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RELATIVISTIC TREATMENT OF STRUCTURE IN THE COULOMB
INTERFERENCE PROBLEM

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A B S T R A C T

The effect of form factors on the Coulomb phase angle in elastic high energy scattering of charged particles is discussed. The influence of the angular dependence of the strong interaction part on the phase is included. Essentially the non-relativistic result of Bethe turns out to be correct.

1. INTRODUCTION

Interference with Coulomb scattering allows the determination of the real part of the strong interaction amplitude in the elastic scattering of charged particles (πp , pp , $\bar{p}p$) near forward direction. Denoting by f the non-flip part of πN scattering the differential cross-section reads (small changes are necessary for pp scattering)

$$\frac{d\sigma}{dt} = \left| \frac{-M}{|t|} e^{2i\delta} + f_{im}(i+\beta) \right|^2 \quad (1)$$

$$M = 2\sqrt{\pi} n, \quad n = \pm \frac{\alpha}{v_L}, \quad \alpha \approx 137^{-1}, \quad \beta = f_{re}/f_{im}$$

(the plus sign in the abbreviation n holds for equal charge scattering, v_L is the laboratory velocity of the incident particle.) The imaginary part f_{im} is known from total cross-section measurements. Therefore, Eq. (1) allows ^{*)} to determine β .

For the Coulomb phase difference δ a formula derived by Bethe ¹⁾ in the framework of potential scattering is

$$\delta_B = n \ln \frac{1.06}{Pa\theta} \quad (2)$$

P = CM momentum
 a = nucleon radius
 θ = CM angle

^{*)} Some experimental analyses multiply M by the product of the form factor of π and p . Because of the extreme smallness of the (over-all) momentum transfer t the effect of structure is practically negligible here.

2.

whereas δ has been given as

$$\delta_s = n \ln \frac{2}{\Theta} \quad (3)$$

by Solov'ev ²⁾ using the methods of point quantum electrodynamics, which are apparently quite different. Both formulae have been used in the recent analysis of Ref. ³⁾. The difference between them gives a 10% theoretical uncertainty in the ratio β .

2. THE SOLOV'EV FORMULA

In any stage of approximation the phase δ is the difference between the phase δ_c of pure Coulomb scattering and some suitable average of this phase over strong interaction, called $\langle \delta_c \rangle_n$ or simply δ_{cn} . We give the values of δ_c and δ_{cn} in the Solov'ev approximation. Using second order perturbation theory the result of Solov'ev arises from the graphs in Fig. 1 (add topological variations). Retaining only the most singular terms at small angles, the amplitude A_c from the graphs 1a and 1b is

$$A_c = \frac{-M}{|t|} \left(1 - i n \ln \frac{P^2 \Theta^2}{\lambda^2} \right) \quad \begin{array}{l} |t| \approx P\Theta \\ \lambda = \text{photon mass} \end{array} \quad (4)$$

Regarding the bracket as the expansion of a phase factor $\exp 2i\delta_c$ one identifies the pure Coulomb phase

$$\delta_c = -n \ln \frac{P\Theta}{\lambda} \quad (5)$$

The leading (logarithmic) contributions to the amplitude A_{cn} corresponding to the graphs 2a and 2b for $t=0$ are

$$A_{cn} = f \left(1 - i n \ln \frac{4P^2}{\lambda^2} \right) \quad (6)$$

The bracket is regarded as the expansion of $\exp 2i\delta_{cn}$, with

$$\delta_{cn} = -n \ln \frac{2P}{\lambda} \quad (7)$$

The net result $A_c + A_{cn}$ can be written in the form (1) with $f = f_{im} \times (1+i\beta)$ and with δ given by

$$\delta_s = \delta_c - \delta_{cn} = n \ln \frac{2}{\theta}$$

3. INCLUSION OF ELECTROMAGNETIC FORM FACTORS

We now discuss the influence of form factors that are inserted at every photon vertex in Fig. 1.

First we note that at near forward direction we can safely neglect form factors in the pure Coulomb amplitude A_c of Eq. (4). A plausibility argument is the following: the appearance of the over-all momentum transfer t in the imaginary part of A_c suggests multiplication by a form factor $G(t)$, which is close to unity for t small. A better argument is this: in impact parameter description forward Coulomb scattering arises almost exclusively from the contributions of large impact parameters, i.e., from large distances (as compared to the charge distribution radius) where the charge distribution acts as a point charge.

For the graph 2b the situation is quite different. The strong interaction amplitude is restricted to low impact parameters, b , because of its short range nature. In fact, in Eq. (6) the amplitude f has been assumed to be independent of angle. Averaging has been reduced to the $b=0$ component of strong interactions.

Precisely in this type of approximation the sum of all contributions of diagrams of the form 2b (we call it the amplitude M_{cn}) has already been discussed for point charges by the authors of Ref. 4) in connection with infra-red divergences. At the electromagnetic vertices of 2b only the convection current contribution B , defined as the part multiplying the electric form factor G_E 5), has been retained *). The amplitude M_{cn} can be factorized $M_{cn} = fB$. By this the amplitude A_{cn} of the graphs 2a and 2b is **) $A_{cn} = f + fB \approx f(1 + i \text{Im} B)$. The evaluation of $\text{Im} B$ in Ref. 4) with $f = \text{const}$ and in point charge approximation leads to Eq. (6). To see this it suffices to consider the typical contribution to the imaginary part of B arising from the graph in Fig. 2:

$$\text{Im} B \sim \alpha p \cdot q \int \frac{d^4 k}{k^2 - \lambda^2} \frac{\Theta(k^0)}{k^2 - \lambda^2} \delta[(p-k)^2 - m^2] \delta[(q+k)^2 - \mu^2] G(k^2) \equiv I \quad (8)$$

*) The corresponding contribution to M_{cn} arising from magnetic interaction is probably small. It does not factorize, because of the spin flip induced by the magnetic operator $[k, \gamma^{\mu}]$. Therefore, the spin flip part of strong interactions enters the calculation too. However, the magnetic contribution is zero for $K^2 = 0$ and strongly suppressed by the product of "strong" and electromagnetic form factors for large K^2 (cf., Section 4).

**) The real part of B , with simultaneous inclusion of real photon emission, leads to radiative corrections to f . These are not treated here. (Cf., the remark at the end of Section 4).

where now we have introduced a form factor $G(k^2)$ which is to be considered as the product of the electric form factor of the proton and of the pion. The δ functions can easily be evaluated in the CM system leading to

$$I = \frac{-\alpha\pi}{2P(p^0 + q^0)} \int_0^{2P} \frac{G(K^2)K dK}{K^2 + \lambda^2} \quad (9)$$

$$P = |\vec{p}| = |\vec{q}|, \quad K = |\vec{k}|$$

Evidently, for $G(K^2) \equiv 1$ we have

$$I \propto -\ln \frac{2P}{\lambda} \equiv I' \quad (10)$$

which is Solov'ev's result, Eq. (6). Let us neglect temporarily the structure of the pion (which corresponds to the treatment of Bethe). G is therefore equal to G_E , the electric proton form factor. For this we shall discuss several analytic forms.

For the exponential

$$G(x) = e^{-\gamma x}, \quad x = -k^2 = K^2, \quad \gamma = \frac{R^2}{6} \quad (11)$$

with r.m.s. radius R the relevant integral I' is

$$I' = -\int_0^{2P} \frac{G dK}{K^2 + \lambda^2} = -\frac{1}{2} \int_0^{4P^2} \frac{e^{-\gamma x} dx}{x + \lambda^2} = -\frac{1}{2} e^{\gamma \lambda^2} \text{Ei}[-\gamma(x + \lambda^2)] \Big|_0^{4P^2} \quad (12)$$

where the integral exponential is defined as in Jahnke-Emde. For large arguments $Ei(-x) \sim \frac{e^{-x}}{x}$ which can be neglected. The contribution of the lower limit leads to

$$I' = \frac{1}{2} (C + \ln \gamma \lambda^2) \quad C = .577 \quad \text{Euler constant} \quad (13)$$

which is to be compared with $-\ln \frac{2P}{\lambda}$. Therefore, we rewrite I' as

$$\begin{aligned} I' &= \frac{1}{2} (C - \ln \frac{4P^2}{\lambda^2} + \ln 4P^2 \gamma) \\ &= \frac{C}{2} - \ln \frac{2P}{\lambda} + \ln \sqrt{\frac{2}{3}} RP \end{aligned} \quad (13')$$

Subtracting this result for the averaged phase: $\delta_{cn} = nI'$ from the pure Coulomb phase shift (5), the phase difference δ of Eq. (1) becomes

$$\delta = n \left(\ln \frac{2}{\theta} - \ln \sqrt{\frac{2}{3}} RP - \frac{C}{2} \right) \quad (14)$$

This resembles very much the Bethe formula (2). In fact, by doing the comparison properly, Bethe's parameter a turns out to be

$$a = \sqrt{\frac{2}{3}} R$$

In addition we must compare (14) with that formula for δ that Bethe would get before averaging over the impact parameters of strong interactions. This formula is [cf., (4.39) and (4.44) of Ref. ¹⁾]

$$\delta_B = n \left(\ln \frac{2}{\theta} - C \right) - n \left(\ln Pa - \frac{C}{2} \right)$$

which is exactly ^{*)} our formula (14).

The effect of averaging over strong interactions is small in the Bethe treatment: $-\frac{C}{2}$ in the second term of the last equation is to be replaced by 0.053. (The term $\ln Pa$ is large, for $a=1F$ and $p_{lab}=10$ GeV/c we have $\ln Pa \approx 2.4$.) For the analogous problem of estimating the influence of the angular dependence of f in our treatment, see Section 4.

Let us state the results for two other simple models of form factors:

For the single pole type

$$G(x) = \frac{m_0^2}{m_0^2 + x}, \quad m_0^{-2} = \frac{R^2}{6} \quad (15)$$

the result is

$$\delta = n \left(\ln \frac{2}{\theta} - \ln \sqrt{\frac{2}{3}} RP \right) \quad (16)$$

which differs from (14) only by the small constant $-\frac{C}{2}$.

For a single pole squared type (dipole)

$$G(x) = \frac{m_0^4}{(m_0^2 + x)^2}, \quad m_0^{-2} = \frac{R^2}{12} \quad (17)$$

*) Alternatively, the Solov'ev formula is obtained in Bethe's treatment if point charge phases are used everywhere:

$$\delta = n \left(\ln \frac{2}{\theta} - C \right) - (-nC) = n \ln \frac{2}{\theta}$$

we have

$$\delta = n \left(\ln \frac{2}{\theta} - \ln \left| \frac{2}{3} R_P - \frac{1}{2} - \ln \sqrt{2} \right| \right) \quad (18)$$

4. AVERAGE OVER STRONG INTERACTION

Let us now improve the averaging over angles in the strong interaction part of Fig. 2. The momentum dependence of πN scattering for $|t'| \lesssim 1(\text{GeV}/c)^2$ is

$$(11) \quad f = f_0 e^{At'} \quad \text{with } A \sim 7.5 (\text{GeV}/c)^{-2} \quad (19)$$

in a very good approximation ⁶⁾. Noting that in the forward direction $t = (p'' - p)^2 = (k + k')^2 = 0$, the momentum transfer $t' = k'^2$ in the strong interaction part of the amplitude in Fig. 2 is equal to the momentum transfer k^2 in the Coulomb part (all particles on shell), we easily include the angular dependence (19) by generalizing

$$G(x) = e^{-Ax} G_E(x) G_\pi(x) \quad (20)$$

where we have simultaneously introduced the π charge form factor $G_\pi(x)$. Choosing the exponential model for both charge form factors we simply have to introduce an effective radius in Eq. (14)

$$R^2 = R_{\text{strong}}^2 + R_P^2 + R_\pi^2 \quad (21)$$

From Eq. (19) the strong interaction radius is $R_{\text{strong}} \approx 0.95F$. The charge radius of the proton from electron scattering ⁷⁾ is $R_p \approx 0.8F$. The pion charge radius is uncertain, but it seems reasonable to assume that it is not much larger than R_p . The important point, however, is to realize that the uncertainty as to what the correct analytic form of the form factors should be is equivalent to an uncertainty of about a factor of 2 in the effective radius [cf., Eqs. (18), (16) and (14)]. This uncertainty cannot be avoided ^{*}). However, from the analysis of the experiments ³⁾ one can hope that a correction of this order ($\ln 2$ v. $\ln PR$) in the phase would give an effect of at most a few percent in the ratio $\beta = f_{\text{re}}/f_{\text{im}}$. Indeed, the Solov'ev phase (3) is very different from the Bethe phase (2), for $p_{\text{lab}} = 10 \text{ GeV/c}$ and $a = 1F$ we have

$\theta_{\text{lab}} (\text{m rad})$	$\ln \frac{1.06}{Pa\theta} = \delta_{\text{E}/n}$	$\ln \frac{2}{\theta} = \delta_{\text{S}/n}$
1	3.05	6.1
10	0.75	3.8
20	0.06	3.1

This difference affects β only by 10%.

The comparison ³⁾ of β with the value β' obtained by using forward dispersion relations slightly favours the Solov'ev phase. If this is not due to systematic errors one should perhaps look for the influence of radiative corrections on the Coulomb and nuclear amplitudes. Their estimation is unfortunately a difficult task, especially for the latter.

^{*}) Furthermore, we must not forget that there is some arbitrariness in the clean separation of strong and electromagnetic interaction as it is drawn in Fig. 2 and consequently some principal uncertainty in the definition of the charge radii.

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R E F E R E N C E S

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FIG.1

FIG.2