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OPTICAL MODEL AND REGGE REPRESENTATION

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A B S T R A C T

The Regge representation for the elastic scattering amplitude is used to describe the differential cross-sections of strongly absorbed projectiles. Apart from the small angles, it leads to very simple but nevertheless accurate expressions. Thus it is shown that the characteristic oscillatory diffraction pattern is very well reproduced by two poles in the complex ℓ plane not too close to the real axis and on each side of it. With more specification (definite relations between the residues and the poles), these two poles provide a good approximation for the classical 3 parameter modified cut-off model of the Wood-Saxon (or Fermi) type.

The model is applied to the case of 18 MeV and 44 MeV alpha scattering on ^{40}A and ^{208}Pb . We also show that the backward diffraction pattern cannot be described by one pole, as would follow from a resonance type of explanation. On the other hand, phase-shifts are poorly reproduced by our model, a trouble related to the non-validity of the representation in the forward direction. A brief discussion is given of the significance of our results in terms of optical potentials, and different lines for further research are sketched.

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I. INTRODUCTION

The elastic scattering of strongly absorbed particles has been described successfully by means of a complex potential, usually the Wood-Saxon potential ¹⁾. Comparisons with experimental data determine the parameters of the potential, which are : the strengths of the real and imaginary parts and the "space" parameters which roughly correspond to the size of the target nucleus and to its surface thickness. However, as Ericson ²⁾ writes, "the massive computer calculations needed to obtain the scattering amplitude make the connection between the potential and the amplitude far from transparent and simple". It is therefore useful, in order to gain a better understanding of the problem, to proceed by steps from the potential to the measurable quantities such as cross-sections and polarizations.

The amplitude provides of course a step of particular interest. Usually, it is expanded in terms of the angular momenta by means of Legendre polynomials :

$$f(\theta) = \frac{i}{2q} \sum_{\ell=0}^{\infty} (2\ell+1) (1 - \eta(\ell)) P_{\ell}(\cos \theta), \quad (1)$$

where q is the centre-of-mass momentum.

The scattering coefficients $\eta(\ell)$, which are simply related to the partial waves and to the phase shifts, have been subjected to a direct parametrization ³⁾⁻⁶⁾; the choice of their general form being dictated by physical considerations such as transparency for large ℓ values and absorption for small ℓ values. Some authors ^{5),6)} have more especially emphasized the interest of obtaining closed and simple expressions for the measurable quantities, in order to get a better feeling of the relationship between the parameters and the experimental results.

A rather different approach for the parametrization of the amplitudes can be made by using, instead of the partial wave expansion (1), the Regge representation ⁷⁾: with certain conditions on the analytical properties of $\eta(\ell)$ in the complex ℓ plane, the amplitude can be expressed, by means of the Sommerfeld-Watson transform, as the sum of a "background integral" and an expansion in terms of poles in the complex ℓ plane, the so-called Regge poles. This representation was reintroduced by Regge in the framework of non-relativistic scattering, and extended to the case of high energy relativistic scattering, where it has been widely discussed and applied. The main reasons for this interest are twofold:

- 1) the rapid convergence of the expansion, so that only a few terms need be considered;
- 2) the existence of simple but accurate asymptotic expressions for the Legendre functions associated to each pole, so that comparisons with experiments are easily done.

At high energy, the Regge representation is applied in the crossed channel, so that the relevant variable for the validity of 1) and 2), $-\cos\theta$, takes very large, positive, evidently non-physical, values. Also, the contribution of the background integral can be shown to decrease with energy, and is therefore neglected.

In the case of non-relativistic scattering the properties 1) and 2) can also occur under certain circumstances ^{*)}:

- a) A rapid convergence of the expansion does occur, except in the forward direction. The relevant parameter is $|\operatorname{Im} L| \theta$, where L is the position of the pole. Only the poles situated nearest to the real axis will give appreciable contributions to the differential cross-sections, for values of θ not too small.

*) They are precisely the original reasons for the introduction of the method. See, Sommerfeld, Partial Differential Equations in Physics, Academic Press N.Y., p.282 (1942).

- b) Asymptotic expressions are obtained for $|L|$ large, but actually their accuracy is very good for values of $\text{Re } L$ corresponding to the cut-off parameter ℓ_0 of Eq. (1). This variable is related, through the impact parameter, to the target radius, and because the surface effects are the most important for strongly absorbed particles, we expect the Regge poles to be situated near ℓ_0 .

Regge poles in non-relativistic scattering have already been used by a number of authors ⁸⁾. In particular, Ceuleneer et al. ⁹⁾ wanted to take advantage of the fact that the Regge representation is more favourable to large angles to fit the backward differential elastic cross-sections of strongly absorbed particles. Ericson ²⁾ and, independently, Inopin ¹⁰⁾ seem, however, to have been the first, at least in nuclear physics, to emphasize the usefulness of properties a) and b), in connection with the diffraction problem. They both start from a smooth cut-off model for $\eta(\ell)$ of the Wood-Saxon (or Fermi) type :

$$\eta(\ell) = \frac{1}{1 + e^{-\frac{\ell - \ell_0}{a}}} \quad (2)$$

They then perform the Sommerfeld-Watson transformation and arrive at a formula involving the two first poles, which are complex conjugate. The form (2) is particularly favourable as it leads to a negligible background integral.

From these results, one feels encouraged to try, instead of the usual partial wave (or phase shift) analysis, a phenomenological fit of cross-sections in terms of Regge poles. The parameters to be determined are the position of the poles and their residue, and, because of the properties a) and b), the analysis is both easy and unique. In doing so, one hopes that the Regge expansion will be more important than the background integral.

There is however, a weakness in this approach. The scattering coefficients $\eta(\ell)$ are expressed by a very slowly converging expansion in terms of Regge poles. That makes it very difficult to determine the

partial waves (or phase shifts) from a few Regge poles. This is related to the fact that $\eta(\ell)$ is an average of the amplitude over all angles within the interval $[0, \pi]$; but, for small θ , the amplitude cannot be described in terms of Regge poles because the Regge representation diverges in the forward direction.

One way to avoid this difficulty is to use a different model in the forward direction, and then combine both approaches to determine the amplitudes.

Another way to use the advantages of the Regge method directly is to study a number of simple models, expressible in different ways, Regge poles, potential theory and/or phase shift analysis. Then, in situations of more complex nature, one could hope that the analysis of cross-sections in terms of Regge poles would reveal one or several of these elementary models so that an immediate understanding of the nature of the phenomena would be achieved.

One of this kind of models is very well known : it is the resonance model, which has been expressed in different ways, and in particular as a Regge pole lying close to a physical value of the angular momentum, with a small positive imaginary part, and a real part increasing with energy ⁷⁾.

The discussion of Ericson and Inopin provides us with another model, the diffraction model; it is quite unexpected a priori that the shape (2) of the scattering coefficient, very classical indeed and with a direct physical meaning, can lead to a two-pole model, with the two poles situated not too close to the real axis and on each side of it. Although from these two poles we are unable to predict in an accurate way the partial waves, if we are in the presence of them, we can use all the knowledge accumulated in the literature on the diffraction picture, relating complex potentials to wave functions and wave functions to phase shifts.

We have thus come to the main object of this paper, which is to discuss in detail the diffraction model in terms of Regge poles. When doing this, we shall always use as a guide the model provided by Eq. (2).

In the next Section, we briefly recall how is obtained the Regge representation, write the approximation formulae which will be used thereafter, and determine the dominant pole. Most of these results are already in Refs. 2) and 10).

In Section III, we discuss the contribution of one pole to the differential cross-section. We show that it cannot describe the experimental backward diffraction patterns. Therefore, instead of a resonance phenomenon, a two-pole description of exchange character is suggested, but not attempted here.

In Section IV, we discuss the contribution of two poles to the differential cross-sections. We see that we can reproduce the well-known oscillating diffraction pattern with two poles on each side of the real axis and not too close to it. The special model defined by Eq. (2) introduces relations between the residues and the positions of the poles. Coulomb effects are included in the analysis, and modify only the residues of the poles.

In Section V, we apply these results to the elastic scattering of 18 MeV and 44 MeV α particles on ^{40}A and ^{208}Pb . We use for that the general two-pole model as well as the three-parameter model obtained from (2) by letting the cut-off parameter ℓ_0 take complex values, following a suggestion of Ericson 2). In both cases, the parameters are determined directly from the experimental data by using the approximated formulae of Section IV, without having to use a computer.

In Section VI, we examine in detail the inversion problem and the unitarity condition. The slow convergence of the $\eta(\ell)$ expressed in terms of Regge poles is illustrated by the case of model (2) without Coulomb effects. A comparison of the scattering coefficients given by different models is made.

In Section VII, we discuss the results, and especially the meaning of the pole below the real axis, in terms of potential theory. This pole actually gives the biggest contribution to the cross-sections. However, it is a well-known property in potential theory that no pole should exist

in the fourth quadrant of the complex ℓ plane in the case of regular potentials. This theorem has been demonstrated only for real potentials, but we extend it in the case of complex potentials in the Appendix. The existence of a pole below the real axis leads therefore to a number of rather varied conclusions concerning the potentials.

II. THE REGGE REPRESENTATION AND ITS APPLICATION TO THE SMOOTH CUT-OFF MODEL

Let us first briefly recall how the Regge representation is obtained. The Sommerfeld-Watson transform consists of two steps ⁷⁾ : first interpret the right member of (1) as a sum over residues corresponding to an integration over the contour C_1 in the complex ℓ plane (see Fig.1). Then deform this contour into the semicircle C_2 , and the vertical line D passing through $-\frac{1}{2}$. If $\eta(\ell)$ remains smaller than an exponential for large values of $|\ell|$, the contribution from the semicircle becomes negligible as its radius increases to infinity. The contribution from D gives the so-called "background integral", which converges under the same conditions, except in the forward direction. The final formula reads :

$$f(\theta) = \frac{1}{2q} \int_{-\infty}^{+\infty} \frac{\eta(-\frac{1}{2} + i p) P_{-\frac{1}{2} + i p}(-\cos \theta)}{\cosh \pi p} p dp + \frac{i\pi}{2q} \sum_{\alpha} \frac{(2\alpha+1) r_{\alpha}}{\sin \pi \alpha} P_{\alpha}(-\cos \theta), \quad (3)$$

where r_{α} represents the residue of $\eta(\ell)$ at the pole α . This is what we call the Regge representation, and α are the Regge poles. For $\theta=0$, formula (3) is not valid.

We now consider $\eta(\ell)$ as composed of a Coulomb part and a nuclear part :

8.

$$\eta(\ell) = e^{2i\sigma(\ell)} \eta_N(\ell) \quad (4)$$

with

$$\begin{cases} \sigma(\ell) = \text{Arg } \Gamma(1 + \ell + i\gamma) \\ \gamma = \frac{Z_1 Z_2}{137} \sqrt{\frac{m_1}{2E_{\text{lab}}}} \end{cases} \quad (5)$$

Here m_1 and E_{lab} are the mass and the energy of the projectile; Z_1 and Z_2 are the number of protons of the projectile and of the target.

We take for the nuclear part the Wood-Saxon shape (2) :

$$\eta_N(\ell) = \frac{1}{1 + e^{-\frac{\ell - \ell_0 + i\ell_1}{a}}} \quad (6)$$

where ℓ_1 is an additional parameter giving a small non-vanishing real part to the phase shift ²⁾. ℓ_0 is the cut-off parameter, and it is roughly proportional to the radius of the target nucleus, while a measures its surface thickness.

One can easily see that $\eta(\ell)$ satisfies the conditions necessary for the validity of the Regge representation. The poles are infinite in number, equidistant and situated on a vertical line passing through $\ell = \ell_0$. Their position is given by

$$L(k) = \ell_0 - i\ell_1 + i\alpha\pi(2k+1) \quad (7)$$

The residues of $\eta_N(\ell)$ are all equal to a , and must be multiplied by the value of $e^{2i\sigma(\ell)}$ at the poles in order to obtain the full residue; we thus get :

$$f(\theta) = I_f + \frac{i\pi a}{2g} \sum_k (2L(k)+1) \frac{\Gamma(1+L(k)+i\sigma)}{\Gamma(1+L(k)-i\sigma)} \frac{P_{L(k)}(-\cos\theta)}{\sin \pi L(k)} \quad (8)$$

where I_f represents the background integral, which is, in the present case, expressed as

$$I_f = \frac{1}{2g} \int_{-\infty}^{+\infty} \rho d\rho \frac{\Gamma(\frac{1}{2} + i\rho + i\sigma)}{\Gamma(\frac{1}{2} + i\rho - i\sigma)} \frac{P_{-\frac{1}{2} + i\rho}(-\cos\theta)}{1 + e^{\frac{2b+1}{2a}} e^{-i\frac{\rho}{a}}} \quad (9)$$

Because the ratio ℓ_0/a is large in the cases considered here, I_f is of the order of $e^{-\ell_0/a}$ and therefore negligible. Thus the form (6) is particularly favourable, as it leads to a negligible background integral, so that Eq. (8) reduces to a sum of Regge poles.

Now we introduce the asymptotic formulae we shall need; we first have ¹¹⁾ :

$$P_L(\cos\theta) = \sqrt{\frac{2}{\pi L \sin\theta}} \left\{ \sin \left[\left(L + \frac{1}{2} \right) \theta + \frac{\pi}{4} \right] + O\left(\frac{1}{|L|}\right) \right\} \quad (10)$$

$$\varepsilon \leq \theta \leq \pi - \varepsilon,$$

where ε is an arbitrarily small positive number. The second term in (10) depends on ε .

For the Coulomb phase shifts, we use ¹¹⁾ :

$$\frac{\Gamma(1+L+i\sigma)}{\Gamma(1+L-i\sigma)} = (1+L)^{2i\sigma} \left[1 + O\left(\frac{1}{|L|}\right) \right] \quad (11)$$

$$\sim e^{2i\sigma \log |1+L| - 2\sigma \arg(1+L)}$$

10.

If now L is not close to the real axis, one of the two exponentials expressing the complex sines in Eqs. (8) and (10) will dominate, and we can write :

$$\frac{P_L(-\cos\theta)}{\sin\pi L} \sim -i\xi \sqrt{\frac{2}{\pi L \sin\theta}} e^{i\xi[(\operatorname{Re} L + \frac{1}{2})\theta - \frac{\pi}{4}] - |\operatorname{Im} L|\theta} \quad (12)$$

where

$$\xi = \frac{\operatorname{Im} L}{|\operatorname{Im} L|} = \pm 1.$$

We see from Eq. (7) that this approximation works well in the case of the specified diffraction model (6), as $a\pi$ is large enough, while we expect ℓ_1 to be small. In Eq. (12), the damping for the contribution of far away poles manifests itself clearly.

We are thus led, quite generally, to the following formula for the amplitude $f(\theta)$

$$f(\theta) = \frac{A}{\sqrt{\sin\theta}} \sum_k \xi_k e^{x_k + i y_k} \quad (13)$$

with

$$\begin{aligned} A &= \frac{\sqrt{2\pi}}{q} \\ \xi_k &= \frac{\operatorname{Im} L_k}{|\operatorname{Im} L_k|} = \pm 1 \\ e^{x_k} &= e^{-|\operatorname{Im} L_k|\theta} \sqrt{|L_k|} |r_k| \\ y_k &= \xi_k \left[(\operatorname{Re} L_k + \frac{1}{2})\theta - \frac{\pi}{4} \right] + \frac{1}{2} \operatorname{Arg} L_k + \operatorname{Arg} r_k \end{aligned} \quad (14)$$

The cross-section is given by

$$\frac{d\sigma}{d\theta} = |f(\theta)|^2$$

$$= \frac{|A|^2}{\sin \theta} \left\{ \sum_k e^{2x_k} + \sum_{k < k'} 2 \xi_k \xi_{k'} e^{x_k + x_{k'}} \cos(\gamma_k - \gamma_{k'}) \right\} \quad (15)$$

The dominant term corresponds to the largest x_k . The x_k are linear functions of θ and therefore the dominant terms are easily determined.

As an illustration, we now consider our specific model (6) :

There, the residues τ_k appearing in (14) are expressed by :

$$\tau(k) = a e^{-2\delta \operatorname{Arg}(1+L(k)) + 2i\delta \log |1+L(k)|} \quad (16)$$

In order to discuss the dominant contribution, we do more drastic approximations, using the fact that, for large ℓ_0 , we expect, at least for the first poles, that

$$\frac{|I_m L|}{|L|} \ll 1 \quad (17)$$

will hold.

We thus write

$$e^{\tau(k)} = \sqrt{\ell_0} a e^{-I_m L(k) \left(\xi(k) \theta + \frac{2\delta}{\ell_0} \right)} \quad (18)$$

The parameter $2\delta/\ell_0$, which represents the classical deflection angle by the Coulomb field²⁾, is seen to play an important role : for the poles situated above the real axis ($\xi(k) > 0$) it helps to the damping.

However, for the poles situated below the real axis ($\text{Im } L < 0$, $\xi(k) < 0$), it appears as a positive contribution in the exponential, and therefore favours greatly the contribution of these poles. This is clearly seen in Fig. 2, where we show the different lines $x(k)$, as expressed by (18).

For $0 < \theta < (2\delta/\ell_0)$, many terms contribute to the amplitude, and we do not expect the two-pole approximation to be valid. For $\theta > (2\delta/\ell_0)$, the first pole below the real axis, which corresponds to $k=-1$ in Eq. (7), is dominant and, starting from a limiting value $\theta_e > (2\delta/\ell_0)$, where $x(-2)$ and $x(0)$ intersect, the first pole above the real axis, corresponding to $k=0$, will be next in importance.

If $\ell_1 = 0$, that is for $\eta_N(\ell)$ real, $x(-1)$ and $x(0)$ are parallel. In this case, $\theta_e = (4\delta/\ell_0)$, and this value is the starting point for the validity of the two-pole approximation.

If ℓ_1 is positive, the two lines $x(0)$ and $x(-1)$ intersect at a critical angle

$$\theta_c = \frac{2\delta}{\ell_0} \frac{a\pi}{\ell_1} \quad (19)$$

It is of importance to know whether this angle is smaller or bigger than π , so that the intersection occurs within or outside the physical region. Indeed, for $\theta > \theta_c$, the pole ($k=0$) begins to dominate over the pole ($k=-1$).

The limiting angle θ_e for the validity of the two-pole approximation is given by the general expression

$$\theta_e = \frac{2\delta}{\ell_0} \frac{2a\pi}{a\pi + \ell_1} \quad (20)$$

From Eq. (20), one sees that it decreases as ℓ_1 increases.

III. THE DIFFERENTIAL CROSS-SECTION CORRESPONDING TO ONE POLE

If a pole is situated near a physical value of ℓ , with a small imaginary positive part, and a real part increasing with energy, it corresponds to a resonance phenomenon. We want to consider this case in detail, because it has been suggested ^{9),12)} that this type of phenomenon is responsible for the backward peak in the cross-section of strongly absorbed particles.

We can no longer use the approximation (12), but the general form for the cross-section is easily worked out from Eq. (10). We find, for a pole situated at L and with a residue r :

$$\frac{d\sigma}{d\theta} = \frac{\pi}{2} \frac{|r|^2}{q^2} \frac{|2L+1|^2}{|L|} \frac{\cosh[2 \operatorname{Im} L (\pi - \theta)] + \sin[(2 \operatorname{Re} L + 1)(\pi - \theta)]}{\sin \theta \left\{ \cosh[2 \pi \operatorname{Im} L] - \sin[(2 \operatorname{Re} L + 1) \pi] \right\}} \quad (21)$$

$$\epsilon \leq \theta \leq \pi - \epsilon,$$

an expression valid for large $|L|$ and θ different from 0 and π .

To study the corresponding behaviour for the differential cross-section, we write :

$$F(\theta) \equiv \frac{d\sigma}{d\theta} \sin \theta = \text{Const.} \left\{ \cosh[2 \operatorname{Im} L (\pi - \theta)] + \sin[(2 \operatorname{Re} L + 1)(\pi - \theta)] \right\} \quad (22)$$

This form indicates a general decrease for $F(\theta)$, according to the first term, with oscillations generated by the second term.

If $|\operatorname{Im} L|$ is small, the maxima of these oscillations are very slowly decreasing. However, the minima correspond approximately to a cancellation of the first term in the expansion of $\cosh(2 \operatorname{Im} L) (\pi - \theta)$ with the value -1 of the sine, and are therefore strongly marked. As $|\operatorname{Im} L|$ increases, the general fall-off accentuates, while the oscillations tend to vanish for small values of θ . When $|\operatorname{Im} L|$ is large enough so that one exponential dominates in the first term, we obtain a pure exponential decrease, with oscillations confined to the larger values of θ .

This general behaviour is shown in Fig. 3 for three increasing values of $\text{Im} L$. The value 7 for the real part has intentionally been chosen rather low, in order to emphasize the accuracy of the approximations used.

The most important result of this discussion is the general decrease of the maxima of the oscillations for $(d\sigma/d\theta)\sin\theta$. The backward peaking of $d\sigma/d\theta$ appears therefore, in this model, to be due only to the factor $\sin\theta^{-1}$. In Fig. 4, however, we show recent experimental results ¹²⁾ for $(d\sigma/d\theta)\sin\theta$. It can be seen that the backward peaking is still very pronounced, so that it appears hopeless to explain it with a single Regge pole. This casts doubts on the validity of the resonance picture for the explanation of the observed data. The general structure of the backward scattering in Fig. 4 looks actually very similar to that of the forward scattering, if one changes θ in $\pi - \theta$. This suggests that an exchange phenomenon type of explanation could be attempted. We do not want, however, to pursue here the discussion of this problem. We shall rather see in the next two Sections how to reproduce a diffraction pattern with two poles.

IV. THE TWO-POLE MODEL

We suppose the two poles situated not too close to the real axis, and far from the origin, so that formulae (13) to (20) apply. From the interference term in (15), it is clear that in order to have reasonable positions of the poles, we must choose them with imaginary parts of opposite sign; then the frequency of the oscillations comes out as the sum over the real parts. (For poles situated in the same quadrant, the frequency comes out as the difference of the real parts, which would necessitate, for our purpose, two poles very far away from each other.) The notations are explained in Fig. 1. From experiment, one can only determine the sum of the real parts, and, for simplicity, we shall take both quantities equal, and call ℓ_0 their common value.

In the three-parameter model defined by Eq. (6), we have

$$\begin{cases} b_1 = a\pi - \ell_1 \\ b_2 = a\pi + \ell_1 \end{cases} \quad (23)$$

We again concentrate the discussion on $(d\sigma/d\theta)\sin\theta$.

$$F(\theta) \equiv \frac{d\sigma}{d\theta} \sin\theta = B^2 \left\{ e^{2X_1} + e^{2X_2} - 2 e^{X_1+X_2} \sin\omega \right\} \quad (24)$$

with

$$\begin{aligned} X_1 &= -b_1\theta + \log|r_1| \\ X_2 &= -b_2\theta + \log|r_2| \\ B &= \sqrt{2\pi\ell_0}/q \\ \omega &= (2\ell_0+1)\theta + \arg r_1 - \arg r_2 \end{aligned} \quad (25)$$

In Eq. (25) we have used the condition

$$|\operatorname{Im} L| \ll |L|, \quad (26)$$

in order to simplify the discussion to the maximum.

In the three-parameter model, the residues are composed of the common factor a and the Coulomb part [see Eqs. (16)-(18)].

The general trend of $F(\theta)$ is an exponential decrease, because one exponential always clearly dominates the other. The critical angle, for which the two lines x_1 and x_2 intersect, is given by

$$\theta_c = \frac{\log |r_2| - \log |r_1|}{b_2 - b_1} \quad (27)$$

If θ_c is smaller than π , then the second exponential starts dominating over the first for $\theta > \theta_c$, and the general slope becomes smaller [we have $b_1 < b_2$; see Eq. (23)]. The first result of Eq. (24), also valid for the more general formula (13), is therefore that the fall-off of $\log F(\theta)$ is linear; if there is a change of slope, the change will be in the direction of a smaller slope.

Around the critical value given by Eq. (27), we expect a minimum which is more pronounced than the others. Indeed, if we expand $F(\theta)$ in $\Delta\theta = \theta - \theta_c$, we get :

$$F(\theta) \sim 2 B^2 e^{x_1 + x_2} \sqrt{1 + \frac{(b_2 - b_1)^2}{2} \Delta\theta^2 - \sin \omega}$$

Within the interval of a half period, $|\Delta\theta|$ is smaller than $\pi/(2\ell_0 + 1)$, and consequently small. If we now consider the nearest value θ for which $\sin \omega$ takes the value $+1$, we get an extra factor of the size $(\pi(b_2 - b_1)/(2\ell_0 + 1))^2$ which greatly reduces the cross-section at this point.

Such a dip does not usually occur on experimental curves, and this is an indication that the critical value is bigger than π .

We now turn to the oscillations. We suppose $x_2 > x_1$, and keep only the two larger terms in (24) :

$$\frac{d\sigma}{d\theta} \sim \frac{B^2}{\sin \theta} e^{2x_2} \left\{ 1 - 2 e^{(x_1 - x_2)} \sin \omega \right\}$$

We study

$$Y = \log \left\{ \frac{d\sigma}{d\theta} \frac{\sin \theta}{B^2} \right\} = 2x_2 + \log \left\{ 1 - 2 e^{(x_1 - x_2)} \sin \omega \right\} \quad (28)$$

$$\sim 2x_2 - 2 e^{(x_1 - x_2)} \sin \omega$$

In the last approximation we have used the fact that x_1 is smaller than x_2 . The derivative of Y is :

$$\frac{dY}{d\theta} \sim -2b_2 - 2 e^{(x_1 - x_2)} \left\{ (b_2 - b_1) \sin \omega + (2\ell_0 + 1) \cos \omega \right\}$$

We neglect the first term in the parenthesis as $(b_2 - b_1)$ is smaller than $(2\ell_0 + 1)$. We must be sure, in doing so, that we shall not consider values of θ for which $\cos(\omega)$ vanishes. One sees immediately that this condition is fulfilled for the extrema of Y , which are approximately given by :

$$\cos \omega_{ex} \simeq - \frac{b_2}{2\ell_0 + 1} e^{(x_2 - x_1)} \quad (29)$$

For small values of θ , this equation has no solutions. The oscillations start from the value θ_0 such that the right member of Eq. (29) is equal to -1 . This gives, for θ_0 , the relation :

$$\theta_0 = \theta_c - \frac{1}{(b_2 - b_1)} \log \frac{2\ell_0 + 1}{b_2} \quad (30)$$

The extrema are obtained for values of $\cos \omega$ not very far from -1 , and we can approximately write that θ_{med} , the median value of θ between a minimum and the next maximum, is obtained for :

$$(2l+1)\theta_{\text{med}} + \text{Arg} \frac{z_1}{z_2} = (2k+1)\pi \quad (31)$$

The extrema are more and more pronounced as θ increases, and the distance between a minimum and the next maximum increases also with θ .

We now determine, from Eqs. (28) and (29), the differences ΔY between the extrema of Y and the values given by the dominant pole at the same angle. ΔY is related to $x_1 - x_2$ by the following equation :

$$\frac{(\Delta Y)^2}{4} + \frac{b_2^2}{(2l+1)^2} = e^{2(x_1 - x_2)} \quad (32)$$

From this one sees that, as θ increases, the magnitude of the oscillations of Y also increases exponentially.

V. APPLICATION TO α SCATTERING

We apply the results of the last Section for the parametrization of the differential elastic cross-sections for the scattering of 18 MeV alpha-particles on $^{40}\text{A}^{13}$, and 44 MeV alpha-particles on $^{208}\text{Pb}^{14}$.

These experimental results have been chosen because of the quasi-absence of a backward peak. We use the two-pole model in its general form, and in its specified form as an approximation for the smooth cut-off model. The parameters are determined from the approximate formula of last Section. In the case of the smooth cut-off model, there are three independent parameters.

This number increases to eight in the case of the general two-pole model. However, the sum $\text{Re } L_1 + \text{Re } L_2$ and an over-all phase of the residues cannot be determined from experiment. Therefore, what we really do is a six-parameter determination. Usually, one uses the unitarity condition on $\eta(\ell)$ to reduce this number^{8),9)}. Here, however, we do not want to impose any restriction due to $\eta(\ell)$, because, as will be discussed in the next Section, we do not consider our model to be good enough for the determination of $\eta(\ell)$.

We do the determination of the parameters on the experimental curves for

$$\log((d\sigma/d\theta)\sin\theta)$$

As there is no sudden deep minimum, we do not expect a change in slope of $Y(\theta)$. This gives the restriction

$$\frac{2\sigma}{b} > \frac{\ell_1}{a} \quad (33)$$

for the three-parameter case.

The determination of ℓ_0 is made by means of Eq. (31), where θ_{med} is the median angle between a minimum and the following maximum. This fixes also the relative phase of the two residues. In ^{40}A , the best value for ℓ_0 is found to be 8, but necessitates a relative phase between r_2 and r_1 as large as 138 degrees. In the three-parameter case, the term $\arg(r_1/r_2)$ is zero in Eq. (31), and the best values compatible with it are $\ell_0 = 7.5$ or 9.7. However, this last value favours the small angles in contradiction with the whole picture, and we thus choose $\ell_0 = 7.5$. In ^{208}Pb the best value is $\ell_0 = 16$ and no relative phase is necessary.

The determination of x_2 comes next, and is made by tracing a straight line through the values of

$$\log((db/d\theta) \sin \theta)$$

corresponding to θ_{med} . In the case of ^{40}A , this is easily done, and gives a value for b_2 around 1.9 or 2.0. In the case of ^{208}Pb , however, the general structure of Y does not correspond to a straight line, but to a curve. In the absence of any marked minimum which would allow for a change of the slope, we must conclude that the background contribution is responsible for this bending. However, to a good approximation, we can still use our formula, and find b_2 around 2.9.

In the three-parameter case, the determination of ℓ_0 and x_2 fixes the solution. The parameters must nevertheless correspond to reasonable values for the critical angle θ_c , which must be larger than π , and for θ_0 , which, together with θ_c , gives a rough idea of the magnitude of the oscillations. These conditions permit an increased precision in the determination of the parameters. In the case of ^{40}A , we obtain a satisfactory fit, as shown in Fig. 5a, with $\ell_0 = 7.5$, $\ell_1 = 0.35$ and $a = 0.5$. These parameters correspond to the first two poles $L_1 = 7.5 + i1.22$ and $L_2 = 7.5 - i1.92$. The fit appears as good as a similar fit done by Alster and Conzett³⁾, with a five-parameter formula for $\eta(\ell)$. This illustrates how the simplicity of our formulae results in a greater efficiency in the parametrization procedure.

The case of ^{208}Pb is more difficult to fit with the three-parameter model. For the best value of ℓ_0 , which is 16, one gets too large cross-sections. To reduce them, one has to increase the value of ℓ_0 , and thus sacrifice seriously the fit of the last oscillations, which quite generally, show a tendency to a larger period than the first oscillations. Figure 6a shows the result of a compromise, with $\ell_0 = 22$, $\ell_1 = 0.5$ and $a = 0.86$, corresponding to the two poles : $L_1 = 22 + i2.2$ and $L_2 = 22 - i3.2$. Again, this example shows how the evidence of the limits of a model can be reached easily because of the simplicity of our formulae.

We also show in Fig. 5a how the approximation of Eq. (6) by two, three and four poles works. The three first poles give essentially the same results as the two first poles, because of the damping suffered by the upper poles. For ^{40}A , the deflection angle $2\chi/\ell_0$ is 41° , and the two-pole approximation is seen to be quite good from 60° onwards. For ^{208}Pb , the deflection angle has approximately the same value. The two-pole approximation is excellent from 60° , and the corresponding curve has not been drawn in Fig. 6a, because it hardly distinguishes from the exact curve.

For the general two-pole model, we still have to determine x_1 . This we do through the relation (32), which shows how $x_1 - x_2$ can be deduced from the magnitude of the oscillations $|\Delta Y|$. We obtain in that way, the following solutions

$$\begin{array}{lll}
 ^{40}\text{A} & : & L_1 = 8.0 + i1.42, \quad \text{with } r_1 = 0.33 \\
 & & L_2 = 8.0 - i1.92, \quad r_2 = -1.3 + i1.21 \\
 \\
 ^{208}\text{Pb} & : & L_1 = 16.0 + i1.8, \quad r_1 = 0.067 \\
 & & L_2 = 16.0 - i2.93, \quad r_2 = 3.57
 \end{array}$$

The common over-all phase for the residues has been chosen quite arbitrarily, so that r_1 is real. In the next Section, we shall see that a consideration of the scattering coefficients can roughly determine this phase.

The results are shown in Figs. 5b and 6b, and are satisfactory, apart from the small angles and in the backward direction. As for small angles, we do not expect the Regge representation to be of any use. On the other hand, the failure in the backward region must be related to the fact that the backward pattern shown in Fig. 4 exists probably also for the two cases studied here, although not so marked.

To conclude this Section, we want to emphasize that no least square fit has been done for the determination of the parameters, so that our results could probably be slightly ameliorated.

VI. THE INVERSION PROBLEM AND THE UNITARITY CONDITION

Given the Regge representation (3), one can come back to the ordinary expansion in Legendre polynomials by calculating the partial waves. We use for this the relation

$$\frac{\pi}{2 \sin \pi \alpha} \int_{-1}^{+1} dx P_{\ell}(x) P_{\alpha}(-x) = \frac{1}{(\alpha - \ell)(\alpha + \ell + 1)} \quad (34)$$

where α is complex and ℓ is an integer.

For $\eta(\ell)$ we obtain

$$\eta(\ell) = 1 + \frac{i}{\pi} \int_{-\infty}^{+\infty} p dp \frac{\eta(-\frac{1}{2} + ip)}{p^2 + (\ell + \frac{1}{2})^2} - \sum_k \frac{\pi_k (2L_k + 1)}{(L_k - \ell)(L_k + \ell + 1)} \quad (35)$$

the second term is the contribution from the background integral. From its form, we expect it to be the most important for small ℓ and then to decrease as ℓ becomes large. In the absence of Legendre functions, there is no damping in the infinite sum, and the whole expansion is very slowly converging.

In the case of the smooth cut-off model (6), the background term is again of the order of $e^{-\ell_0/a}$ for large values of ℓ_0/a , and therefore negligible.

We now discuss the contribution to the scattering coefficient coming from one pole, and for that we write :

$$-2 \frac{2L + 1}{(L - \ell)(L + \ell + 1)} = 2 \left\{ \frac{1}{\ell - L} - \frac{1}{L + \ell + 1} \right\} \quad (36)$$

For large values of $\text{Re } L$ the second term is smaller than the first. If we neglect it, we easily determine the path of $1/(\ell-L)$ in the complex η plane. As seen in Fig. 7b, it starts, for $\ell=0$, at the value $-L^{-1}$, which is close to the origin. Then, as ℓ increases, the path goes around the imaginary axis, which it reaches for $\ell = \text{Re } L$, at the value $(\text{Im } L)^{-1}$.

The path is then symmetrical with respect to the imaginary axis and leads back to zero for ℓ infinite. The path is more or less flattened according to the magnitude of $(\text{Im } L)^{-1}$. The residue factor r introduces a change of size and a rotation around the origin. Finally, the scattering coefficient $\eta(\ell)$ is obtained by shifting the origin by one unit along the real axis.

The unitarity condition expresses that $|\eta(\ell)|$ should not be greater than 1. To fulfil this condition, one sees that the residue must be purely imaginary, with the same sign as $\text{Im } L$, and that we must have :

$$\left| \frac{r}{\text{Im } L} \right| < 2 \quad (37)$$

The quantity $|r/\text{Im } L|$ roughly measures the diameter of the path, that is, the importance of the contribution from the pole L to the expansion (35).

Let us first examine the case of model (6) in the absence of Coulomb forces. Then the residues are all real, and the unitarity is violated at each step of the pole approximation. This is illustrated in Fig. 7. We first represent, in Fig. 7a the scattering coefficients $\eta(\ell)$ in the complex η plane for the case of ^{40}A . The quantities determined by the fit of the preceding Section are compared with those determined by the Alster-Conzett parametrization of the same experimental data, where the following form of $\eta(\ell)$ was used ³⁾:

$$\eta(\ell) = e^{2i\delta} \left[1 + e^{\frac{\ell - \ell_0}{4\ell_0}} \right]^{-1} \left[1 + e^{-\frac{\ell - \ell_1}{4\ell_1}} \right]^{-1}$$

Both of these forms clearly respect unitarity. In Fig. 7b are represented the contributions to $\eta(\ell)$ arising from the two first poles separately, from the two first poles simultaneously, and from the four first poles simultaneously. We see that, even if four poles were very good to determine the cross-section, they give contributions to $\eta(\ell)$ that look very different from the total result.

One can understand what is going on if one notices that, in the absence of Coulomb forces, the expansion in Eq. (35) represents a hyperbolic tangent¹⁵⁾. With $\ell_1 = 0$, we write - neglecting the background term and using (36) :

$$\begin{aligned} \eta(\ell) &= 1 - a \left[\sum_k \frac{1}{\ell - \ell_1 + ia\pi(2k+1)} + \frac{1}{\ell_0 + \ell + 1 + ia\pi(2k+1)} \right] \\ &= 1 - \frac{1}{2} \left(\tanh\left(\frac{\ell - \ell_1}{2a}\right) + \tanh\left(\frac{\ell_0 + \ell + 1}{2a}\right) \right) \end{aligned}$$

For ℓ_0 large, the last term gives 1, and we obtain :

$$\eta(\ell) = \frac{1}{2} \left(1 + \tanh\left(\frac{\ell - \ell_1}{2a}\right) \right)$$

which is precisely (6).

We therefore see that, even if the last term in (36) is small compared to the first one, it belongs to an infinite slowly converging sum, the total value of which is of the same order of magnitude as the sum of the dominant terms.

The main difficulty lies here in the infinite number of poles. With a reasonable potential, however, we expect the number of poles to be finite, so that the contributions to $\eta(\ell)$ from two of them should have more physical meaning than in this particular case.

Already when we include the Coulomb effects in the smooth cut-off model, the situation improves dramatically, as can be seen in the next Figures, and especially in Fig. 8. The second pole, which gave the largest contribution for the cross-section, plays here also the major role. Condition (37) for unitarity reads, for the two-pole approximation :

$$\begin{cases} \frac{a e^{2\gamma \frac{b_2}{b}}}{b_2} < 2 \\ 2\gamma \log |1+b| \simeq (2k - \frac{1}{2})\pi \end{cases} \quad (38)$$

In the case of ^{40}A , shown in Fig. 8, we have, with the parameters of last Section :

$$\left| \frac{r_2}{\text{Im } L_2} \right| = 1.015, \quad \text{Arg } r_2 = (2k+1)180^\circ + 117^\circ$$

so that the unitarity conditions are fulfilled. The two-pole approximation leads to scattering coefficients which roughly show the same behaviour at the nucleus surface as the total $\eta(l)$ [Figs. 8a and 8b]. The four-pole approximation has not been represented, because it departs very much from this behaviour, and leads to a severe violation of unitarity.

For ^{208}Pb , we have

$$\left| \frac{r_2}{\text{Im } L_2} \right| = 2.6, \quad \text{Arg } r_2 = (2k+1)180^\circ + 108^\circ$$

and the two-pole approximation leads to a violation of unitarity of approximately 30%. This is due to the fact that the residue is too large, which was also the cause of the failure of the model for the cross-sections. The scattering coefficients resulting from the two-pole approximation do not resemble very much their starting point (6), as shown in Fig. 9. However, the results for the cross-sections were the same in both cases. It is clear that, as for the non-Coulombian case of Fig. 7, these two poles have no physical meaning, and must be considered as a mathematical trick.

We now turn to the general two-pole model. The condition on the diameter of the path is fulfilled in both cases. The ratio $|r_2/\text{Im } L_2|$ takes the values 0.93 for ^{40}A , and 1.22 for ^{208}Pb . These values are not far from 1, which is what is needed for a total absorption inside the nucleus. In the last Section, the common phase of the residues was chosen quite arbitrarily so that r_1 was real. This turns out not to be good from the unitarity condition point of view. Therefore, as this parameter is still free, we have fixed it so that the shape of $\eta(\ell)$ resembles roughly the smooth cut-off model shape. This does not lead to a very accurate determination, especially because the value ℓ_0 is not the same for both cases, and we only refer the reader to Figs. 8a and 9, where the additional phases given to the residues of the last Section are indicated as bent arrows.

For ^{40}A , where the smooth cut-off model was successful, the two poles defined by it were not very different from the two poles of the more general model. It is therefore not surprising that, as seen in Fig. 8, their scattering coefficients have a rather similar aspect.

For ^{208}Pb , the smooth cut-off model was not successful, and the two corresponding poles were rather different from the poles of the more general model. From the consideration of the scattering coefficients, we have already concluded that the two-pole approximation to $\eta(\ell)$ given by Eq. (6) was mainly of mathematical nature and did not seem to have very much physical reality. It is interesting to observe in Fig. 9 that the general two-pole model, successful in the description of the cross-sections, leads to scattering coefficients which again bear a rough resemblance to those of the smooth cut-off model.

Thus, in the two cases considered here, the scattering coefficients $\eta(\ell)$ given by the general two-pole model have roughly the shape we expect them to have from physical considerations at the surface of the target. In the case of a finite number of poles, we evidently expect the background effects to correct $\eta(\ell)$ inside the nucleus. On the other hand, the poles situated further from the real axis, which can be

neglected for the cross-section, should have a more important effect on the surface part. Anyhow, after the paradox discussed in the absence of Coulomb forces, and illustrated in Figs. 7 and 9, where a correct description of the cross-sections could be obtained from quite unphysical phase shifts, we feel that the general two-pole model has brought us back rather close to a certain physical reality.

This discussion emphasizes also the role played by the whole scattering coefficient $\eta(\ell)$, with Coulomb and nuclear effects included. Indeed, from the examination of Figs. 8 and 9, a separation of these two effects does not appear to be recommended in the Regge approach. In this picture, the oscillatory character of the Coulomb phase shift is lost, its contribution being frozen at its value at the poles.

VII. CONCLUSIONS

The main object of this paper was to show the convenience of the Regge representation for parametrizing differential cross-sections. The parameters to be determined are the position and the residue of the poles, and this can be done easily by using approximation formulae, and in a rather unique way. The existence of a background integral (the estimation of which is difficult) does not disturb these results very much and this is encouraging.

When a model defined by Regge poles can be related to other considerations, as for example the resonance or the diffraction models, the Regge pole analysis can provide a direct understanding of the physical phenomena involved. Therefore it may be hoped that other scattering processes will be analyzed in this way. In particular, for those showing backward diffraction patterns, it will be of great interest to know if one can find independent Regge parametrizations of the forward and backward patterns which could lead to the observed incoherent medium phenomena (see Fig. 4).

Although we do not expect the Regge representation to be accurate in the determination of the scattering coefficients, we have seen that Coulomb effects play an important role in bringing these quantities into a more physical form. Therefore, in the case where few poles are present, one can hope that the Regge representation can give a not too deformed idea for $\eta(\ell)$, at least at the surface of the target. Anyhow it does not look advisable to use the unitarity condition in the determination of the parameters. However, if this condition turns out to be approximately fulfilled in most other cases, as has happened for the processes considered here, it would be a great encouragement for believing in the physical reality of these Regge poles.

The diffraction picture is related to strong absorption effects. In their extreme form, these effects show up in the pure imaginary character of the phase shift. The scattering coefficients are, in these conditions, real and it can easily be seen that another pole situated at

ℓ_0^* corresponds to a pole in the complex ℓ plane situated at ℓ_0 . We expect this result not to be completely destroyed by the introduction of a small real part in the phase shift; therefore it is not surprising that we find the diffraction pattern expressed by means of two poles on each side of the real axis.

The existence of a pole in the fourth quadrant raises very important questions in terms of potentials, because there exists a theorem ¹⁶⁾ which forbids poles in this quadrant for the most regular potentials, and in particular for Yukawian potentials (sums of Yukawa potentials). This theorem has been demonstrated only in the case of real potentials, but we show, in the Appendix, how it can be extended to the case of complex potentials.

The conclusion is that, for regular potentials (see in the Appendix for a more precise definition) which satisfy the conditions for the validity of the Regge representation, the occurrence of a Regge pole in the fourth quadrant implies that the imaginary part of the complex potential should change sign. This is not a priori void of any physical meaning. One could imagine optical potentials which are strongly absorptive in the interior of the nucleus and emitting at the surface of the nucleus.

The difficulty is that potentials of the Wood-Saxon type ¹⁷⁾, which are quite successful in the description of the diffraction phenomena, have an imaginary part which is always negative. If the poles are going to have a physical reality, and are not only a useful mathematical trick, and if we want potentials with a negative imaginary part, we must then drop the condition of regularity at the origin. This essentially means potentials more singular than the centrifugal term at the origin, and their physical meaning is a hard core type of phenomenon. The theory of the analytical properties in the complex ℓ plane of such potentials is not yet very advanced ¹⁸⁾, and in particular nothing has been done on complex potentials.

Again by comparing with ordinary Wood-Saxon potentials, we find that there is no need to include hard core phenomena to have results that fit the experiments. What happens in the case of these potentials

is that the analytical properties are not regular enough to permit the Sommerfeld-Watson transformation. It might well be that this is the case for general optical potentials. If we look what happens at high energy, we find similar oscillatory diffraction patterns for protons of 20 GeV/c laboratory momentum scattered on complex nuclei ¹⁹⁾, whereas, for the elastic scattering of elementary particles ²⁰⁾, the diffraction picture plays again an important role, with almost pure imaginary phase shifts and even a backward peak. At these energies, however, we expect the poles in the right half-plane to vanish for regular potentials ¹⁶⁾. Moreover, in several models at high energies ²⁰⁾, the scattering coefficients show a Gaussian behaviour, which forbids a Sommerfeld-Watson transformation.

If the physical basic reality is the same at high and low energy, this would suggest that optical potentials must be so as to forbid a Regge representation. This would thus reject Yukawian potentials. It is quite difficult to understand why this type of potential, which, it seems, could easily reproduce the main physical properties of the Wood-Saxon type of potential, would fail in the description of diffraction patterns.

Another way to direct the discussion is to suppose that the pole in the fourth quadrant does not really exist, and that its contribution represents part of the background integral. If this were the case, it will be quite encouraging that the background term can be approximated by a pole contribution. Because of the symmetry properties of the Legendre functions with respect to the origin of the λ plane (where $\lambda = \ell + \frac{1}{2}$), a pole at λ_0 in the fourth quadrant gives the same contribution as a pole in the second quadrant situated at $-\lambda_0$. Then, it is clear that a way to get a pole in the second quadrant, as part of the background integral, will be to shift the contour of integration further to the left. This means that the $\eta(\ell)$ could be analytically continued into the left plane. Such analytic continuation has been considered mostly in the case of singular potentials, which brings us back to a previous conclusion.

We are very aware that the discussion is somewhat inconclusive. We feel that, because of the very strong implications of analytical properties, diffraction patterns could be related to precise space properties of the optical potential. The present discussion suggests a number of possible, though sometimes contradictory, properties, but no choice between them can be made before more work is done, of mathematical as well as phenomenological character.

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A P P E N D I X

We follow here the demonstration given by de Alfaro and Regge ¹⁶⁾, on the position of the poles in the ℓ complex plane, in the case of real and continuous potentials, which are more regular at the origin than the centrifugal term, and satisfy

$$\left\{ \begin{array}{l} \int_c^\infty |V(x)| dx = M(c) < \infty \\ \int_0^{c'} x |V(x)| dx = N(c') < \infty \end{array} \right. \quad (c, c') > 0 \quad (\text{A.1})$$

Under these conditions, the partial wave functions are analytical functions of λ in the right half-plane (here $\lambda = \ell + \frac{1}{2}$). Taken at a fixed positive energy k^2 , the wave functions $\varphi(\lambda, k, x)$, where x is the radial variable, determine, for $x \rightarrow \infty$, the Jost functions $f(\lambda, \pm k)$:

$$\varphi(\lambda, x) \underset{x \rightarrow \infty}{\sim} \frac{1}{2ik} \left[f(\lambda, k) e^{ikx} - f(\lambda, -k) e^{-ikx} \right] \quad (\text{A.2})$$

The Jost functions are related to the S matrix [identical here to the scattering coefficients $\eta(\ell)$], and to the phase shifts $\delta(\lambda, k)$ by :

$$S(\lambda, k) = e^{2i\delta(\lambda, k)} = \frac{f(\lambda, k)}{f(\lambda, -k)} e^{i\pi(\lambda - \frac{1}{2})} \quad (\text{A.3})$$

This allows the following parametrization of the Jost functions :

$$\begin{aligned} f(\lambda, k) &= \tau(\lambda, k) \exp. i \left\{ \delta(\lambda, k) - \frac{\pi}{2} \left(\lambda - \frac{1}{2} \right) \right\} \\ f(\lambda, -k) &= \tau(\lambda, k) \exp(-i) \left\{ \delta(\lambda, k) - \frac{\pi}{2} \left(\lambda - \frac{1}{2} \right) \right\} \end{aligned} \quad (\text{A.4})$$

We now write the Schrödinger equation and its complex conjugate :

$$\begin{cases} \varphi'' + k^2 \varphi - \frac{\lambda^2 - \frac{1}{4}}{x^2} \varphi - V \varphi = 0 \\ \varphi^{*''} + k^2 \varphi^* - \frac{\lambda^{*2} - \frac{1}{4}}{x^2} \varphi^* - V^* \varphi^* = 0 \end{cases}$$

We then multiply the first equation by φ^* , the second by φ , and subtract. We get

$$\frac{d}{dx} (\varphi^* \varphi' - \varphi'^* \varphi) - \frac{4i \operatorname{Re} \lambda \operatorname{Im} \lambda}{x^2} - 2i \operatorname{Im} V |\varphi|^2 = 0 \quad (\text{A.5})$$

Expressed for the partial waves, this is just the ordinary equation for flux conservation.

We now integrate (A.5) from 0 to ∞ , and use the condition of regularity on $\varphi(\lambda, k, x)$ which makes it vanish at the origin. For $x \rightarrow \infty$ on the other hand, we use the asymptotic form (A.2). We obtain :

$$\int_0^\infty \left(\frac{2 \operatorname{Re} \lambda \operatorname{Im} \lambda}{x^2} + \operatorname{Im} V \right) |\varphi|^2 dx = -\frac{1}{2k} |\tau(\lambda, k)|^2 \sinh 2 \left(\operatorname{Im} \delta - \frac{\pi}{2} \operatorname{Im} \lambda \right) \quad (\text{A.6})$$

Equation (A.6) is our starting point for the discussion. Consider first

$$\operatorname{Im} \lambda = 0$$

For physical values of λ , (A.6) reduces to

$$\int_0^\infty \operatorname{Im} V |\varphi|^2 dx = -\frac{1}{2k} |\tau(\lambda, k)|^2 \sinh 2 \operatorname{Im} \delta, \quad \lambda \text{ real}$$

The unitarity condition requires $\operatorname{Im} \delta > 0$, so that we have the condition on the potential :

$$\int_0^{\infty} \text{Im} V |\varphi|^2 dx < 0 \quad (\text{A.7})$$

for physical values of λ .

We next consider under which conditions we have a pole of the S matrix. Near a pole λ_0 of order n :

$$S \sim \frac{g}{(\lambda - \lambda_0)^n}, \quad S = e^{2i \text{Re} \delta - 2 \text{Im} \delta}$$

thus,

$$e^{-2 \text{Im} \delta} \sim \left| \frac{g}{(\lambda - \lambda_0)^n} \right|, \quad \text{Im} \delta \xrightarrow{\lambda \rightarrow \lambda_0} -\infty \quad (\text{A.8})$$

This corresponds obviously to an emission of particles, and cannot occur for physical values of ℓ .

Condition (A.8) implies that near a pole the right member of (A.6) is positive. A necessary condition to have a pole is therefore that the left member of (A.6) be positive, which reads:

$$2 \text{Re} \lambda \text{Im} \lambda \int_0^{\infty} \frac{|\varphi(\lambda, k, x)|^2}{x^2} dx > - \int_0^{\infty} \text{Im} V |\varphi(\lambda, k, x)|^2 dx \quad (\text{A.9})$$

If the potential is real, this condition reduces to

$$\text{Im} \lambda > 0 \quad (\text{A.10})$$

as we always restrict λ to the right half-plane. This is the well-known result that poles of the S matrix are situated in the first quadrant of the λ plane.

If the potential is such that condition (A.7) holds everywhere in the λ plane, we again arrive to the same result, with even more restrictions, as we have :

$$\operatorname{Im} \lambda > \frac{- \int_0^{\infty} \operatorname{Im} V |\varphi(\lambda, k, x)|^2 dx}{2 \operatorname{Re} \lambda \int_0^{\infty} \frac{|\varphi(\lambda, k, x)|^2}{x^2} dx} \quad (\text{A.11})$$

This is easily understood : a negative imaginary potential has an absorptive effect, and it pushes up the poles which correspond to emission, until the centrifugal term begins to dominate [see also Eq. (A.5)].

If, however, the imaginary part of the potential changes sign, so that Eq. (A.7) is valid for physical values of λ , but not everywhere in the λ plane, and if the right member of (A.9) becomes negative in some cases, then we might have poles below the real axis. We however know from (A.11) that they cannot lie too low down.

The conclusion is that, for potentials more regular at the origin than x^{-2} , and satisfying (A.1) and (A.7), a necessary condition to obtain poles in the fourth quadrant is that the imaginary part of the potential changes sign.

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FIGURE CAPTIONS

Figure 1 - The contour integration in the ℓ complex plane and the two first poles.

Figure 2 - The dominant pole in the smooth cut-off model.

Figure 3 - The differential cross-section from one pole, multiplied by $\sin\theta$. The dashed line corresponds to approximation (21).

Figure 4 - The experimental elastic differential cross-section for α particles on ^{40}Ca at 30.5 MeV, multiplied by $\sin\theta$ (data from Ref. ¹²).

Figure 5 - The ratio of the elastic differential cross-section to the Rutherford cross-section for α particles on ^{40}A at 18 MeV. The points are the experimental results (from Ref. ¹³). The curves are the theoretical results.

a) for the smooth cut-off model, with $\ell_0 = 7.5$; $\ell_1 = 0.35$; $a = 0.5$. The dashed lines represent the contribution from two and four poles.

b) for the general two pole model with $L_1 = 8.0 + i 1.42$; $r_1 = 0.33$; $L_2 = 8.0 - i 1.92$; $r_2 = -1.3 + i 1.21$. The dashed lines represent the separate contributions from each pole.

Figure 6 - The elastic differential cross-section for α particles on ^{208}Pb at 44 MeV. The points are the experimental results (from Ref. ¹⁴). The curves are the theoretical results

a) for the smooth cut-off model, with $\ell_0 = 22$; $\ell_1 = 0.5$; $a = 0.86$.

b) for the general two pole model with $L_1 = 16 + i 1.8$; $r_1 = 0.067$; $L_2 = 16 - i 2.93$; $r_2 = 3.57$. The dashed lines represent the separate contributions from each pole.

Figure 7 - The scattering coefficients for α scattering on ^{40}A , without Coulomb effects

- a) the smooth cut-off model with the Alster-Conzett parametrization.
- b) the poles approximating the smooth cut-off model.

Figure 8 - The scattering coefficients for α scattering on ^{40}A , with Coulomb effects

- a) the smooth cut-off model compared with the Alster-Conzett parametrization and the general two pole model. (The arrow indicates the change of phase for the residues.)
- b) the poles approximating the smooth cut-off model.

Figure 9 - The scattering coefficients for α scattering on ^{208}Pb , with Coulomb effects, for the smooth cut-off model, its two-pole approximation, and the general two-pole model. (The arrow indicates the change of phase for the residues.)

FIG. 1

FIG. 2

FIG. 3

FIG. 4

FIG. 5a

FIG. 5b

FIG. 6a

FIG. 6b

FIG. 7

- $\eta_N(l) e^{2i\sigma(l)}$
- - - 2 pole approx.
- · - general 2pole model
- · · · unitary circle

FIG. 9