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UNIQUENESS AND SYMMETRY BREAKING IN S MATRIX THEORY

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A B S T R A C T

The hypothesis of dynamical generation of symmetries can be put to experimental test, within the framework of partial wave dispersion relations, and fails. This is a summary of a long rigorous paper using non-linear functional analysis ¹⁾.

The equations of S matrix theory are non-linear integral equations. Most mathematical work on them has assumed the left-hand cut to be fixed. They can then be reduced to linear N/D equations which are amenable to classical analysis ²⁾. Because crossing symmetry is not included in this approach, important questions regarding the uniqueness of bootstraps and dynamical generation of symmetries cannot be answered.

We have shown ¹⁾ that non-linear functional analysis can be applied to partial wave dispersion relations directly, without assuming any N/D decomposition, and including crossing symmetry. Within this framework in which the number of waves is assumed finite, we can answer rigorously the uniqueness and dynamical symmetry questions. Here we shall state the conclusion, and also outline the general principles of the mathematical techniques, since we think they are of considerable importance for the future of S matrix theory.

Suppose we are given a solution of a set of non-linear integral equations like those of S matrix theory. Can we tell by looking at the solution whether it is unique or not? If S matrix theory is correct at all, then the observed universe must be a solution of the S matrix equations. Therefore, if we can answer this question, we can subject the bootstrap and dynamical symmetry hypotheses to experimental tests.

The mathematical answer is as follows : we classify the solutions of the non-linear integral equation into manifolds of various dimensions. The dimension will be the number of arbitrary parameters on which this class of solution depends. There may be several such manifolds, since not all solutions can necessarily be continuously deformed into each other. In the vicinity of any given solution, we approximate the non-linear equation by a linear one. This determines a linear operator called the Fréchet differential. It can be constructed from the given solution. Belonging to the Fréchet differential is an

integer called its index, which can also be calculated from the given solution. If the index is positive, then it is the dimension of the manifold to which the given solution belongs. If the index is negative or zero, then the given solution is at least locally unique. We emphasize that though the index is calculated from the linear approximation, its relation to the uniqueness question does not depend on the linear approximation, but is true of the exact equation.

The Fréchet differential is constructed by perturbing the given solution slightly, and then deriving a linear equation for the first order perturbation. This is obviously very closely related to the Dashen-Frautschi perturbation theory ³⁾. Our results also throw light on the conditions under which the latter is valid. In practice, Dashen and Frautschi always assume that the perturbation can be adequately represented by changes in the masses and coupling constants. They then get linear algebraic equations for these, whose kernel they call the A matrix. Under this approximation, $1 - A$ will be the Fréchet differential. The first order perturbation resulting from a given driving term is obtained by inverting the Fréchet differential. Spontaneous symmetry breaking corresponds to the Fréchet differential having a zero eigenvalue. The inhomogeneous equation

$$(1 - A)x = y \quad (1)$$

will then have no finite solution unless the driving term y is orthogonal to this eigenvector. In the linear approximation, the homogeneous equation

$$(1 - A)x = 0 \quad (2)$$

has a continuum of solutions - any multiplet of the eigenvector. It has been argued ⁴⁾ from an examination of non-linear algebraic equations for mass shifts that this continuum will be reduced when higher order perturbations are considered, and that the exact equation will have a

discrete perturbed solution, a little way away from the unperturbed one. This indeed is true, but it is not the whole truth. For the original equation was not algebraic, but a singular integral equation, and singular integral equations have a peculiar property - the homogeneous and inhomogeneous equations can have solutions at the same time. If this happens, then the continuum is not eliminated by higher order perturbations, but goes through into the exact solution.

Let us call α the number of linearly independent solutions of the homogeneous equation (2), and β the number of linearly independent driving terms for which the inhomogeneous equation (1) blows up. Then the index of the Fréchet differential is $\kappa = \alpha - \beta$. The Fredholm alternative says that the index is zero if A is a compact operator. More generally, a Φ -operator is one for which the two numbers α and β are finite but not necessarily equal, and which satisfies certain other technical requirements. There is a quite extensive theory of such operators ⁵⁾. A special case will be familiar to readers of Muskhelishvili's book ⁶⁾. Their relevance to the uniqueness problem arises from an implicit function theorem for non-linear equations $f(x) = 0$ in Banach space, whose Fréchet differentials $f'(x)$ are Φ -operators. This is due to Vainberg and Trenogin ⁷⁾ and says essentially that in some neighbourhood of the given solution x , the equation can be exactly reduced to β scalar equations in α real scalar variables, where α and β are the numbers defined above belonging to the Fréchet differential $f'(x)$. Thus, after the integral equation has been reduced to algebraic ones, the index κ gives the difference between the number of variables and the number of equations left to determine them. Apart from accidental degeneracies, which can be shown unlikely, it will give the dimension of the manifold of solutions to which the given one belongs. Stable bootstraps correspond to $\alpha = \beta = 0$. For spontaneous symmetry breaking of the type considered by previous authors ⁴⁾ we have $\alpha = \beta = 1$ giving one equation in one variable, so we can expect a discrete solution. However, for $\alpha \neq \beta$ funny things happen, as we shall see.

The next step is to use the stability theorem for $\bar{\Phi}$ -operators ⁵⁾. This says that if T is a $\bar{\Phi}$ -operator, and U is a compact operator, then $T + U$ is also a $\bar{\Phi}$ -operator with the same index as T . (A compact operator is the sort occurring in the kernels of non-singular integral equations, and possessing all the nice properties expected from finite-dimensional analogies). In Ref. ¹⁾ we prove that the left-hand cut contribution to the Fréchet differential is compact, so that it can be dropped from the calculation of the index. It is easy to show that the contribution of any cut which does not touch the physical region will be compact - the difficulty in the proof comes from the point at infinity, where the left and right-hand cuts touch. For this also to be compact, we require (a) the crossing relation for the imaginary part to be bounded, and (b) the perturbation to die out at infinity. The Chew-Mandelstam equations are obviously inconsistent without a left-hand cut-off if particles of spin greater than 1 are exchanged. A more subtle analysis ⁸⁾ shows that they require a cut-off for vector meson exchange also. It turns out that the minimal left-hand cut-off needed to make the equations consistent, will also give the required boundedness of the crossing operator. To be generous to bootstraps, we suppose this left-hand cut-off to be given - it is not included in the count of arbitrary parameters. The requirement that the perturbation die out at infinity is met by supposing the real part to vanish there, and assuming the contribution of many-particle states to the unitarity equation to be also given and not perturbed. This is the most obvious point that would need generalizing in the strip approximation. We note that these conditions are already general enough to include practically all bootstrap models.

Thus the index of the Fréchet differential will be unchanged if the left-hand cut contribution is omitted. (Whether we also omit bound-state poles makes no difference.) The resulting equation is then simple enough to be reduced to a Hilbert problem and solved by the techniques of Muskhelishvili's book ⁶⁾. Let us label the independent partial waves by j , with N_j being the number of coupled channels for each, and n_B^j the number of bound states. Then the index is, in terms of the experimental S matrix,

$$\kappa = \sum_j (N_j + 1) n_c^j \quad (3)$$

where

$$n_c^j = n_B^j + \left[\arg \det \| S_{\mu\nu}^j(s) \| \right]_{s=s_1}^{s=\infty} \quad (4)$$

s_1 being the lowest right-hand threshold. (This makes certain assumptions about asymptotic behaviour, for which see Ref. ¹). This formula does not yet include the external mass perturbations and centrifugal sum rules, because these are not continuously differentiable. The external mass perturbations will not affect the uniqueness either way (apart from accidental degeneracies), because they involve only a finite number of parameters and give an equal number of equations to determine them if the system is closed. The centrifugal sum rules arise from the requirement that higher partial waves vanish at threshold. They will give a finite number of additional conditions, modifying the index to

$$\bar{\kappa} = \kappa - \sum_j \sum_{\mu=1}^{N_j} \left[\frac{1}{2} (N_j + 1) \ell_{\mu}(j) \right] \quad (5)$$

where $\ell_{\mu}(j)$ is the orbital angular momentum in the μ^{th} channel of the j^{th} partial wave.

Unlike N/D results ²⁾, this includes crossing symmetry. However a comparison is enlightening. It shows that n_c^j as given by (4) is the number of CDD poles. Each contributes $N_j + 1$ arbitrary parameters - the mass of the corresponding elementary particle, and its coupling to each of the N_j channels. The centrifugal barrier may fix some of them. Thus we have shown firstly that CDD ambiguities are not eliminated by crossing symmetry (this was previously known only in soluble models). However, and more important, the index has been shown

to be independent of the left-hand cut contribution and of the external mass terms. But these are just the two places where group-theoretical factors enter the Dashen-Frautschi equation ³⁾. If only the right-hand cut were taken into account, then all different modes of symmetry breaking (corresponding to spurions in different representations) would look alike. Therefore, if there are CDD ambiguities in the unperturbed equation, they will get much worse in the perturbed one, for there will then be separate CDD parameters for every mode of symmetry breaking. Thus, in the positive index case, there is no dynamical symmetry and we are back in the arbitrary situation of Lagrangian field theory.

The precise mechanism by which this happens in the Dashen-Frautschi approach is as follows : their equation involves multiplication by $D^2(s)$. This must have no poles or the Dashen-Frautschi equation will also acquire them (with undetermined residues). It can then be shown, in the one-channel case, that its asymptotic behaviour will be $D^2(s) \sim s^{2n}$, where n is the experimental number by which Levinson's theorem is violated [Eq. (4)]. This does not involve any assumption as to whether "elementary" particles are present. Therefore, the Dashen-Frautschi equation will require $2n$ subtractions, with arbitrary subtraction constants differing for each mode of symmetry breaking. There is a similar but more complicated many-channel result. A statement in one of the Dashen-Frautschi papers that CDD poles would not affect their argument is wrong.

Thank goodness, the reader will say at this point, that received orthodox opinion holds CDD poles to be theologically impossible. However, let us imitate Galileo and put this dogma to experimental test. This can be done in the case of the $N-N^*$ reciprocal bootstrap, where there are several πp phase-shift analyses extending up to high energies ⁹⁾, which can be used to evaluate Eq. (3). They all agree that $\kappa \geq 4$, so that, even allowing for the centrifugal barrier in the resonant waves, we will have at least two arbitrary parameters. The evidence that $\delta(P_{33})$ goes to 180° at high energies is particularly clear and unanimous, implying a CDD pole even in this the best understood of all resonances ¹⁰⁾. Therefore, according to our uniqueness theorem, the $SU(2)$ symmetry is not sustained dynamically by S matrix

theory. In particular, it is impossible to get a unique result for the n-p mass difference without additional assumptions. The Dashen calculation, as has been pointed out by Shaw and Wong¹¹⁾, did not use the experimental phase shifts at all.

The evidence for CDD poles has also been noticed recently by Atkinson and Halpern¹²⁾, who suggest that they might be due to the coupled channels required by SU(6). In this case, Dashen-Frautschi perturbation theory would be valid for SU(6) breaking, but not for its SU(2) subset alone. It should soon be possible to test this experimentally, using the coupled channel form of (4). However, an argument against this explanation is that it would not stop Regge trajectories from turning back. This is very closely connected to Levinson's theorem¹³⁾ and the experimental situation is very similar - in potential scattering Regge trajectories turn back quickly, just as $\delta(P_{33})$ does in the Chew-Low model, but experimentally they go up and up.

It is more plausible that both effects may arise from many-particle contributions to the Mandelstam representation. However, we can only escape along this path (or by assuming the $\mathcal{L}$ terms in (5) eventually counterbalance the CDD ambiguities), if we jettison all the existing bootstrap calculations claimed as evidence for dynamical symmetries, since this is plainly not how they were performed.

There are two obvious subjects for further investigation : (a) to generalize the index formula to the strip approximation. This does not look all that hard - the non-linear functional analysis techniques are extremely powerful, and they may even make possible an S matrix theory of many-particle states ; (b) to find out what supplementary conditions, when added to S matrix theory, will give the correct answer for the n-p mass difference and other symmetry-breaking effects.

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