1}(2qb),$$

$$l > N_0, \quad s \geq 4\mu^2. \quad (49)$$

For the lower partial waves, we have additional contributions from the subtraction terms.

An analogous procedure for $f_l^2(s)$ gives

$$f_e^2(s) = 2q \int_0^\infty db h_2(b, s) J_{2e+1}(2qb),$$

$$l > N_0, \quad s \geq 4\mu^2. \quad (49')$$

For the frequently used $f_l^\pm(s)$ amplitudes,

$$f_e^\pm(s) = f_e^1(s) \pm f_e^2(s)$$

we have as a consequence of (49)

$$f_e^\pm(s) = 2q \int_0^\infty db h^\pm(b, s) J_{2e+1}(2qb),$$

$$h^\pm(b, s) = h_1(b, s) \pm h_2(b, s).$$

Formulae (49) are similar to expressions given in Refs. 5) and 8), however, the spectral functions are different. The spectral functions introduced by these authors are given by von Neumann expansions which can be shown to converge in the whole b plane as a consequence of analyticity and unitarity. Consequently, they are entire functions of b ; furthermore, in general they do not show the typical "impact properties" discussed above. On the contrary, Eqs. (49) cannot be inverted with the help of von Neumann techniques because of the possible singularities of the functions $h_1(b, s)$.

If the Sommerfeld-Watson representations ²²⁾ for $f_1(s, t)$ and $f_2(s, u)$ are introduced into (31) and (31'), the spectral functions h_1 and h_2 become directly related to the Carlson interpolations ¹²⁾ $f_1(\ell, s)$ and $f_2(\ell, s)$. This provides the inversion of formulae (49) and (49').

If it is possible to analyse a scattering amplitude directly in terms of a representation like (32), i.e., to find the spectral functions h_1 and h_2 , formulae (49) and (49') yield the partial wave amplitudes for $\ell > N_0$. This might present a practical tool to calculate partial wave amplitudes.

Concluding this Section, we remark that analytic properties of the spectral functions $h_1(b, s)$ in s may be deduced also from (31) and (31'). The unitarity condition implies an integral equation for these functions. This is of course the main disadvantage of the form (32) of the scattering amplitude, as long as arbitrary energies are considered. In this context, we do not enter into further details concerning these questions. As already remarked before, the high energy properties of the parametrization (32) of the scattering amplitude will lead to a simple picture which we discuss in the next Section.

5. PROPERTIES OF THE FUNCTIONS $h_1(b, s)$ FOR HIGH ENERGY

Now we wish to establish the relationship

$$f_2(\ell = bq - \frac{1}{2}, s) \rightarrow h_2(b, s), \quad q \rightarrow \infty, \quad (50)$$

between the Carlson interpolations of $f_1^1(s)$ and $f_2^2(s)$ and the spectral functions $h_2(b, s)$. As already mentioned before, the reasoning within the set of introduced quantities cannot be made completely rigorous. This, as will be seen in the following, because the very general assumptions we start with permit to derive some necessary properties of the partial wave amplitudes and their interpolations only for $\ell + \frac{1}{2} > C_0 \log q$, where C_0 is a suitable positive constant. However, with a slight redefinition of the used quantities this difficulty may be circumvented. In the following both variants will be considered. For reasons of analogy, we may again restrict the discussion to f_1 and h_1 .

a) Relation between f_1 and h_1

We proceed by direct comparison between

$$h_1(b, s) = \frac{1}{2q\sqrt{s}} \frac{1}{2\pi i} \int_{C_1} dt K_0(b\sqrt{t}) f_1(s, t) \quad (51)$$

and the Carlson interpolation of $f_1^1(s)$, defined by

$$f_1(\ell, s) = \frac{1}{2q\sqrt{s}} \frac{1}{2\pi i} \int_{C_1} dt Q_\ell(1 + \frac{t}{2q^2}) f_1(s, t), \quad (52)$$

$\text{Re } \ell > N_0,$

where Q_ℓ is the Legendre function of the second kind. Formula (52) is found by the same method as applied in Ref. ¹²⁾ in the case of Mandelstam analyticity. The interpolation (52) of $f_\ell^1(s)$ is holomorphic in the half-plane $\text{Re } \ell > N_0$ and obeys the Carlson bound $|f_1(\ell, s)| < \text{const.} \exp \{a \text{Re } \ell + b |\text{Im } \ell|\}$, a real and $b < \pi$, in this region. Thus, it is the unique interpolation of $f_\ell^1(s)$ with these properties.

Since in (51) and (52) the only difference comes from the K_0 and Q_ℓ functions, we have first to compare these two explicitly known quantities. More exactly, we want to show that for large energy, i.e., large bq ,

$$Q_{\ell=bq-\frac{1}{2}} \left(1 + \frac{1}{2} \left[\frac{b\sqrt{t}}{bq} \right]^2 \right) \rightarrow K_0(b\sqrt{t}) \quad (53)$$

and how the Legendre function approaches this limit.

We shall proceed in two steps. First, we show that

$$Q_{\ell=bq-\frac{1}{2}} \left(1 + \frac{1}{2} \left[\frac{b\sqrt{t}}{bq} \right]^2 \right) = K_0(b\sqrt{t}) + \mathcal{O} \left(\frac{[\log bq]^3}{[bq]^2} \right) K_0(b\sqrt{t}), \quad (54)$$

$$\begin{cases} |t| \leq \frac{C^2}{\sin^2 \varepsilon/2} b^{-2} (\log bq)^2, \\ t \in C_1. \end{cases} \quad (55)$$

The error estimation holds uniformly in this region. C is a positive constant.

Secondly, we show that

$$\left| Q_{\ell=bq-\frac{1}{2}} \left(1 + \frac{1}{2} \left[\frac{b\sqrt{t}}{bq} \right] \right) \right| < \begin{cases} -C' e^{-b\sqrt{|t|} \sin \frac{\pi}{2}} ; \\ |t| \leq \frac{\bar{C}^2}{\sin^2 \frac{\pi}{2}} q^2, t \in C_1 \\ -C'' e^{-bq \log g(\frac{t}{q^2})} ; \\ |t| \geq \frac{\bar{C}^2}{\sin^2 \frac{\pi}{2}} q^2, t \in C_1 \end{cases} \quad (56)$$

where

$$g\left(\frac{t}{q^2}\right) = \left| 1 + \frac{1}{2} \frac{t}{q^2} + \sqrt{\frac{t}{q^2} + \frac{1}{4} \left(\frac{t}{q^2}\right)^2} \right|$$

and

$$\log g\left(\frac{t}{q^2}\right) \geq \varepsilon > 0, \quad t \in C_1;$$

ε independent of q .

For $K_0(b\sqrt{t})$ holds

$$\left| K_0(b\sqrt{t}) \right| < C''' e^{-b\sqrt{|t|} \sin \frac{\pi}{2}}, \quad t \in C_1. \quad (57)$$

This shows that the values of these functions for

$$|t| \geq t_1 = \frac{C^2}{\sin^2 \tau_2} b^{-2} (\log bq)^2, \quad t \in C_1, \quad (58)$$

are at most of the order $O([bq]^{-C})$, where C can be made arbitrarily large. The statements (54), (56) and (57) will be proved with the help of a suitable integral representation for $Q_\ell(z)$, which looks rather similar to the representation (36) for $K_0(w)$. The details are given in the Appendix.

Now we use these results to compare $f_2(\ell=bq-\frac{1}{2}, s)$ and $h_2(b, s)$ as given by (51) and (52). We first state the result,

$$f_2(\ell=bq-\frac{1}{2}, s) = h_2(b, s) + O([bq]^{-2} q^{\varepsilon_1}), \quad (59)$$

$$q \rightarrow \infty, \quad b \geq B q^{-1+\varepsilon_2},$$

where B is a positive constant and $\varepsilon_2 > 0$ may be chosen arbitrarily and independent of q . $\varepsilon_1 > 0$ is independent of q and can be made arbitrarily small if one chooses for the parameters t_0 and τ , which characterize the boundary C_1 , sufficiently small values (note that in all our considerations we only use the existence of energy independent values of t_0 and τ different from zero). The second term on the right-hand side has to be understood in the sense of being at most of the order of $[bq]^{-2} q^{\varepsilon_1}$ since it is derived from estimates involving upper bounds of the relevant quantities. For sufficiently large ε_2 , this term goes to zero

for $q \rightarrow \infty$ for all values of $b \geq Bq^{-1+\varepsilon_2}$.

Some care has to be taken in interpreting Eq. (59) since a priori it is impossible to say if the first term on the right hand side is really dominant for $q \rightarrow \infty$. However, if it would not dominate at least for certain values of b , the scattering amplitude itself would at most be of order q^α , where $\alpha > 0$ can be made arbitrarily small with ε_2 and ε_1 . This may easily be estimated by the Froissart method²³). Since the observed scattering amplitudes are of order q^2 near the forward direction, this implies that $h_1(b, s)$ is of order q^0 at least for a certain minimal set of values of b . In the following, we limit the consideration to functions for which h_1 dominates, among which there are certainly the observed scattering amplitudes. (Strictly speaking, this restriction should also figure among what we take as basis properties of the scattering amplitude.)

Equation (59) gives the high energy relationship between f_2 and h_1 discussed in Section 3 and shows the typical impact character of the parameter b . As was already mentioned earlier, there is in addition to our basic assumptions a further assumption necessary for the validity of this result. In the next subsection, we will discuss how to eliminate this difficulty.

To verify Eq. (59), the integral in (52) has to be broken up into two terms, one containing only contributions from $|t| \leq t_1$, the other from $|t| \geq t_1$, where t_1 is given by (58). As a consequence of (56) and the temperedness properties c) and d) of $f_2(s, t)$ in the variables t, u and s , the contribution from $|t| \geq t_1$ can be made $O([bq]^{-C'} \cdot q^M)$; M is a certain positive integer and $C' > 0$ can be made arbitrarily large, by a suitable choice of C in the definition of t_1 . Thus, for $b \geq Bq^{-1+\varepsilon_2}$ this term is at most $O(q^{-M'})$, $M' > 0$ and arbitrarily large by a suitable choice of C .

Into the term with $|t| \leq t_1$ we introduce (54) and use the explicit expression for the error defined by formulae (A-10, 14). In (A-17) the explicit form of the dominating contribution

to this deviation of $Q_{\ell=bq^{-\frac{1}{2}}}$ from K_0 is given. All terms of this deviation of order higher than $(bq)^{-C}$ are of the form of a polynomial in $w = b\sqrt{t} \times k_n(w) \times K_0(w) \times [bq]^{-k}$, $k \geq 2$, where $k_n(w)$ is defined in (A.18). For illustration, we just consider the dominating part of the deviation; for the others, the argument is analogous. The term of order $O(bq)^{-C}$ in the deviation of Q_ℓ from K_0 contributes only to the error $O(q^{-M'})$ discussed before. With (A.17) we have in (52)

$$-\frac{1}{3}(bq)^{-2} \frac{1}{2q\sqrt{s}} \frac{1}{2\pi i} \int_{t_1'}^{t_1''} dt \left\{ w^2(w + \frac{1}{4}) + w(w + \frac{1}{4})k_1(w) + \frac{1}{8}k_1(w) \right\} K_0(w) f_1(s, t), \quad (60)$$

where $|t_1'| = |t_1''| = t_1$ and $t_1', t_1'' \in C_1$ and the integration goes over the part of C_1 between t_1' and t_1'' . We have to put $w = b\sqrt{t}$ and $|w| \leq C \log(bq)$ for the considered values of t .

Now, if we assume that for the partial wave amplitudes $f_\ell^1(s)$ of $f_1(s, t)$ holds

$$|f_\ell^1(s)| < \text{const. (indep. of } q) \quad (61)$$

for

$$0 \leq \ell \leq C_0 \log q - \frac{1}{2}, \quad (61')$$

where C_0 is a suitable positive constant, it is possible to show within our framework that there exists for $f_1(s, t)$ the bound

$$|f_1(s, t)| < \text{const.} \frac{q^{2+\varepsilon_1}}{|t|} \quad (62)$$

for high q and

$$t \in C_1, \quad |t| < t_1.$$

ε_1 is the number defined before in the context of equation (59).

We come back to the question of determining bounds for $f_1(s, t)$ later (subsection c) and remark here only that (61) is in fact an additional hypothesis which does not follow from unitarity since the partial amplitudes $f_1^1(s)$ do not fulfil unitarity relations of the type (18). On the other hand, properties like (61) follow within our framework for $f_1^1(s)$ for $\ell > C_0 \log q^{-\frac{1}{2}}$. As already mentioned before, these difficulties can be avoided by the method presented in the next subsection. However, from the practical point of view (61) is certainly not a very stringent condition.

With (62) and the known properties of $k_1(w)$, $k_2(w)$ and $K_0(w)$ the expression (60) can now be majorized by

$$\begin{aligned} & \text{const. } (bq)^{-2} (\log bq)^3 q^{\varepsilon_1} \int_{t_1'}^{t_1''} |dt| \frac{1}{|t|} = \\ & = \text{const. } (bq)^{-2} (\log bq)^3 q^{\varepsilon_1} \{ \text{const.} + \log t_1 \}, \end{aligned}$$

where t_1 is given by (58). This shows that the expression (60) is bounded by $O([\log bq]^{-2} \cdot q^{\varepsilon_1})$ for $b \geq Bq^{-1+\varepsilon_2}$. It should be noted that the t dependence of the bound (62) is essential for the validity of this result. In the same way, we see that the lower order deviations of $Q_{\ell=bq-\frac{1}{2}}$ from K_0 give lower order corrections to the investigated term.

Finally, in the integral which contains only $K_0(b\sqrt{t})$ after the substitution of (54) into (52) and only values of t with $|t| < t_1$, the limits of integration can be put to infinity. The corresponding error is again at most $O(q^{-M'})$ for $b \geq Bq^{-1+\varepsilon_2}$ for the same reasons as before. Thus, this term gives directly $h_1(b,s)$ in (59). It is worthwhile to recall that all listed basic properties of the scattering amplitude are necessary for the derivation of Eq. (59).

b) Rigorous argument with redefined quantities

We show now that it is possible to avoid the boundedness assumption (61) for the lower partial wave amplitudes by working with redefined quantities. Again we discuss only f_1 explicitly.

We decompose the function $f_1(s,t)$ according to

$$f_1(s,t) = p_1^L(s,t) + f_1^L(s,t), \quad (63)$$

where

$$p_1^L(s,t) = \frac{\sqrt{s}}{q} \sum_{\ell=0}^L (2\ell+1) f_\ell^1(s) P_\ell\left(1 + \frac{t}{2q^2}\right).$$

$p_1^L(s,t)$ is a polynomial in t with degree L . Now, all preceding considerations can be carried through for the functions $f_1^L(s,t)$ and $f_2^L(s,u)$. However, if we choose $L \geq N_0$, where N_0 is the power of t which characterizes the behaviour of $f_1(s,t)$ for $t \rightarrow \infty$ according to Eq. (41), the condition $N \geq L+1$ has to hold for the exponent N in the Cauchy representation, which is introduced for $f_1^L(s,t)$ in analogy to Eq. (19). This because $p_1^L(s,t)$ might behave as t^L for $t \rightarrow \infty$ and thus $f_1^L(s,t)$ ought to behave similarly in order to give the right behaviour of $f_1(s,t)$.

Thus, in the preceding formulae we can simply replace

$$\left. \begin{aligned} f_1(s, t) &\longrightarrow f_1^L(s, t) \\ f_2(s, u) &\longrightarrow f_2^L(s, u) \\ h_{\frac{1}{2}}(b, s) &\longrightarrow h_{\frac{1}{2}}^L(b, s) \\ N_0 &\longrightarrow L+1, \text{ if } L \geq N_0 \end{aligned} \right\} \quad (64)$$

and take $N \geq L+1$ if $L \geq N_0$.

However, the partial wave interpolations corresponding to $f_{1,2}^L$ are the same as before and give their partial wave amplitudes for $\operatorname{Re} \ell > L$ if $L \geq N_0$.

Now, for $f_{1,2}^L$ the bound (62) holds without requiring the boundedness (61) of the lower partial waves (see next subsection) if we choose

$$L = C_0 \log q - \frac{1}{2} \quad (65)$$

The entire reasoning leading to the result (59) applies to the new quantities with the only difference that for the value t_1 , which served to decompose the integral over C_1 into the parts $|t| \leq t_1$ and $|t| \geq t_1$, we have to choose $C^2/\sin^2 \tau/2 \cdot b^{-2}(\log bq)^4$ instead of $C^2/\sin^2 \tau/2 \cdot b^{-2}(\log bq)^2$. This because we have to compensate for the q dependence (65) of the maximal power with which $f_1^L(s, t)$ or $f_2^L(s, u)$ may increase for increasing t or u . However, the result is the same as (59)

$$f_2(\ell = bq - \frac{1}{2}, s) = h_2^L(b, s) + \quad (66)$$

$$+ o'([bq]^{-2} q^{\varepsilon_1}),$$

$$q \rightarrow \infty, \quad b \geq Bq^{-1} + \varepsilon_2$$

This result is now derived from the basic properties of the scattering amplitude only. It will be seen in the next section that the high energy form of the impact parametrization of the scattering amplitude is not influenced by the new choice of quantities.

c) Remarks on high energy bounds for $f_1(s, t)$ and $f_2(s, u)$ and some properties of the Carlson interpolations

In the same way like Kinoshita, Loeffel and Martin ¹³⁾, it is possible to deduce from the properties a)-e), with the weaker analyticity assumption replacing Mandelstam analyticity, the following properties of the Carlson interpolations $f_2(\ell, s)$ for high energy.

$$|f_2(\ell, s)| < C'' \quad , \quad (67)$$

for all complex ℓ with

$$|\arg(\ell + \frac{1}{2} - C_0 \log q)| < C' \frac{1}{\log q} \quad , \quad (67')$$

where C_0 , C' and C'' are suitable positive constants independent of s . With the help of a Cauchy representation in ℓ for $f_2(\ell, s)$

(67) and (67') give a smoothness property of these interpolations for real $l + \frac{1}{2} > C_0 \log q$ of the form

$$\left| \frac{d f_2(l, s)}{d l} \right| < \frac{C''' \log q}{l + \frac{1}{2} - C_0 \log q}, \quad (67'')$$

$$l + \frac{1}{2} > C_0 \log q.$$

From (59) follows with (67)

$$|h_2(b, s)| < C^{\text{IV}}, \quad (68)$$

$$b \geq B q^{-1 + \varepsilon_2}.$$

The same holds for $h_2^{\text{L}}(b, s)$. The smoothness property (67'') will be used in the next section, when we discuss the high energy behaviour of the impact parametrization (32) of the scattering amplitude.

As in Ref. ¹³⁾, the Sommerfeld-Watson representation of $f(s, t, u)$ is used to derive an upper bound for it for $\theta \neq 0, \pi$. we may apply the same technique to find upper bounds like (62) for $f_1^{\text{L}}(s, t)$ and $f_2^{\text{L}}(s, u)$ for the values of t and u we are interested in (e.g., $t \in C_1$, $|t| < t_1$). If we add the boundedness condition (61) for the lower partial wave amplitudes, we get bounds of type (62) for $f_1(s, t)$ and $f_2(s, u)$.

As a side remark, it may be mentioned that particularly the Kinoshita-Loeffel-Martin bound ¹³⁾ on $f(s, t, u)$ for $\theta \neq 0, \pi$ is also obtained from the properties a)-e).

- d) Remark concerning the high energy relationship between f_1^2 and h_1 according to Ref. 5)

Before ending this section, we add a remark with respect to a procedure proposed by Cottingham and Peierls ⁵⁾ to establish a relation of type (59) between partial wave amplitudes and spectral functions, which are related to the partial wave amplitudes by expressions like (49) and (49'). They apply a technique developed by Curtis ²⁴⁾ to find asymptotic expansions for integrals of the type (49), (49'). This gives for example for $h_1(b,s)$ and $f_1^1(s)$

$$f_1^1(s) = h_1\left(b = \frac{\ell + \frac{1}{2}}{q}, s\right) - \frac{1}{8q^2} \frac{d^2 h_1(b,s)}{db^2} \bigg|_{b = \frac{\ell + \frac{1}{2}}{q}} + \dots \quad (69)$$

This holds only for spectral functions h which may be represented as a product of a power series in b and an exponential $e^{-\alpha b}$, $\alpha > 0$, for $0 \leq b \leq \infty$. It seems to be difficult to justify such a property of the spectral functions h on the basis of general properties of the scattering amplitude. Furthermore, this is not sufficient to guarantee that (69) really gives an asymptotic expansion for $q \rightarrow \infty$ as may be easily seen from an example, e.g., $h(b,s) = \text{const} \cdot e^{-\alpha b} \cdot \cos(qb)$. This clearly shows that the lack of a minimum set of dynamical assumptions about the scattering amplitude to begin with, requires the introduction of assumptions at later stages for which there is only ad hoc justification.

We will take up the consideration of the high energy properties of the spectral functions $h_1(b,s)$ again in Section 7. There we shall discuss briefly the consequences of unitarity for $h_1(b,s)$ as they result from Eqs. (59) or (66), as well as the problem² of finding from them an optical potential producing the scattering.

6. IMPACT PARAMETRIZATION OF $f(s,t,u)$ AT HIGH ENERGY

In this section, we study the high energy limit of equation (32) when combined with the high energy form (59) of the spectral functions $h_1(b,s)$. Since the discussion applies equally to both versions of the formalism, we consider only the first. Again, we restrict the discussion to $f_1(s,t)$ whenever the procedure for $f_2(s,u)$ is analogous.

We start with a representation of $f_1(s,t)$ like (30) for physical values of t and consider $N \geq N_0$ as a parameter which will be adjusted later in a specific way. As already remarked before, the spectral function $h_1(b,s)$ does not depend on the way we choose N .

We separate the integral into two terms $\int_0^\infty = \int_0^{b_0} + \int_{b_0}^\infty$, and we choose b_0 as

$$b_0 = Bq^{-1+\varepsilon_2}, \quad (70)$$

where B and ε_2 have been introduced in the context of Eq. (59). Let us now briefly indicate the procedure how we treat the different terms of equations (30). First, we shall show that $\int_0^{b_0}$ can be made $O(e^{-\alpha q^{\varepsilon_2}})$, $\alpha > 0$ if we choose $N > B_1 q^{\varepsilon_2}$ with an appropriate $B_1 > B > 0$.

Into the integral $\int_{b_0}^\infty$ we then introduce the expression (29) for $J_0^N(b\sqrt{-t})$ and interchange integration and summations, since the integrand is no longer singular in the considered region. This results in a modification of the original subtraction polynomial.

Next, the entire expression (32) for $f(s,t,u)$ has to be considered to get an estimate of the subtraction terms. The two

modified subtraction polynomials coming from $f_1(s, t)$ and $f_2(s, u)$ yield an over-all polynomial in t . Using unitarity, formulae (49) and (49') for the connection between $f_2^1(s)$ and $h_2(b, s)$ as well as the high energy form (59) for $h_2(b, s)$ together with the properties (61) and (61') of the Carlson interpolations, this over-all polynomial will be estimated to be at most $O(q^{\varepsilon_2^1})$; $\varepsilon_2^1 > 0$ can be made arbitrarily small with ε_2 .

Finally, introducing the high energy form of $h_2(b, s)$ into the integrals containing $J_0(b\sqrt{-t})$ and $J_0(b\sqrt{-u})$, the characteristic high energy expression for $f(s, t, u)$ analogous to (1) will be found.

a) Investigation of $\int_0^{b_0}$

From (36) follows for $\int_0^{b_0}$

$$\begin{aligned} & \left| 2q\sqrt{s} \int_0^{b_0} db b h_2(b, s) J_0^N(b\sqrt{-t}) \right| \leq \\ & 2q\sqrt{s} \int_0^{b_0} db b |h_2(b, s)| |J_0^N(b\sqrt{-t})| \end{aligned} \quad (71)$$

For $|h_2(b, s)|$, we use the bound (43). With the definition (29), a bound for $|J_0^N(b\sqrt{-t})|$ can be derived. From

$$J_0^N(b\sqrt{-t}) = (-1)^N \left(\frac{1}{2} b\sqrt{-t}\right)^{2N} \frac{1}{[(N-1)!]^2}.$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{[N(N+1) \cdots (N+n)]^2} \left(\frac{1}{2} b\sqrt{-t}\right)^{2n}$$

follows

$$\begin{aligned} |J_0^N(b\sqrt{-t})| & < K q^{2N_0} (B q^{\varepsilon_2})^{2(N-N_0)} \\ & \cdot N e^{-2N(\log N - 1)} b^{2N_0} \end{aligned} \quad (72)$$

for

$$0 \leq b \leq b_0 = Bq^{-1+\varepsilon_2}, \quad -4q^2 \leq t \leq 0, \quad (73)$$

$$N > B_1 q^{\varepsilon_2}, \quad B_1 > 0.$$

To find (72), we have used Stirling's formula to estimate the factorial and $\frac{1}{2}b\sqrt{-t} \leq Bq^{\varepsilon_2}$ for the values (73) of b and t . K is a constant independent of b , q and N . (72) implies

$$|J_0^N(b\sqrt{-t})| < K_1 q^{2N_0} (Bq^{\varepsilon_2})^{-2N_0} B_1 q^{\varepsilon_2} \cdot e^{-2B_1 q^{\varepsilon_2} (\log \frac{B_1}{B} - 1)} b^{2N_0}$$

Now, with (43) and (71), we get

$$\left| 2q\sqrt{s} \int_0^{b_0} db b \, h_1(b, s) \cdot J_0^N(b\sqrt{-t}) \right| < K_1' b_0^\varepsilon C_1(s) q\sqrt{s} q^{2N_0+\varepsilon} (Bq^{\varepsilon_2})^{-2N_0} e^{-2B_1 q^{\varepsilon_2} (\log \frac{B_1}{B} - 1)}$$

This shows, as already remarked before, that the contribution from $\int_0^{b_0}$ is $O(e^{-\alpha' q^{\varepsilon_2}})$, $\alpha' > 0$ and $\varepsilon_2 > 0$, if we choose $B_1 > B$ appropriately. The same result holds also in the rigorous version of the formalism, where we have to replace $N_0 \rightarrow L \sim \log q$.

With this, we get now from (30) for $f_1(s, t)$

$$\begin{aligned}
f_1(s, t) = & \sum_{k=0}^{N-1} a'_k(s) t^k + \\
& + 2q\sqrt{s} \int_{b_0}^{\infty} db b h_1(b, s) J_0(b\sqrt{-t}) + \\
& + O(e^{-\alpha q^2 \varepsilon_2}) , \\
& -4q^2 \leq t \leq 0 ,
\end{aligned} \tag{74}$$

where

$$a'_k(s) = a_k(s) - \frac{2q\sqrt{s}}{2^{2k}(k!)^2} \int_{b_0}^{\infty} db b^{2k+1} h_1(b, s). \tag{75}$$

We have used

$$J_0^N(b\sqrt{-t}) = J_0(b\sqrt{-t}) - \sum_{k=0}^{N-1} \frac{(-1)^k}{(k!)^2} \left(\frac{1}{2} b\sqrt{-t}\right)^{2k}$$

and interchanged summation and integration. This is now allowed since the lower limit of the integral is $b_0 > 0$; furthermore, the exponential decrease of $h_1(b, s)$, as established in Section 4-b), for $b \rightarrow \infty$, guarantees the convergence of the integrals in (75).

Up to this point, the discussion of the expression (30') for $f_2(s, u)$ is analogous.

b) Estimation of the polynomial terms in t

To get an estimate of the polynomial terms in t , we have to consider the entire expression $f(s, t, u)$. With (74) and the analogous result for $f_2(s, u)$, we get

$$\begin{aligned}
f(s, t, u) = & \sum_{k=0}^{N-1} C_k(s) t^k + \\
& + 2q\sqrt{s} \int_{b_0}^{\infty} db b \left\{ h_1(b, s) J_0(b\sqrt{-t}) + h_2(b, s) J_0(b\sqrt{-u}) \right\} + \\
& + O(e^{-\alpha q^2 \varepsilon_2}) , \\
& -4q^2 \leq t \leq 0 ,
\end{aligned} \tag{76}$$

where the polynomial terms are now rewritten as a polynomial in t , using $s + t + u = 4\mu^2$.

The polynomial in t may be written as a sum over Legendre polynomials

$$\sum_{k=0}^{N-1} C_k(s) t^k = \frac{\sqrt{s}}{q} \sum_{l=0}^{N-1} (2l+1) f_l'(s) P_l\left(1 + \frac{t}{2q^2}\right). \tag{77}$$

The total l component $f_l(s)$, $0 \leq l \leq N-1$, of $f(s, t, u)$ is

$$f_l(s) = f_l^{1'}(s) + f_l^{2'}(s) + (-1)^l f_l^{2'}(s) + O(e^{-\alpha q^2 \varepsilon_2}) \tag{78}$$

Here, $f_l^{1'}(s)$ and $(-1)^l f_l^{2'}(s)$ are the l components of the integral terms corresponding to h_1 and h_2 in (76). In analogy to the considerations in Section 4-d), they may be expressed in terms of $h_1(b, s)$ and $h_2(b, s)$ as

$$f_l^{2'}(s) = 2q \int_{b_0}^{\infty} db h_1(b, s) J_{2l+1}(2qb) ,$$

$$0 \leq l \leq \infty. \tag{79}$$

With help of the unitarity constraint (18) for $f_e(s)$ and the high energy properties of $h_1(b,s)$ established in the last section, it becomes now possible to find an upper bound for the ℓ components $f_\ell(s)$ of the polynomial in t in (76). By the Froissart technique used to derive high energy bounds of scattering amplitudes²³⁾, an upper bound for this polynomial can then be found.

From (73) follows

$$|f_e'(s)| \leq |f_e(s)| + |f_e^{1'}(s)| + |f_e^{2'}(s)| + o(e^{-\alpha' q^{\varepsilon_2}}) \quad (80)$$

Because of unitarity, the first term is bounded by 1. The two other unknown terms may be investigated with the help of (79), combined with the high energy properties of $h_1(b,s)$ and the high b information of Section 4-c). Again, we concentrate on $f_\ell^{1'}(s)$. We find

$$|f_\ell^{1'}(s)| < \left| 2q \int_{b_0}^{b_2} db h_1(b,s) J_{2\ell+1}(2qb) \right| + o(q^{-\beta}),$$

$$b_2 = k b_1, \quad b_1 = \frac{\log \tilde{C} + N' \log q}{\sin \frac{\pi}{2} \cdot \sqrt{t_0}} \quad \text{and} \quad k > 1 \quad (81)$$

For this, the exponential decrease of $h_1(b,s)$ for $b \rightarrow \infty$, derived in Section 4-c), and $|J_{2\ell+1}(2qb)| \leq 1$ have been used. $\beta > 0$ is a constant, which can be made arbitrarily large by a suitable choice of the number k .

Introducing now (59) into (81), one finds for the contribution from the error term

$$\left| 2q \int_{b_0}^{b_2} db \sigma([bq]^{-2} q^{\varepsilon_1}) J_{2\ell+1}(2qb) \right| < \\ < \text{const. } q^{\varepsilon_1 - \varepsilon_2} \quad (82)$$

In the following, we choose $\varepsilon_2 > \varepsilon_1$.

For the contribution from the main term we perform first a partial integration

$$\begin{aligned} 2q \int_{b_0}^{b_2} db f_1(\ell = bq - \frac{1}{2}, s) J_{2\ell+1}(2qb) = \\ = 2q f_1(b_2 q - \frac{1}{2}, s) \int_{b_0}^{b_2} db J_{2\ell+1}(2qb) - \\ - 2q^2 \int_{b_0}^{b_2} db \frac{df_1(bq - \frac{1}{2}, s)}{d\ell} \cdot \int_{b_0}^b db' J_{2\ell+1}(2qb'). \end{aligned} \quad (83)$$

Here, the first term is $O(q^{-\beta})$ because of the well-known high ℓ damping of the partial wave amplitudes, which in our discussion is implied by (59) and the high b behaviour of $h_1(b, s)$.

Using the relation

$$J_{2\ell+1}(x) = x^{-2(\ell+1)} \frac{d}{dx} \left[x^{2(\ell+1)} J_{2(\ell+1)}(x) \right],$$

we derive

$$\int_{x_0}^x dx' J_{2\ell+1}(x') = J_{2(\ell+1)}(x) - J_{2(\ell+1)}(x_0) + 2(\ell+1) \int_{x_0}^x dx' x'^{-1} J_{2(\ell+1)}(x')$$

Applying the upper bound

$$|J_n(x)| < 1, \quad n \geq 0 \quad (84)$$

and integer, we obtain

$$\left| \int_{x_0}^x dx' J_{2\ell+1}(x') \right| < A_1 \ell \log \frac{x}{x_0}, \quad (85)$$

where A_1 is a constant independent of ℓ .

Now, we use the smoothness property (67'') together with (85) to find a bound for the integral on the left-hand side of (83)

$$\begin{aligned} & \left| \frac{1}{2q} \int_{b_0}^{b_2} db f_1\left(bq - \frac{1}{2}, s\right) J_{2\ell+1}(2qb) \right| < \\ & < q \int_{b_0}^{b_2} db \frac{C''' \log q}{bq - C_0 \log q} A_1 \ell \log \frac{b}{b_0} + o(q^{-\beta}) < \\ & < A_2 \ell (\log q)^3. \end{aligned} \quad (86)$$

With (81) and (86), we have for $|f_\ell^{1'}(s)|$

$$|f_\ell^{1'}(s)| < A_3 \ell (\log q)^3, \quad (87)$$

where the constant A_3 is independent of l . The same result holds for $f_l^{2'}(s)$.

From (80), we find now for $|f_l'(s)|$

$$|f_l'(s)| < A_4 l (\log q)^3. \quad (88)$$

Now we use (77), taking into account $|P_l(1+t/2q^2)| \leq 1$, $-4q^2 \leq t \leq 0$, to estimate the polynomial in t as

$$\left| \sum_{k=0}^{N-1} c_k(s) t^k \right| < A_5 q^{\varepsilon_2'}, \quad (89)$$

where $\varepsilon_2' > 0$ can be made arbitrarily small with ε_2 .

It may be remarked here that using a more refined bound for J_{2l+1} than (84), it might be possible to find a somewhat smaller exponent in (88). However, in view of (89), this is not essential for our discussion.

c) High energy form of (32)

Finally, introducing the high energy form (59) of $h_1(b, s)$ into (76) and taking again into account the exponential damping of these functions for large b , (76) becomes

$$f(s, t, u) = 2q\sqrt{s} \int_{b_0}^{b_2} db b \left\{ f_1(l=bq-\frac{1}{2}, s) J_0(b\sqrt{-t}) + \right. \\ \left. + f_2(l=bq-\frac{1}{2}, s) J_0(b\sqrt{-u}) \right\} + o(q^{\varepsilon_2'}),$$

$$-4q^2 \leq t \leq 0, \quad q \rightarrow \infty. \quad (90)$$

We recall here that in this context the symbol $\mathcal{O}(q^{\varepsilon_2^1})$ has to be understood as defining a contribution which is at most of the order $q^{\varepsilon_2^1}$; this is slightly different from the usual meaning, describing the order according to which the quantity actually behaves. Thus, it follows from the general properties of the scattering amplitude we started with that the difference between $f(s, t, u)$ and the impact integral in (90) is at most proportional to $q^{\varepsilon_2^1}$. However, this error term might well be much smaller due to more specific properties of the scattering amplitude, which are implied by additional dynamical restrictions. To make this explicit is certainly a very difficult task and amounts probably almost to solve the entire dynamical problem. On the other hand, one might try to put in from the beginning, in addition to the general properties, detailed information on $f(s, t, u)$ from experiment.

To discuss the approximation of the scattering amplitude given by relation (90), we first recall the known high energy bounds for $f(s, t, u)$, which are consequences of assumptions a)-e) ¹⁵⁾

$$t = 0, -4q^2: |f(s, t, u)| < C_3 q^2 (\log q)^2 \quad (91)$$

$$t \neq 0, -4q^2: |f(s, t, u)| < C_4 \frac{q^2 (\log q)^{3/2}}{-t(1 + t/4q^2)} \quad (91')$$

where C_3 and C_4 are independent of t and q . (91) shows that the order of the error term in (90) relative to the maximal allowed value for the impact term in forward direction is at most $\mathcal{O}(q^{-2+\varepsilon_2^1})$ (forgetting about powers of $\log q$). The relative magnitude of the same two quantities for momentum transfers

$$t \approx \text{const. } q \quad (92)$$

is at most $O(q^{-1+\varepsilon_2})$, as may be seen from (91'). Of course, this says nothing about the relative order of these two terms for the scattering amplitudes actually observed in experiment, which are far from saturating the bound (91') for the t values (92). On the other hand, for $t = 0$ (91) is saturated by the observed scattering amplitudes (apart from logarithmic factors) and the above statement is consequently a statement about the relative order of the error term and the impact term.

It is interesting to compare the measure of quality for the approximation (90) introduced above and the corresponding statement in the high energy Schrödinger theory indicated in the introduction. The resulting high energy form (90) of the scattering amplitude may also be visualized in the following way. It is formally obtained from the partial wave expansion by replacing the summation $\sum_{\ell=0}^{\infty} \ell_0 \rightarrow \ell$ $\ell_0 = b_0 \cdot q^{-\frac{1}{2}}$, through an integration $\int_0^\infty d\ell$ and substituting $\ell = bq - \frac{1}{2}$. Furthermore, the partial wave amplitudes $f_\ell^1(s)$ have to be replaced by the interpolations $f_2(\ell, s)$ and for the Legendre polynomials, the Mac Donald approximation ²⁵⁾

$$P_\ell \left(1 - \frac{1}{2} \left[\frac{x}{\ell + 1/2} \right]^2 \right) = J_0(x) + \dots,$$

valid for large ℓ , has to be substituted. In this way, the use of an expression like (90) for the scattering amplitude is often (formally) motivated in treatments of high energy scattering. It may also be asked how to justify such a transition directly. The condition for this is that the individual terms of the sum (or at least a number of terms which give a dominating contribution to the sum) may be interpolated by a slowly varying function; this can be seen for instance with the help of the sum formula of Euler ²⁶⁾. It is

obvious that the conditions (67") on the derivatives of the Carlson interpolations $f_1(\ell, s)$ provide such properties at high energy for sufficiently large ℓ .

Finally, it may be remarked that the first term in (90) describes a peak of scattering in forward direction, while the second describes backward peaking. Some more physical aspects of Eq. (90) will be considered in the next Section.

7. FURTHER PROPERTIES OF $h_1(b,s)$ AT HIGH ENERGY

We add in this section a few remarks concerning the question of combining more detailed information from unitarity with the impact parametrization of the scattering amplitude at high energy. These properties play an essential role in, for example, Van Hove type ^{3),5),6)} calculations where they are used to determine the high energy scattering amplitude in the form (90) as shadow of inelastic processes. Furthermore, we briefly treat the question of finding an optical potential reproducing the observed scattering.

a) Consequences of unitarity

As already remarked earlier, the unitarity properties of the spectral functions $h_1(b,s)$ in the general impact parametrization (32) of the scattering² amplitude for arbitrary energy are expressed as non-linear integral equations. However, working at high energy with the approximation (90), the simple unitarity properties of the (interpolated) partial wave amplitudes come directly into the game. For the non-interpolated (even, odd) partial waves, the unitarity relation reads

$$\text{Im } f_e^{\pm}(s) = \frac{1}{4} g_e^{\pm}(s) + |f_e^{\pm}(s)|^2, \quad (93)$$

$$0 \leq g_e^{\pm}(s) \leq 1, \quad (94)$$

where $1/4g_e^{\pm}(s)$ comes from the overlap term in the unitarity sum ³⁾. (Here, the index $\pm$ just indicates that we shall concentrate on the even and odd partial waves separately.) Writing (93) as

$$\begin{aligned}
\frac{1}{2i} (f^{\pm}(l,s) - f^{\pm*}(l^*,s)) &= \\
&= \frac{1}{4} g^{\pm}(l,s) + f^{\pm}(l,s) f^{\pm*}(l^*,s) , \quad (95)
\end{aligned}$$

$$\text{Re } l > N_0 ,$$

where the star denotes complex conjugation, interpolations $g^{\pm}(l,s)$ of the l components $g_l^{\pm}(s)$ of the overlap function are defined through the Carlson interpolations $f^{\pm}(l,s)$ of the partial wave amplitudes $f_l^{\pm}(s)$. From this follows that also the $g^{\pm}(l,s)$ are of Carlson type ; of course one has

$$\begin{aligned}
\text{even: } g^{+}(l,s) &= g_e^{+}(s) , \\
\text{odd: } g^{-}(l,s) &= g_e^{-}(s) , \\
&\text{Re } l > N_0 \quad (96)
\end{aligned}$$

For real l , (95) becomes

$$\begin{aligned}
\text{Im } f^{\pm}(l,s) &= \frac{1}{4} g^{\pm}(l,s) + |f^{\pm}(l,s)|^2 , \\
&l > N_0 . \quad (97)
\end{aligned}$$

Now, these unitarity conditions may be expressed in terms of $f_1(l,s)$ with the help of

$$f^{\pm}(l,s) = f_1(l,s) \pm f_2(l,s) , \quad (98)$$

$$g^{\pm}(l,s) = g_1(l,s) \pm g_2(l,s) \quad (98')$$

and we get

$$\text{Im } f_1(\ell, s) = \frac{1}{4} g_1(\ell, s) + |f_2(\ell, s)|^2 + |f_1(\ell, s)|^2, \quad (99)$$

$$\text{Im } f_2(\ell, s) = \frac{1}{4} g_2(\ell, s) + f_1(\ell, s) f_2^*(\ell, s) + f_1^*(\ell, s) f_2(\ell, s), \quad (99')$$

$$\ell > N.$$

With the assumption that $|f_1| \gg |f_2|$ and f_1 purely imaginary in first order, Eq. (99) determines f_1 in terms of g_1 as in the calculations for instance of Refs.^{5), 6)}. In the same approximation, (99') leads to a determination of $\text{Im } f_2(\ell, s)$ in terms of g_2 and f_1 . Consistency then requires that $|\text{Im } f_2| \sim |g_2| \ll |g_1|$, a property of the overlap part g_2 which is seen to be fulfilled in model calculations²⁷⁾. Thus, this way of solving the Eqs. (99) and (99') gives (qualitatively) a large forward and a smaller backward peak, as observed, e.g., in $\pi^\pm p$ scattering.

b) Optical potential

We introduce now the phase shifts according to

$$f_e^\pm(s) = \frac{1}{2i} (e^{2i\delta_e^\pm(s)} - 1), \quad (100)$$

here again the index $\pm$ means that we consider in the following the even and odd partial waves separately.

At high energy and for large ℓ , a potential

$$V(|\vec{x}|) = V_1(|\vec{x}|) + V_2(|\vec{x}|) P,$$

P = exchange operator,

which reproduces the observed scattering, is connected to the phase shifts $\delta_{\ell}^{\pm}(s)$ according to the quasi-classical approximation to the solution of the radial wave equation, as was already indicated in the introduction. This connection reads ²⁾ (up to orders q^{-1}) in the case considered

$$\begin{aligned} \delta_{\ell}^{\pm}(s) &= -\frac{m}{q} \int_{\frac{\ell+1/2}{q}}^{\infty} d|\vec{x}| \frac{|\vec{x}|}{\sqrt{x^2 - \left(\frac{\ell+1/2}{q}\right)^2}} V^{\pm}(|\vec{x}|) \\ &= -\frac{m}{q} \int_0^{\infty} dz V^{\pm}\left(\sqrt{z^2 + \left(\frac{\ell+1/2}{q}\right)^2}\right), \quad (101) \end{aligned}$$

$$V^{\pm}(|\vec{x}|) = V_1(|\vec{x}|) \pm V_2(|\vec{x}|),$$

m is the reduced mass of the particles. The high energy approximation (101) is also valid for continuously varying ℓ values. If we write

$$\delta_{\ell}^{\pm}(\ell, s) = \frac{1}{2i} \left(e^{2i\delta_{\ell}^{\pm}(\ell, s)} - 1 \right), \quad (102)$$

we find for $\ell = bq - \frac{1}{2}$ from (101)

$$\delta^{\pm}(\ell = bq - \frac{1}{2}, s) = -\frac{m}{q} \int_0^{\infty} dz V^{\pm}\left(\sqrt{z^2 + b^2}\right) \quad (103)$$

With this result, the complete analogy of the considered high energy limit of a scattering amplitude, having the basic properties of Section 2 to a Schrödinger amplitude is established. Equation (103)

may be considered as an integral equation determining the potentials V_1 and V_2 in terms of the phase shift interpolations $\delta^\pm(\ell, s)$ defined by Eq. (102). These in turn are determined through (98) in terms of the interpolations $f_1(\ell, s)$, which are the weight functions in the high energy form (90) of the scattering amplitude.

To determine the optical potential from an impact parameter description, the characteristic high energy limit $h_2(b, s) \rightarrow f_2(\ell = bq - \frac{1}{2}, s)$, which follows from the basic dynamical assumptions, is necessary. It is not possible to proceed by identifying an impact type weight function directly with the eikonal weight function (2) in Schrödinger theory and to apply (3) to determine the potential. This because the weight functions in a representation in terms of Bessel functions $J_0(b\sqrt{-t})$ and $J_0(b\sqrt{-u})$ are not uniquely defined by the scattering amplitude, as long as only finite parts of the intervals $-\infty \leq t \leq 0$ and $-\infty \leq u \leq 0$ are considered. There may, e.g., always be added a function whose Hankel transform²⁸⁾ is zero in the considered interval. It may thus be said that the postulated dynamical properties of the scattering amplitude select the physically significant weight function with the characteristic impact properties of the corresponding parameter b .

Equations (90) and (103) show explicitly that the optical potential has to grow proportionally to q , i.e., to the centre of mass energy, in order that the scattering amplitude behaves like $s \sim q^2$ (for fixed t) as observed in experiment.

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A P P E N D I X

COMPARISON OF $Q_\ell(1 + \frac{1}{2} [\frac{w}{\ell + \frac{1}{2}}]^2)$ AND $K_0(w)$ FOR $\ell \rightarrow \infty$

First we establish an integral representation for Q_ℓ similar to representation (36) for $K_0(w)$. This representation is used to compare the two functions.

a) Integral representation for $Q_\ell(z)$, $z = \cos \xi$

We start with the Laplacian integral for $Q_\ell(z)$ ²⁹⁾

$$Q_\ell(z) = \int_0^\infty dx e^{-(\ell+1) \log [z + \sqrt{z^2-1} \cos x]} \quad (A-1)$$

z is in the plane cut from $-\infty$ to $+1$.

Making the substitution, with $z = \cos \xi$,

$$\log [\cos \xi + \sqrt{\cos^2 \xi - 1} \cos x] = v + \xi, \quad (A-2)$$

we get

$$Q_\ell(\cos \xi) = \frac{1}{\sqrt{2}} e^{-(\ell+1/2)\xi} \cdot \int_0^\infty dv e^{-(\ell+1/2)v} \frac{1}{\sqrt{\cos(v+\xi) - \cos \xi}} \quad (A-3)$$

where the integration is along a path

$$v = \log [\cos \xi + \sqrt{\cos^2 \xi - 1} \cos x] - \xi, \quad 0 \leq x \leq \infty, \quad (A-4)$$

in the v plane. We have

$$\zeta = \cos^{-1} z = \log(z + \sqrt{z^2 - 1}) \quad , \quad (\text{A-5})$$

where the logarithm is defined through $\log x > 0$, $x > 1$. The domain of the values of ζ corresponding to the cut z plane is shown in Fig. 3.

We show now that it is allowed to deform the path of integration in (A-3) to the positive real axis $0 \leq v \leq \infty$. For given ζ , the integrand in (A-3) becomes singular (besides at $v = 0$) for $v = -2\zeta + 2k\pi i$, $k = 0, 1, \dots$. Since $\zeta \in S$ (see Fig. 3), for these singular points always $\text{Re } v < 0$. But for the values of v along the path $\text{Re } v \geq 0$. This may be seen as follows. (A-4) reads

$$\begin{aligned} v &= \log \frac{r'}{r} \quad , \\ r' &= z + k\sqrt{z^2 - 1} \quad , \quad 1 \leq k \leq \infty \quad , \\ r &= z + \sqrt{z^2 - 1} \quad \rightarrow \quad z = \frac{1}{2}(r + r^{-1}) \quad . \quad (\text{A-6}) \end{aligned}$$

This gives

$$r' = \frac{1}{2} \left[(k+1)r - (k-1)r^{-1} \right] \quad . \quad (\text{A-7})$$

From this follows

$$\left| \frac{r'}{r} \right| = \frac{1}{2} \left| k+1 - (k-1)r^{-2} \right| \geq \frac{1}{2} (k+1 - (k-1)|r|^{-2}) \geq 1 \quad ,$$

because $|r| \geq 1$. Thus

$$\operatorname{Re} v = \log \left| \frac{r'}{r} \right| \geq 0.$$

As a consequence we may deform the path to the real v axis because for $r \rightarrow \infty$ the integrand vanishes exponentially and the singularity at $v = 0$ is integrable.

As result, we find for $Q_\ell(\cos \zeta)$ the integral representation

$$Q_\ell(\cos \zeta) = \frac{1}{\sqrt{2}} e^{-(\ell + \frac{1}{2})\zeta} \int_0^\infty dv e^{-(\ell + \frac{1}{2})v} \frac{1}{\sqrt{\cos(v+\zeta) - \cos \zeta}} \quad (A-8)$$

$\zeta \in S.$

b) Relation between $Q_\ell(1 + \frac{1}{2} [\frac{w}{\ell + \frac{1}{2}}]^2)$ and $K_0(w)$ for values of w such that $(\frac{w}{\ell + \frac{1}{2}}) \rightarrow 0, \ell \rightarrow \infty$

We put

$$z = 1 + \frac{1}{2} \chi^2, \quad \chi = \frac{w}{\ell + \frac{1}{2}} \quad (A-9)$$

$w \in$ right half plane. Then follows

$$\zeta = \log(z + \sqrt{z^2 - 1})$$

$$= \chi a(\chi),$$

$$a(\chi) = 1 + \frac{1}{3} \chi^2 + o(\chi^3). \quad (A-10)$$

Now, we find from (A-8)

$$Q_\ell(z) = e^{-w} q_\ell(z), \quad (\text{A-11})$$

$$q_\ell(z) = \frac{1}{\sqrt{2}} \frac{a(x)}{\ell + 1/2} \int_0^\infty dv e^{-v} e^{-(v+w)\sigma(x^2)} \cdot \frac{1}{\sqrt{\cos\left(\frac{v+w}{\ell+1/2} a(x)\right) - \cos\left(\frac{w}{\ell+1/2} a(x)\right)}} \quad (\text{A-12})$$

To discuss the integral (A-12), we divide it into two parts

$$\int_0^\infty = \int_0^{v_0} + \int_{v_0}^\infty$$

and choose $v_0 = C \cdot \log(\ell + \frac{1}{2})$. The second integral is $O([\ell + \frac{1}{2}]^{-C})$, $C > 0$ and may be chosen arbitrarily. In the first integral we develop the square root term in powers of the arguments of the cos functions. We use

$$\cos(x+y) - \cos y = \sum_{k=1}^{\infty} \frac{1}{(2k)!} \left\{ (x+y)^{2k} - y^{2k} \right\} \quad (\text{A-13})$$

and apply

$$(x+y)^{2k} - y^{2k} = \left([x+y]^2 - y^2 \right) \sum_{m=1}^k (x+y)^{2(k-m)} y^{2(m-1)}.$$

We get

$$\cos(x+y) - \cos y = \frac{1}{2} x[x+2y] \left(1 + O[(x+y)^2; y^2]\right).$$

Now, $q_\ell(z)$ becomes

$$\begin{aligned} q_\ell(z) &= \int_0^{v_0} dv e^{-v} \frac{1}{\sqrt{v(v+2w)}} \cdot \\ &\quad \frac{e^{-(v+w)O(x^2)}}{\sqrt{1 + O\left(\left[\frac{v+w}{\ell+1/2}a\right]^2; \left[\frac{w}{\ell+1/2}a\right]^2\right)}} + O\left(\left[\ell+\frac{1}{2}\right]^{-c}\right) = \\ &= \int_0^{v_0} dv \frac{1}{\sqrt{v(v+2w)}} \left\{ 1 + O_1\left(\frac{[\log(\ell+1/2)]^3}{[\ell+1/2]^2}\right) \right\} + O\left(\left[\ell+\frac{1}{2}\right]^{-c}\right), \end{aligned} \quad (\text{A-14})$$

if $|w|$ varies over a region $|w| \leq C \cdot \log(\ell + \frac{1}{2})$. As a consequence of (A-14), (A-11) and formula (36) for $K_0(w)$, we find for $Q_\ell(z)$

$$Q_\ell(z) = K_0(w) + O_2\left(\frac{[\log(\ell+1/2)]^3}{[\ell+1/2]^2}\right) K_0(w). \quad (\text{A-15})$$

The error estimation holds uniformly in a region $|w| \leq C \cdot \log(\ell + \frac{1}{2})$, $|w| \geq \varepsilon$, $\varepsilon > 0$. This gives the formula (54). For the errors O_1 and O_2 in (A-14) and (A-15) one finds more explicitly

$$\begin{aligned} O_1 &= -\frac{1}{3} (\ell + \frac{1}{2})^{-2} \left[w(w + \frac{1}{4}) + w(w + \frac{1}{4})v + \frac{1}{8} v^2 \right] + \\ &\quad + O_1' \left(\frac{[\log(\ell+1/2)]^4}{[\ell+1/2]^3} \right), \end{aligned} \quad (\text{A-16})$$

$$|v|, |w| \leq C \log(\ell + \frac{1}{2})$$

and putting this into (A-14), with (A-11) and (29)

$$\begin{aligned}
 Q_e(z) = & K_0(w) - \frac{1}{3} \left(l + \frac{1}{2} \right)^{-2} \left[w^2 \left(w + \frac{1}{4} \right) + \right. \\
 & + w \left(w + \frac{1}{4} \right) k_1(w) + \frac{1}{8} k_2(w) \left. \right] K_0(w) + \\
 & + \sigma_2' \left[\frac{[\log(l + \frac{1}{2})]^5}{[l + \frac{1}{2}]^3} \right] K_0(w), \\
 & |w| \leq C \log(l + \frac{1}{2}), \quad (A-17)
 \end{aligned}$$

$$\begin{aligned}
 k_n(w) = & \frac{\int_0^{v_0} dv e^{-v} (v(v+2w))^{-\frac{1}{2}} v^n}{\int_0^\infty dv e^{-v} (v(v+2w))^{-\frac{1}{2}}}, \quad n = 1, 2, \dots, \\
 & v_0 = C \log(l + \frac{1}{2}). \quad (A-18)
 \end{aligned}$$

The deviation terms of Q_e from K_0 of orders higher than $[l + \frac{1}{2}]^{-C}$ are all of the form of a polynomial in $w \times k_n(w) \times K_0 \times [l + \frac{1}{2}]^{-k}$, $k \geq 2$.

c) Upper bounds for $|Q_e(z)|$, $z = 1 + \frac{1}{2} \left[\frac{w}{l + \frac{1}{2}} \right]^2$ and $|K_0(w)|$

The representation (36) for $K_0(w)$ gives immediately

$$|K_0(w)| < C' e^{-\operatorname{Re} w}$$

For the Legendre function we use again (A-8). From the results of b) follows

$$|Q_e(z)| < C'' e^{-\operatorname{Re} w'}$$

for $|w| \leq C_1 \cdot (\ell + \frac{1}{2})$, $|w| \geq \varepsilon$, $\varepsilon > 0$, where the constant $C_1 > 0$ and sufficiently small.

For $|w| \geq C_1(\ell + \frac{1}{2})$ (A-8) gives

$$|Q_e(z)| < C''' e^{-(\ell + \frac{1}{2}) \operatorname{Re} \zeta}$$

with

$$\operatorname{Re} \zeta = \log / z + \sqrt{z^2 - 1} /.$$

For the values of w with $|w| \geq C_1 \cdot (\ell + \frac{1}{2})$ and $\operatorname{Re} w > \varepsilon_1' > 0$, we have always $\operatorname{Re} \zeta \geq \varepsilon_1'' > 0$.

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Fig. 1 The analyticity domain D_1 of $f_1(s,t)$
 M_1 = Mandelstam cut

FIG. 2 The analyticity domain D_2 of $f_2(s, u = -4q^2(s) - t)$
 M_2 = Mandelstam cut

$$\tau = \text{Cos}^{-1} z = \log (z + \sqrt{z^2 - 1})$$

FIG. 3 The strip S is the domain of values of τ corresponding to z in the cut plane