CM-P00057329

GEOMETRICAL RESTRICTIONS ON THE PION-PION PARTIAL WAVES

André Martin

CERN - Geneva

A B S T R A C T

Various rigorous inequalities on pion-pion partial waves in the interval $0 \leq s \leq 4$ are obtained from crossing symmetry and positivity properties due to unitarity. In particular, the position of the minimum of the $\pi_0 \pi_0$ S wave amplitude is located between close limits. The threshold behaviour of higher waves is established, with the help of some additional assumptions. The positivity of the $T=0$ $\ell \geq 2$ partial waves is used to prove the existence of an essential singularity at $s = 0$ on the second sheet, or more precisely, that poles accumulate at $s = 0$.

I. INTRODUCTION

If we look back at the progress made in understanding the pion-pion problem, we see that we are still on a rather uncertain ground. The self-consistency requirements on individual partial waves do not lead to unique solutions for the P wave. Furthermore, solutions which exhibit the ρ resonance do not necessarily give rise to the same S waves. In particular the sign of the S waves at threshold is unknown. We do not know, so far, whether full use of unitarity, crossing symmetry, and unitarity will yield a unique solution. In fact, my own opinion is that this is not the case. However, it is clear that full use of crossing symmetry, once unitarity of the partial waves is satisfied, would certainly reduce considerably the ambiguities. Let us take a very simple example : consider the $\pi_0 \pi_0 \rightarrow \pi_0 \pi_0$ amplitude. Then the forward scattering amplitude can be written as

$$\sum (2\ell+1) f_\ell^{00}(s)$$

Now, wherever this expansion converges, we have to satisfy the identity :

$$\sum (2\ell+1) f_\ell^{00}(s) \equiv \sum (2\ell+1) f_\ell^{00}(4-s)$$

So if we perform for each partial wave a N/D calculation, we see that the CDD poles we introduce in each partial wave are tied together. However, these a posteriori restrictions are not very convenient because they can be used only after the partial wave calculation has been done and since anyway the number of partial waves taken into account is limited, one can always argue on how much consistency should be required. We think that it might be more useful to derive restrictions from crossing symmetry on individual partial waves and this is what we shall try to do here in a very unsystematic way.

As analyticity assumption we do not need Mandelstam representation but just the results which have been shown recently to follow from axiomatic field theory ¹⁾. In fact, we do not see at the moment how to use the Mandelstam representation to derive conditions which would be independent of the over-all number of subtractions, while, fortunately, the maximum number of subtractions in the axiomatic domain is known ²⁾.

Section II is devoted to the S wave in $\pi_0 n_0 \rightarrow \pi_0 \pi_0$ scattering. Section III deals with the problem of threshold behaviour of higher waves. In Section IV we present some inequalities on the higher waves. Section V contains an explicit and rigorous proof of the existence of essential singularities on the second sheet. In Section VI, the possible use of these restrictions is discussed.

II. RESTRICTIONS ON THE $\pi_0 \pi_0 \rightarrow \pi_0 \pi_0$ S WAVE

We define the $\pi_0 \pi_0 \rightarrow \pi_0 \pi_0$ S wave

$$f_0^{00}(s) = \frac{\sqrt{s}}{2k} \left[\frac{1}{3} e^{i\delta_0^0} \sin \delta_0^0 + \frac{2}{3} e^{i\delta_0^2} \sin \delta_0^2 \right]$$

where $k = \sqrt{(s-4)/4}$ and δ_0^0 and δ_0^2 are the S wave phase shifts for $T=0$ and $T=2$. We shall obtain a number of restrictions on the S wave behaviour for $0 \leq s \leq 4$, i.e., in the interval between the two cuts. We list here the results:

$$f_0^{00}(s) < f_0^{00}(4), \quad 0 \leq s < 4 \quad (1)$$

$$\frac{df_0^{00}(s)}{ds} > 0, \quad 2 \leq s \leq 4 \quad (2)$$

$$f_0^{00}(0) \geq \frac{1}{2} \int_2^4 f_0^{00}(s) ds \quad (3)$$

$$f_0^{00}(0) \geq f_0^{00}\left(2\left[1 + \frac{1}{\sqrt{3}}\right]\right) \simeq f_0^{00}(3.1) \quad (4)$$

$$\frac{df_0^{00}(s)}{ds} < 0, \quad 0 \leq s \leq 1.29 \quad (5)$$

$$\frac{df_0^{00}(s)}{ds} > 0, \quad 1.7 \leq s \leq 1.76 \quad (6)$$

The results have been classified by order of increasing complexity of the proof. Conditions (1) and (2) have been derived earlier by Jin and the author³⁾. Both are just consequences of the convexity of $F(s, t, u)$ for $0 < t$ fixed < 4 , in the interval $-t < s < 4$, and circular permutations. This convexity follows from :

- i) the validity of twice subtracted dispersion relations²⁾ for $|t| < 4$;
- ii) the positivity of the absorptive part $A_s(s, t)$ for $0 \leq t \leq 4$ which is a consequence of unitarity and of the convergence of the Legendre polynomial expansion.

Condition (3) is rather obvious. The S partial wave is defined as

$$\frac{1}{4-s} \int_0^{4-s} F^{00}(s, t, 4-s-t) dt$$

which, for $s=0$, reduces to

$$f_0^{00}(0) = \frac{1}{4} \int_0^4 F^{00}(0, t, 4-t) dt = \frac{1}{2} \int_2^4 F^{00}(0, t, 4-t) dt \quad (7)$$

if $t \rightleftharpoons u$ crossing symmetry is used. We can rewrite this as

$$f_0^{00}(0) = \frac{1}{2} \int_2^4 \left[f_0^{00}(s) + \sum_{\ell=2}^{\infty} (2\ell+1) f_{\ell}^{00}(s) \right] ds \quad (8)$$

Now for $\ell \geq 2$ the Froissart-Gribov representation

$$f_{\ell}^{00}(s) = \frac{2}{4-s} \int_4^{\infty} Q_{\ell}\left(\frac{2t}{4-s} - 1\right) A_t^{00}(s, t) dt \quad (9)$$

holds if $|s| < 4$ ²⁾, and due to the positivity of A_t , $f_{\ell}^{00}(s)$ is positive for $0 < s < 4$. So it follows that Eq. (8) can be turned into inequality (3).

Condition (4) is slightly more difficult to get. First we shall prove that

$$f_0(0) > F(s=0, t=2(1+\frac{1}{\sqrt{3}}), u=2(1-\frac{1}{\sqrt{3}})) \quad (10)$$

We shall change variables and consider the function

$$F(s, \cos\theta) \quad \cos\theta = 1 + \frac{2t}{s-4}$$

for $0 \leq s \leq 4$. This function is analytic in a cut plane with a positive discontinuity across the right-hand cut and is symmetric in $\cos\theta \rightarrow -\cos\theta$. For the partial waves with $\ell \geq 2$ the Froissart-Gribov representation holds :

$$F(s, \cos\theta) = f_0(s) + \frac{1}{\pi} \sum_{\ell \text{ even} \geq 2}^{\infty} (2\ell+1) P_{\ell}(\cos\theta) \int_{z_0(s)}^{\infty} A(s, z) Q_{\ell}(z) dz \quad (11)$$

Using the Darboux-Christoffel formula

$$\begin{aligned} & \sum_{\ell \text{ even} \geq L} (2\ell+1) P_{\ell}(x) Q_{\ell}(z) \\ &= L \frac{z Q_{L-1}(z) P_L(x) - x P_{L-1}(x) Q_L(z)}{z^2 - x^2} \end{aligned} \quad (12)$$

we get

$$F(s, \cos\theta) = f_0(s) + \frac{2}{\pi} \int_{z_0}^{\infty} \frac{z Q_1(z) P_2(\cos\theta) - \cos\theta P_1(\cos\theta) Q_2(z)}{z^2 - \cos^2\theta} A(z) dz \quad (13)$$

For $-1/\sqrt{3} < \cos\theta < +1/\sqrt{3}$ we have $P_2(\cos\theta) < 0$ and $(\cos\theta)^2 > 0$, and hence, due to the positivity of $A(z)$,

$$F(s, \cos\theta = \frac{1}{\sqrt{3}}) < f_0(s) \quad (14)$$

from which (10) follows as a particular case $s=0$. Then the chain of inequalities

$$f_0(0) > F(s=0, t=2(1+\frac{1}{\sqrt{3}})) > f_0(2(1+\frac{1}{\sqrt{3}}))$$

leads to inequality (4). Of course, one could, if one wishes, write a more sophisticated inequality involving $L=0$ and $L=2$ partial waves.

Property (5) results from a more refined use of crossing symmetry. For $0 \leq s \leq 4$ we again isolate the S wave contribution :

$$F^{00}(s, t, u) = f_0^{00}(s) + \frac{1}{\pi} \int_4^{\infty} A(x, t) \left[\frac{1}{x-t} + \frac{1}{x-u} - \frac{2}{4-s} \log \frac{x}{x+s-4} \right] dx \quad (15)$$

This expression is obviously crossing symmetric in $t \rightleftharpoons u$ but not necessarily in $s \rightleftharpoons u$. This should be required. In particular, we impose $(d/ds)F(s, t, 4-s-t) = 0$ for $s=u$, i.e., $t=4-2s$. This gives

$$\begin{aligned}
& \frac{df_0^{00}(s)}{ds} + \frac{1}{\pi} \int_4^\infty \frac{dA(x,s)}{ds} \left[\frac{1}{x-4+2s} + \frac{1}{x-s} - \frac{2}{4-s} \log \left(\frac{x}{x+s-4} \right) \right] dx \\
& + \frac{1}{\pi} \int_4^\infty A(x,s) \left[-\frac{1}{(x-s)^2} - \frac{2}{(4-s)^2} \log \left(\frac{x}{x+s-4} \right) + \frac{2}{4-s} \frac{1}{x+s-4} \right] dx = 0
\end{aligned} \tag{16}$$

This equation is interesting in itself, as it shows that the knowledge of the absorptive part fixes the S wave up to a constant.

The bracket appearing in the first integral can be shown to be positive for $0 \leq s \leq 0.62$. The bracket appearing in the second integral can be shown to be positive for $0 \leq s \leq 1.6$. Since by unitarity $A(x,s)$ and $(d/ds)A(x,s)$ are positive, we conclude

$$\frac{df_0^{00}(s)}{ds} < 0 \quad \text{for} \quad 0 \leq s \leq 0.62. \tag{17}$$

It is also easy to see, noticing that the bracket of the first integral is nothing but the expression appearing in Eq. (13), that this bracket is negative for

$$4 \frac{\sqrt{3}-1}{3\sqrt{3}-1} \simeq 0.698 \leq s \leq 4 \frac{\sqrt{3}+1}{3\sqrt{3}+1} \simeq 1.76$$

which corresponds to $|\cos \theta_s| = 1/\sqrt{3}$.

8.

The second bracket, on the other hand, may be shown to be negative for $1.7 \leq s \leq 4$ (for $1.6 \leq s \leq 1.7$ it changes sign when x varies from 4 to infinity).

These two intervals overlap and we conclude therefore that

$$\frac{df_0^{00}(s)}{ds} > 0 \quad \text{for } 1.7 < s < 1.76 \quad (18)$$

Now, we would like to improve (17) in order to locate more precisely the minimum of $f_0^{00}(s)$. This can be done but the technique is more sophisticated because we are investigating a region where the two integrals appearing in (15) have opposite sign. The trick will consist in eliminating $(df_0^{00}(s))/ds$ by using Eq. (16) from the expression of

$$\frac{d}{dt} F(s, t, 4-s-t)$$

Then we get :

$$\begin{aligned} & \frac{dF}{ds}(s, t, 4-s-t) \\ &= \frac{1}{\pi} \int_4^\infty \frac{dA(x, s)}{ds} \left[\frac{1}{x-t} + \frac{1}{x-u} - \frac{1}{x-4+2s} - \frac{1}{x-s} \right] dx \\ &= \frac{1}{\pi} \int_4^\infty A(x, s) \left[\frac{1}{(x-u)^2} - \frac{1}{(x-s)^2} \right] dx \\ &= (s-u) \left[\frac{1}{\pi} \int_4^\infty A(x, s) \frac{2x-4+t}{(x-u)^2(x-s)^2} dx \right. \\ & \quad \left. - \frac{1}{\pi} \int_4^\infty \frac{dA(x, s)}{ds} \frac{2x(s-t) + t(4-2s) - us}{(x-s)(x-t)(x-u)(x-4+2s)} dx \right] \end{aligned} \quad (19)$$

Since we know that for $4 > t \geq 0$, $4 > s$, $4 > u$, dF/ds is positive for $s > u$ and negative for $s < u$, we get

$$\frac{1}{\pi} \int_4^{\infty} A(x,s) \frac{2x-4+t}{(x-u)^2(x-s)^2} dx \geq \frac{1}{\pi} \int_4^{\infty} \frac{dA(x,s)}{ds} \frac{2x(s-t)+t(4-2s)-us}{(x-s)(x-t)(x-u)(x-4+2s)} dx \quad (20)$$

Now, if we compare the second integral appearing in (20) with

$$\frac{1}{\pi} \int_4^{\infty} \frac{dA(x,s)}{ds} \left[\frac{2}{4-s} \log\left(\frac{x}{x+s-4}\right) - \frac{1}{x-s} - \frac{1}{x-4+2s} \right] dx$$

we see that the ratio of the two integrands stays between finite limits. We can then optimize with respect to t to get proportionality of the first two coefficients of the expansion of both integrals in powers of $1/x$. One easily finds that the optimum is obtained for $t=0$. Then (20) reduces to :

$$\frac{1}{\pi} \int_4^{\infty} A(x,s) \frac{2x-4}{(x-4+s)^2(x-s)^2} dx \geq \frac{5}{\pi} \int_4^{\infty} \frac{dA(x,s)}{ds} \frac{2x-4+s}{x(x-4+s)(x-s)(x-4+2s)} dx \quad (21)$$

and one finds that for $1 \leq s \leq 1.33$ at least one has

$$\begin{aligned}
& \int_4^{\infty} \frac{dA(x,s)}{ds} \left[\frac{2}{4-s} \log\left(\frac{x}{x+s-4}\right) - \frac{1}{x-s} - \frac{1}{x-4+2s} \right] \\
& \leq \frac{2/3(4-s)^2 - s^2(4-2s)^2}{2s} \int_4^{\infty} A(x,s) \frac{2x-4}{(x-4+s)^2(x-s)^2} dx \quad (22)
\end{aligned}$$

Then, inserting this inequality in Eq. (16), we get, for
 1.33 > s > 1 :

$$\begin{aligned}
\frac{df_0^{(0)}(s)}{ds} & \leq -\frac{1}{\pi} \int_4^{\infty} A(x,s) \left[\frac{2}{4-s} \frac{1}{x+s-4} - \frac{1}{(x-s)^2} - \frac{2}{(4-s)^2} \frac{\log\left(\frac{x}{x+s-4}\right)}{x+s-4} \right. \\
& \quad \left. - \frac{2/3(4-s)^2 - s^2(4-2s)^2}{2s} \frac{2x-4}{(x-4+s)^2(x-s)^2} \right] dx \\
& \quad (23)
\end{aligned}$$

It is found that in the range $1 \leq s \leq 1.33$ the leading term of the $1/x$ expansion of the bracket for $x \rightarrow \infty$ is always positive. This is, of course, not sufficient. One finds that the most stringent condition is to require that the bracket be positive at the lower limit of integration $x=4$. Then one finds, numerically

$$\frac{df_0^{(0)}(s)}{ds} < 0 \quad \text{for} \quad 1 \leq s \leq 1.29 \quad (24)$$

To fill the gap between $s=0.698$ and $s=1$, we have to replace inequality (22) by something else which is valid below $s=1$. From the inequality

$$\frac{2}{4-s} \log \frac{x}{x+s-4} \leq \frac{1}{4} \left[\frac{1}{x} + \frac{1}{x+s-4} \right] + \frac{3}{4} \left[\frac{1}{x-s} + \frac{1}{x-4+2s} \right] \quad (25)$$

valid for $0 < s < 1$, we can get

$$\begin{aligned} & \int_4^{\infty} \frac{dA(x,s)}{ds} \left[\frac{2}{4-s} \log \left(\frac{x}{x+s-4} \right) - \frac{1}{x-s} - \frac{1}{x-4+2s} \right] dx \\ & \leq \frac{2-s}{2} \int_4^{\infty} A(x,s) \frac{2x-4}{(x-4+s)^2 (x-s)^2} dx \end{aligned} \quad (26)$$

which connects exactly with (22) at $s=1$. Hence

$$\frac{df_0^{00}(s)}{ds} \leq -\frac{1}{\pi} \int_4^{\infty} A(x,s) \left[\frac{2}{4-s} \frac{1}{x-4+s} - \frac{1}{(x-s)^2} - \frac{2}{(4-s)^2} \log \frac{x}{x+s-4} - (2-s) \frac{x-2}{(x-4+s)^2 (x-s)^2} \right] dx \quad (27)$$

One can check that the bracket is positive for $\frac{1}{2} < s \leq 1$, which overlaps with the interval of validity of (17). To summarize the inequalities obtained on $(df_0^{00}(s))/ds$ we have now :

- i) $df_0^{00}(s)/ds > 0$ for $4 \geq s \geq 2$
 - ii) $df_0^{00}(s)/ds > 0$ for $1.76 \geq s > 1.7$
 - iii) $df_0^{00}(s)/ds < 0$ for $1.29 \geq s \geq 0$
- (28)

Therefore $f_0^{00}(s)$ has at least one minimum between $s=1.29$ and $s=1.7$. We believe that by working hard enough, one could manage to fill the gap between (28.i) and (28.ii). It is very probable that $f_0^{00}(s)$ has a unique minimum. Whether this can be proved, we do not know. One remarkable fact is that this minimum is shifted to the left at $s=2$ while the full forward amplitude has its minimum at $s=2$ exactly. Once more this is the evidence that higher partial waves are needed to fulfil crossing symmetry.

We think that the restrictions we have obtained on the $\pi_0\pi_0$ S wave might be useful to those who carry effective calculations of the $\pi\pi$ amplitude. It would be a mistake to disregard them, especially when one wants to calculate things like the $\pi\pi$ S wave scattering lengths which, after all, should be at least as sensitive to what happens nearby, in the range $0 < s < 4$, as to the distant ρ exchange which starts only at $s=-26$. We hope that these new conditions might help in resolving the ambiguities in sign of the scattering lengths. Of course, their weakness lies in the fact that they hold only for the $1/3T=0+2/3T=2$ combination. By nature they are linear, i.e., if $f_0^{00}(s)$ satisfies all of them, $A+Bf_0(s)$ satisfies them too. What remains to do, is to include unitarity in its non-linear form, while here only the positivity aspect of unitarity has been used.

III. THRESHOLD BEHAVIOUR OF $\ell \geq 2$ PARTIAL WAVES

We recall first that the Froissart-Gribov representation holds for partial waves with $\ell \geq 2$ for $0 \leq s < 4$. Second, for the $\pi_0 \pi_0 \rightarrow \pi_0 \pi_0$ amplitude and for the $T=0$ amplitude they are positive :

$$f_\ell^{00}(s) = \frac{2}{4-s} \int_4^\infty Q_\ell\left(\frac{2t}{4-s} - 1\right) \left[\frac{1}{3} A_t^{T=0}(s,t) + \frac{2}{3} A_t^{T=2}(s,t) \right] dt \quad (29)$$

$$f_\ell^{T=0}(s) = \frac{2}{4-s} \int_4^\infty Q_\ell\left(\frac{2t}{4-s} - 1\right) \left[\frac{1}{3} A_t^{T=0} + \frac{5}{3} A_t^{T=2} + A_t^{T=1} \right] dt \quad (30)$$

We now consider the scattering amplitude for fixed t (say the $\pi_0 \pi_0 \rightarrow \pi_0 \pi_0$, for instance). We have, in the neighbourhood of $s=4$, $s < 4$:

$$F(s,t) = \frac{1}{\pi} \int_4^{s_0} \frac{A_s(s',t) ds'}{s' - s} + O(1) \quad (31)$$

and from the positivity of A

$$\int_4^{s_0} \frac{A_s(s',t) ds'}{s' - s} < \frac{4-s_1}{4-s} \int_4^{s_0} \frac{A(s',t) ds'}{s' - s_1} \quad (32)$$

for $s_1 < s < 4$.

So we have, for $s_1 < s < 4$:

$$|F(s, t)| < \frac{C_1}{4-s} + C_2$$

Now, the S wave contribution to $F(s, t)$, $f_0(s)$ can be eliminated by considering the difference between $F(s, t)$ and $F(s, 0)$. We have clearly for $s_1 < s < 4$:

$$|F(s, t) - F(s, 0)| < \frac{D_1}{4-s} + D_2$$

Now

$$F(s, t) - F(s, 0) = \sum_{\ell=2}^{\infty} (2\ell+1) p_{\ell}(s) \left[P_{\ell}\left(\frac{2t}{4-s} - 1\right) - 1 \right] \quad (33)$$

if we take $t > 4-s_1$ we have $(2t/4-s) - 1 > 1$, and then this is a sum of positive terms. Hence from (33), we get :

$$0 < p_{\ell}(s) < \frac{D_1/(4-s) + D_2}{(2\ell+1) \left(P_{\ell}\left(\frac{2t}{4-s} - 1\right) - 1 \right)} \quad (34)$$

for $s_1 < s < 4$. This inequality holds both for the $1/3(T=0) + 2/3(T=2)$ combination and for the $(T=0)$ alone. Therefore

$$\left| p_{\ell}^{T=0,2}(s) \right| < \frac{D'_1/(4-s) + D'_2}{(2\ell+1) \left[P_{\ell}\left(1 + \frac{2t}{s-4}\right) - 1 \right]} \quad \begin{matrix} s_1 < s < 4 \\ 4-s_1 < t < 4 \end{matrix} \quad (35)$$

Hence

$$\lim_{s \rightarrow 4} \frac{|f_\ell(s)|}{|s-4|^{\ell-1}} \leq \frac{C}{(2\ell+1)} \frac{1}{t^\ell} \frac{(\ell!)^2}{(2\ell)!} \quad (36)$$

where t can be taken as close as one wishes to $t=4$.

Now, at this point, comes a very important postulate which, to our knowledge, has never been established from first principles : we assume that the point $s=4$ is not an essential singularity of the amplitude, considered as a function of $k = \sqrt{(s-4)/4}$. It is generally said that through unitarity one can write

$$f_\ell^{T=0,2} = \frac{1}{\phi_\ell^{T=0,2}(s) - ik}$$

where ϕ_ℓ is a meromorphic function around $s=4$. Now what is only known is that ϕ_ℓ is single valued around $s=4$, as a function of s . However, nothing forbids $s=4$, or $s=4+x$, or all the points of the elastic cut from being essential singularities of ϕ_ℓ . This will be discussed elsewhere.

For the time being, we assume that $s=4$ is not an essential singularity. Then, inequality (36), first established for $s < 4$, and in fact, also valid for $2\eta - \varepsilon > \text{Arg}(s-4) > \varepsilon$, holds also for $\text{Arg}(s-4) = 0$ and $\text{Arg}(s-4) = 2\pi$.

Further, we assume some uniformity for the limit : for $4 < s \text{ real} < 4 + \eta$:

$$|f_\ell(s)| < \frac{|s-4|^{\ell-1}}{2\ell+1} \frac{C}{t^\ell} \frac{(\ell!)^2}{(2\ell)!} x^\ell, \quad t \leq 4$$

where x is some finite number larger than 1. Under these conditions, it is easy to prove that

$$|A_s^{T=0,2}(s,t) - 2\text{Im}f_0^{0,2}(s)| < O(1)$$

for $4 < s < 4+\eta$ [for instance $\text{Im}f_2(s)P_2(1+(2t/s-4)) \sim (s-4)^{\frac{3}{2}}$], and hence $A_s^{T=0,2}(s,t) < C_1 + (C_2/\sqrt{s-4})$, from unitarity of the S wave. Hence, re-injecting in

$$F(s,t) - F(s,0) = \frac{1}{\pi} \int_4^{s_0} \frac{A_s(s',t) - A_s(s',0)}{s' - s} ds' + O(1)$$

for the $\Pi_0 \Pi_0$, or $T=0$ amplitude, we get

$$F(s,t) - F(s,0) = O\left(\log \frac{1}{|s-4|}\right)$$

and hence

$$\lim_{s \rightarrow 4} \frac{f_e^{T=0,2}(s)}{|4-s|^e} \frac{1}{\log \frac{1}{|s-4|}} \quad \text{finite}$$

but then, since $f_e(s)$, as a function of $\sqrt{s-4}$, is assumed to be meromorphic around $s=4$, we get that

$$\lim_{s \rightarrow 4} \frac{f_e^{T=0,2}(s)}{|4-s|^e}$$

exists and is finite, for $e \gg 2$.

For the $T=1$ partial waves $e \gg 3$ one can proceed as follows : consider the $T=1$ amplitude. In the s channel, its absorptive part is positive. In the t channel, it is a certain combination $\alpha(T=0) + \beta(T=2) + \gamma(T=1)$, $\gamma > 0$, and in the u channel the same combination with opposite signs. The $T=0$ and $T=2$ contributions to $f_e^{T=1}$ are given by

$$\frac{2}{4-s} \int A_t^{T=0,2}(s,t) Q_\ell\left(\frac{2t}{4-s}-1\right) dt$$

with ℓ odd. Now

$$\left| Q_\ell\left(\frac{2t}{4-s}-1\right) \right| \leq \left| Q_{\ell-1}\left(\frac{2t}{4-s}-1\right) \right| \frac{Q_\ell\left(\frac{4+s}{4-s}\right)}{Q_{\ell-1}\left(\frac{4+s}{4-s}\right)}$$

Then we have already, from the normal threshold behaviour for $\pi_0 \pi_0 \rightarrow \pi_0 \pi_0$:

$$\frac{2}{4-s} \int A_t^{T=0,2}(s,t') Q_{\ell-1}\left(-1+\frac{2t'}{4-s}\right) dt' < C_\ell (s-4)^{\ell-1}$$

Hence, summing these terms, we have

$$f_\ell^{T=1} + (\alpha + \beta) C_\ell (s-4)^\ell > 0$$

for $0 < s < 4$, and then we can easily establish

$$f_\ell^{T=1}(s) < \text{const } (s-4)^{\ell-1}$$

following the same method as before. Then we recycle as before, using elastic unitarity and arrive at the same conclusion ⁵⁾.

IV. INEQUALITIES ON HIGHER WAVES

A direct consequence of the results of Section III is the validity of the Froissart-Gribov representation for $s=4$, $\ell \geq 2$. It has already been stated elsewhere ^{2),4)} that this implies that the $T=0$ and $1/3(T=0)+2/3(T=2)$ scattering lengths, for $\ell \geq 2$ are positive. Further, defining α_ℓ as

$$\lim_{s \rightarrow 4} \frac{f_\ell(s)}{(s-4)^\ell}$$

one can prove

$$\frac{1}{3} \alpha_\ell^{T=0} + \frac{2}{3} \alpha_\ell^{T=2} \geq \frac{\Gamma(\ell+1)}{\Gamma(\ell+3/2)} \frac{1}{2\sqrt{\pi}} \int_4^\infty \frac{\frac{1}{3} A_s^{T=0}(s',0) + \frac{2}{3} A_s^{T=2}(s',0)}{(2s')^{\ell+1}} ds' \quad (37)$$

$$\alpha_\ell^{T=0} \geq \frac{\Gamma(\ell+1)}{\Gamma(\ell+3/2)} \frac{1}{2\sqrt{\pi}} \int_4^\infty \frac{\frac{1}{3} A_s^{T=0}(s',0) + \frac{5}{3} A_s^{T=2}(s',0) + \left(1 + \frac{8}{s-4}\right) A_s^{T=1}(s',0)}{(2s')^{\ell+1}} ds'$$

Other inequalities can be obtained : both for the $1/3(T=0)+2/3(T=2)$ we have for $0 < s < 4$

$$f_{\ell+n}(s) < \frac{Q_{\ell+n}\left(\frac{4+s}{4-s}\right)}{Q_\ell\left(\frac{4+s}{4-s}\right)} f_\ell(s) \quad \begin{matrix} n \geq 0 \\ \ell \geq 2 \end{matrix} \quad (38)$$

and, in particular

$$q_{l+2} \leq \frac{(l+1)(l+2)}{(l+3/2)(l+5/2)} \frac{1}{64\mu^2} q_l \quad (39)$$

These inequalities show that for $l \geq 2$, $T=0,2$, $0 < s < 4$, the amount of higher waves is essentially controlled by the amount of D waves. To conclude this Section, let us indicate that, using inequality (21), it must be possible to get inequalities on df_2/ds for the $\pi_0 \pi_0 \rightarrow \pi_0 \pi_0$ case as was done for the S wave. We expect to find $(df_2/ds) < 0$, but this has to be confirmed.

V. AN ESSENTIAL SINGULARITY OF THE SCATTERING AMPLITUDE ON THE SECOND SHEET

The existence of a line of essential singularities on $s < 0$ of the second sheet has been made plausible some time ago ⁶⁾. Here we want to present a rigorous proof that the point $s=0$ is an essential singularity of the $T=0$ $\pi\pi$ scattering amplitude.

We recall that $f_{\ell}^{T=0}(s)$ is positive for $0 \leq s < 4$, $\ell \gg 2$. It is essential to remark that $f_{\ell}^{T=0}(0)$ cannot vanish because it is defined as an integral over cross-sections with positive coefficients. If this integral vanishes, the cross-section is zero almost everywhere and the scattering amplitude is identically zero. Now, using unitarity, we can write

$$f_{\ell I}^{T=0}(s) = \frac{\sqrt{s}}{2 [\phi_{\ell}(s) - ik]} \quad (40)$$

where I designates the first sheet and $\phi_{\ell}(s)$ is meromorphic in $|s-4| < 4$ at least ¹⁾, and real for s real. The expression of f_{ℓ} on the second sheet is given by

$$f_{\ell II}^{T=0}(s) = \frac{\sqrt{s}}{2 [\phi_{\ell}(s) + ik]} \quad (41)$$

Since $f_{\ell I}^{T=0}$ does not vanish for $0 \leq s < 4$, we conclude that $\phi_{\ell}(s)$ has no pole on $0 < s < 4$, and that

$$\lim_{s \rightarrow 0} \phi_{\ell}(s) = -1$$

(Note the importance of the $\sqrt{s}$ factor.)

On the other hand, $\lim_{s \rightarrow 4} \phi_\ell(s) = +\infty$. So the denominator of (41) varies from -2 to $+\infty$ as s varies from 0 to 4 . So it vanishes at least once, which means that $f_\ell^{T=0}(s)$ for $\ell \gg 2$ has at least one pole on the second sheet located between $s=0$ and $s=4$. At the pole

$$f_{\ell I}(s) = \frac{\sqrt{s}}{2\sqrt{4-s}}$$

Now let us show that the poles of different partial waves do not coincide. Indeed, if

$$f_{\ell I}^{T=0}(s_e) = \frac{\sqrt{s_e}}{2\sqrt{4-s_e}}$$

we know that, from the positivity of the absorptive part and the Froissart-Gribov formula :

$$f_{\ell+n I}^{T=0}(s_e) < f_{\ell I}^{T=0}(s_e) < f_{\ell-n I}^{T=0}(s_e).$$

If we consider now the sequence of partial waves, where do poles accumulate ? We have the inequality

$$f_{\ell I}^{T=0}(s) < \frac{Q_\ell\left(\frac{4+s}{4-s}\right)}{Q_2\left(\frac{4+s}{4-s}\right)} f_{2 I}^{T=0}(s)$$

so for sufficiently large ℓ , $4 \gg s > 0$, we have

$$f_{\ell I}(s) < \frac{\sqrt{s}}{2\sqrt{4-s}}$$

and poles cannot accumulate. The only possible and necessary accumulation point is $s=0$. In fact these poles form a Regge trajectory (or several arcs of Regge trajectory), since in $|s| < 4$, $-\pi < \text{Arg } s < +\pi$, $\text{Re } \ell \geq 2$ $f_{\ell I}(s)$ is an analytic function of ℓ and s .

Now if we consider the full amplitude or more precisely

$$F_{II}(s, \cos\theta=1) - F_{II}(s, \cos\theta), -1 \leq \cos\theta \leq +1$$

we can easily prove that on any interval $0 < s_1 < |s| < s_2 < 4$, $-(\pi - \varepsilon) < \text{Arg } s < \pi - \varepsilon$, it consists of a finite number of individual partial wave contributions which might have poles and of a regular part : this is done by noticing that in this domain, for $\ell > L(s_1, s_2)$, $|f_{\ell I}(s)| < \frac{1}{4}(\sqrt{s}/\sqrt{4-s})$, and then the convergence of the series on the second sheet is a consequence of the absolute convergence of the series on the first sheet. Therefore in $|s| < 4$, $-\pi < \text{Arg } s < +\pi$, $F_{II}(s, \cos\theta=1) - F_{II}(s, \cos\theta)$ is a meromorphic function of s with an accumulation point of poles at $s=0$. Notice that we have eliminated the S wave because we know nothing about the $T=0$ S wave. In a neutral theory, the results of Section II could be used to prove that the $f_0(s)$ has at most one pole in $0 < s < 1.29$. What matters is that poles of different individual partial waves cannot cancel each other.

Let us conclude this Section by one remark which might or might not be important. If one could prove that for any $\ell \gg 2$, $df_{\ell}/ds < 0$ in a neutral theory, for $0 < s < 4$ one could prove that each partial wave has only one pole on the second sheet for $0 < s < 4$ and it would not be difficult to prove that this pole $s(\ell)$ is such that as ℓ increases, s decreases. Now, suppose we are given the forward scattering amplitude exactly. Then, by analytic continuation, we know its poles on the second sheet and we can attribute each pole in the second sheet to a given partial wave. This means that if $s_2, s_4, \dots, s_{\ell}$ are the poles we know on the first sheet $f_2(s_2)$ and $(d/ds)f_2(s_2)$, $f_4(s_4)$ and $(d/ds)f_4(s_4)$, etc. This constitutes additional constraints on the amplitude and then freedom to choose the non-forward scattering amplitude is much restricted. By being clever enough, one might reach the conclusion that the forward scattering amplitude fixes the amplitude for any transfer. This may be wrong. If, however, one could prove it, it would be a very serious improvement of the S matrix programme.

VI. CONCLUDING REMARKS AND HOPES

Obviously the work we have presented here is no more than the sketch of a new line of approach to crossing symmetry. It lacks generality. Each result is obtained by a particular method. However, we think that the general idea, which is to try to exhaust the consequences of crossing symmetry and positivity (which is only one part of unitarity) on given partial waves, should be retained. We already have a pretty good idea of the shape of the $\pi_0 \pi_0$ S wave. In a neutral theory, one can try to meet the requirement of unitarity by taking :

$$f_0(s) = \frac{\sqrt{s}}{2[\phi(s) - i'k]}$$

Then, take some form of $\phi(s)$ which has the required analyticity properties depending on a number of parameters. One then finds that if the number of parameters is not too large, the conditions obtained fix all parameters except one, for instance the scattering length $f_0(4)$. Suppose that the form thus obtained is analytic in s . Then using inequality (37) and generalization, one can get a lower limit for the D wave contribution and higher wave contributions in $0 < s < 4$. Then, taking these contributions to be correct, one can readjust the S wave to have better crossing symmetry. One can then try to improve the higher waves in $0 < s < 4$ by prolongating the previously obtained D waves beyond $s=4$ up to the point where they attain the unitarity limit. Then one will be able to insert this in the Froissart-Gribov formula and get improved D waves. However, probably, before doing that kind of thing, one has to find more restrictions. Maybe one could treat at once a finite number of partial waves, let us say S and D waves, provided that the final results apply to separate partial waves. We think that one should investigate this field more deeply if one wants to obtain dispersion-theoretical results on the pion-pion system which have some reliability.

ACKNOWLEDGEMENTS

I thank Professor J. Hamilton for encouraging me to publish this paper.

REFERENCES

- 1) A. Martin, Nuovo Cimento 42, 930 (1966).
- 2) A. Martin and Y.S. Jin, Phys.Rev. 135, B1375 (1964).
- 3) A. Martin and Y.S. Jin, Phys.Rev. 135, B1369 (1964).
- 4) L. Łukaszuk and A. Martin, Phys.Letters 21, 96 (1966).
- 5) After this work was completed, we received two preprints, one by G. Roepstorff and J.L. Uretsky (Argonne, June 1966), and one by Y.S. Jin (Brown University) which also establish threshold behaviour. However, in both papers, as well as in ours, some assumptions are made to get the normal threshold behaviour. In the first one, one assumes that the off-shell amplitude is finite at $E=E$ threshold, and in the second one that the absorptive part inside the extended large Lehmann ellipse stays finite as one approaches threshold.
- 6) P.G.O. Freund and R. Karplus, Conference on Elementary Particles, Aix-en-Provence, Vol.1, 327 (1961). Here the essential singularity appears in a different way, as an accumulation of cuts in $\cos\theta$ for $s < 0$.

FIG. 1