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NON-RESONANT ZEROS OF THE REAL PART OF THE DENOMINATOR
FUNCTION OF THREE-PARTICLE AMPLITUDES

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ABSTRACT

We point out that the definition of a three-particle resonance used by some authors is inadequate, since it will not lead to an observable bump in the cross-section. This is so in some of the models which predict a $T = 5/2$ nucleon isobar.

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I. INTRODUCTION

In a field theoretic framework, the scattering functions can be analytically continued away from the physical region to complex values of the energy w . A resonance is generally understood as a pole of the analytically continued t matrix on a "nearby" Riemann sheet. Such a pole is assumed to be relevant if it is caused by a zero of the Fredholm type denominator $D(w)$ of the t matrix, i.e.,

$$D^{\Pi}(w) = 0 \quad (1)$$

Such a zero does actually lead to resonant behaviour of the cross-section.

In general, however, such a (complex valued) zero is hard to locate because all one has is an integral equation or, at best, an integral representation for $D(w)$. Numerous authors therefore have begun to speak of a resonance if only the condition

$$\operatorname{Re} D(w) = 0 \quad (2)$$

is fulfilled for w (or $s=w^2$) on the real axis. Clearly (2) is weaker than (1). The purpose of this work is to point out a few interesting situations where (2) is true but not (1)¹⁾. In particular, this is relevant to models which predict a $T=5/2$ isobar²⁾.

Firstly, we wish to give a trivial example. Consider a $D(w)$ which has a pole or, more precisely, is of the form

$$D_1(w) = [w - w_R + i\Gamma]^{-1} \quad (3)$$

This denominator function clearly cannot give rise to a resonance. Nevertheless, it has $\operatorname{Re} D_1(w_R) = 0$.

At this point one might believe that the distinction between (1) and (2) and our example is academic. This is the case for two-body amplitudes, as long as one does not consider three-particle intermediate

states ³⁾. This changes drastically if one considers production and three-particle amplitudes which have become of great interest since the work of Faddeev ⁴⁾ and Lovelace ⁵⁾. Poles of D at certain positions are actually required by unitarity. Three-particle amplitudes always have factors which represent the final state interactions. If two of their singularities coincide, the amplitude would blow up. This is prevented by a pole of D . At least two authors ^{2), 6)} have considered as resonance a zero of $\text{Re } D$ which, as a more careful analysis shows, is accompanied by a nearby pole of $D(w)$, and hence cannot produce a peak in the cross-section.

II. THE KÄLLEN-GLASER MODEL

The simplest model of interest in this context which we have at hand is the Källén-Glaser ⁷⁾ version of the Lee model. In this model the V particle is unstable. We have given the solution of its three-particle sector elsewhere ⁸⁾. The important results for us are that the two-body $(N\theta)$ amplitude has a denominator $H(w)$ which is an analytic function except for the elastic cut ($\mu < w < +\infty$). It is non-zero on the entire first sheet and its analytic continuation below the real axis shows a resonant zero at the point $w^{\text{II}} = w_r - i\Gamma = Z_0$. In a field theoretic context we may assume $\text{Im } H(w)$ to be an analytic function, and we can use the expression

$$\text{Disc.}[H^-(w)] = 2i \text{Im}[H^-(w)] = \frac{1}{H^+(w)} - \frac{1}{H^-(w)}, \quad (4)$$

for the analytic continuation of the imaginary part. Note that $H^{\text{II}}(w)$ has the resonance zero on its first sheet, where it factors as

$$H^{\text{II}}(w) = (w - Z_0) G^{\text{II}}(w). \quad (5)$$

$G^{\text{II}}(Z_0) = g^{-1}$, where g is the effective $N^*\theta$ coupling.

The three-particle ($N\theta\theta$) amplitude has a characteristic denominator ⁸⁾

$$D_2(w) = \frac{1}{\pi} \int_{\mu}^{\infty} \frac{dw'}{H(w-w')} \operatorname{Im} \frac{1}{H(w')}. \quad (6)$$

$D_2(w)$ has a three-particle branch point at $w=2\mu$ and after continuation through this cut, the familiar $N^*\theta$ cut starting at $w^{\text{II}} = Z_0 + \mu$. In addition, however, it has a singularity at the point $w=2Z_0$ (Fig. 1). We shall now establish this singularity at $2Z_0$ and show that it is a pole by computing its residue. To do this we have to find the analytic continuation of $D_2(w)$ through the inelastic cut and onto the sheet reached by passing to the right of the $N^*\theta$ branch point (Fig. 1). For real w we have $D(w+i\varepsilon) - D(w-i\varepsilon) = 2i \operatorname{Im} D(w+i\varepsilon)$, where

$$\operatorname{Im} D(w) = \frac{1}{\pi} \int_{\mu}^{\infty} [\operatorname{Im} H^{-1}(w')] [\operatorname{Im} H^{-1}(w-w')] dw'. \quad (7)$$

Using Eq. (4), we define the analytic continuation of $2i \operatorname{Im} D_2(w)$ by

$$C(w) = \frac{1}{2\pi i} \int_{\mu}^{w-\mu} \left[\frac{1}{H^{\text{I}}(w')} - \frac{1}{H^{\text{II}}(w')} \right] \left[\frac{1}{H^{\text{I}}(w-w')} - \frac{1}{H^{\text{II}}(w-w')} \right] dw'. \quad (7')$$

The analytic continuation of $D_2(w)$ itself is

$$\bar{D}_2^{\text{II}}(w) = D_2^{\text{I}}(w) - C(w). \quad (8)$$

$D^{\text{II}}(w)$ has in addition to the singularities of $D^{\text{I}}(w)$ all the singularities of $C(w)$. $C(w)$ is analytic with a cut from 2μ to $+\infty$, the $N^*\theta$ branch point and a singularity at $w=2Z_0$ as seen from (7'). This last singularity is only present in the product of the last two terms from the brackets which we write

$$I = \frac{-1}{2\pi i} \int_{\mu}^{w-\mu} dw' / [H^{\Pi}(w') H^{\Pi}(w-w')]. \quad (9)$$

Near $w = 2Z_0$ this will be the dominant part of D_2 . Following our programme (Fig. 1), we start from a value of w on the first sheet: $w \gg 2w_r$. Then $\text{Re } w > \text{Re}(Z_0 + \mu)$ and in the w' plane we have the situation of Fig. 2a. The end-point of integration and the pole in the upper half plane will move as we vary w . We now take w straight down near $2Z_0$ as indicated in Fig. 1. In the w' plane, this will bring the end-point of integration to $w' \sim 2Z_0 - \mu$ and the moving pole will pinch the contour against the fixed one (Fig. 2b). Using Eq. (5), we rewrite (9) as:

$$I = (2\pi i)^{-1} \int_{\mu}^{2Z_0 - \mu} \frac{dw'}{(w' - Z_0 + i\varepsilon)(w' - w + Z_0 - i\varepsilon)} [G^{\Pi}(w') G^{\Pi}(2Z_0 - w')]^{-1}. \quad (10)$$

The $\pm i\varepsilon$ have been inserted to maintain the relative position of poles and contour. We now make partial fractions in (10), observe that the principal values cancel, and use the δ functions to do the integrals. This gives for $w_r > \mu$

$$I = \frac{g^2}{w - 2Z_0} \quad (11)$$

which was to be shown ⁹⁾.

The function $D_2(w)$, (6), can be split into real and imaginary parts for which we write

$$\operatorname{Re} D_2(w) = \frac{-1}{\pi} \int_{\mu}^{\infty} \frac{\operatorname{Im} H(w')}{|H(w')|^2} \cdot \frac{\operatorname{Re} H(w-w')}{|H(w-w')|^2} dw' \quad (12)$$

$$\operatorname{Im} D_2(w) = \frac{1}{\pi} \int_{\mu}^{w-\mu} \frac{\operatorname{Im} H(w')}{|H(w')|^2} \cdot \frac{\operatorname{Im} H(w-w')}{|H(w-w')|^2} dw' \quad (13)$$

Now, for a narrow $N\Theta$ resonance, the first factor $|H(w')|^{-2}$ in (12) acts practically as a δ function. For $w' \sim w_r$, the second factor is well behaved because there $\operatorname{Re} H(w-w') \sim 0$. The main contribution to this integral therefore comes from $w' \sim w_r$ and at least over the width of the $N\Theta$ resonance the other factors vary slowly. We can therefore approximate this integral well by taking the second factor outside. We get

$$\operatorname{Re} D_2(w) \sim - \frac{\operatorname{Re} H(w-w_r)}{|H(w-w_r)|^2} \int_{\mu}^{\infty} \operatorname{Im} \frac{1}{H(w')} dw' \quad (14)$$

Clearly, this shows a zero at $w = 2w_r$ due to $\operatorname{Re} H(w_r) = 0$ which comes from the input resonance. That $\operatorname{Re} D_2$ has a zero somewhere between $\mu < w < \infty$ can be seen from the fact that $\operatorname{Re} D_2(w)$ is non-singular for w real and $\operatorname{Re} D_2(\mu) > 0$, while $\operatorname{Re} D_2(+\infty) < 0$. The $N\Theta$ resonance only has fixed the location of this zero. The minus sign in front of (14) is just right so that the phase shift would increase through 90° as one might expect for a resonance. Up to a real factor, we can namely write for the essential three-body part of the amplitude

$$T(w) \sim D_2^{-1}(w) = e^{i\delta(w)} \sin \delta(w) \quad (15)$$

and find for the so-defined phase shift $\delta(w)$

$$\cot \delta(w) = \frac{\operatorname{Re} T(w)}{\operatorname{Im} T(w)} = - \frac{\operatorname{Re} D(w)}{\operatorname{Im} D(w)}. \quad (16)$$

An equivalent discussion has been considered by several authors as evidence for a resonance. If we now look at the imaginary part of D_2 , (13), we see that it does not behave smoothly near $w = 2w_r$ because there the two peaks coming from the denominators coincide and there is no numerator which could cancel one of these contributions since $\operatorname{Im} H(w')$ does not vanish in the considered region. The coincidence of the two peaks evidently gives a much larger value for the integral, which is proportional to $|H(w_r)|^{-4}$ for a narrow $N\Theta$ resonance, and we see that $\operatorname{Im} D_2(w)$ has a sharp peak at $w \sim 2w_r$. This situation is shown in Fig. 3. The total cross-section cannot show a peak.

We may note that this does not depend on the magnitude of $\operatorname{Im} D_2$ or the coupling constant but only on the fact that $\operatorname{Im} D_2$ varies rapidly just at the point where $\operatorname{Re} D_2$ vanishes. In fact, the general situation here is independent of the parameters of the model. This can be seen by observing that $\operatorname{Re} D_2$ and $\operatorname{Im} D_2$ are related via the unsubtracted dispersion relation

$$\operatorname{Re} D_2(w) = \frac{1}{\pi} P \int_{2\mu}^{\infty} \frac{\operatorname{Im} D_2(w')}{w' - w} dw'. \quad (17)$$

This holds because $D_2(w)$ has a cut from 2μ to $+\infty$ and, since it is a real analytic function, its discontinuity is just its imaginary part. From (13) follows that $\operatorname{Im} D(w) = 0$ for $w < 2\mu$ and comparison with (6) shows that we need no subtraction. Therefore any change in $\operatorname{Im} D_2$ will not move the zero of $\operatorname{Re} D_2(w)$ outside of the peak of $\operatorname{Im} D_2$. This would be necessary to observe a resonance.

III. MULTIPLE LEE MODELS

We have also solved for the essential scattering and bound state functions in a Lee model with several V-type particles. Of special interest here is the one where V_1 is stable, while V_2 is unstable ^{10),11)}. The three-body amplitudes ($V_1\theta \rightarrow V_1\theta$, $V_1\theta \rightarrow N_2\theta$, $N_2\theta \rightarrow N_2\theta$, $V_1\theta \rightarrow V_2\theta$) have the common denominator

$$D_3(w) = I_1 - w_0^2 H_1(w) (w - w_0) (I_1 I_2 - I_3 I_4), \quad (18)$$

where $H_1(w)$ is the denominator of the $N\theta$ amplitude, given in Ref. ¹⁰⁾. It has the form $H_1(w) = wG(w)$, where $G(w)$ now contains a CDD pole at $w = w_0$. We introduce the pole-free functions $g(w) = (w - w_0)G(w)$ and $h(w) = (w - w_0)H_2(w)$. Then we have

$$\begin{aligned} I_1(w) &= \frac{1}{\pi} \int_{\mu}^{\infty} \frac{dw'}{(w' - w_0) g(w - w')} \operatorname{Im} \frac{1}{G(w')} \\ I_2(w) &= \frac{1}{\pi} \int_{\mu}^{\infty} \frac{dw'}{(w' - w_0) h(w - w')} \operatorname{Im} \frac{1}{H(w')} \\ I_3(w) &= \frac{1}{\pi} \int_{\mu}^{\infty} \frac{dw'}{(w' - w_0) g(w - w')} \operatorname{Im} \frac{1}{H(w')} \\ I_4(w) &= \frac{1}{\pi} \int_{\mu}^{\infty} \frac{dw'/w'}{(w' - w_0) h(w - w')} \operatorname{Im} \frac{1}{H(w')} \end{aligned}$$

At the point in question $w^{\text{II}} = 2Z_0$, the residue of the brackets on the right-hand side of (12) vanishes. In this neighbourhood, the analytic properties of $D_3(w)$ are given by $I_1(w)$. This has the same structure as $D_2(w)$, (6), and our entire analysis holds with no essential changes. Hence even in these models there is a pole at $w^{\text{II}} = 2Z_0$ and we have verified numerically that $\operatorname{Re} D_3(w \sim 2w_r) = 0$, while again there is a sharp peak in $\operatorname{Im} D_3$ at this energy. The cross-section shows no resonance with this mass value.

The model just discussed is equivalent to Aitchison's ^{6),12)}. If we insert in addition to the cut-off function, implicitly present above, a momentum factor k , we get a model with derivative coupling which results in p wave scattering. To put in physical numbers, we may take the input resonance to be the $(3.3) \quad \bar{N}N$ resonance at 1240 MeV. Since we are in a static model, we have to subtract the nucleon mass giving $w_r = 300$ MeV. Our analysis shows that in this model there is a pole of $D(w)$ near the physical region at $Z_0 = 600 \text{ MeV} - 2i\Gamma$, where Γ is the width of the N^* , and a non-resonant zero of $\text{Re } D_3$ at $2w_r = 600$ MeV which corresponds to a mass of 1540 MeV. The model just described has been discussed by Trueman ²⁾.

IV. DISCUSSION

While the above are very special models, we have reason to believe that a similar situation will prevail once one admits recoil or even in relativistic situations. The main point is that our argument depends on the right-hand structure of D only, and this is quite realistic in the Lee model. In relativistic theories, the simple relation $w = 2w_r$ will be replaced by the Landau-Cutkosky rules as applied to diagrams representing the denominator function $D(s)$. The difficulty is that in general one will not have an explicit expression for D ¹³⁾.

As a side-step, we may mention that the Källén-Glaser model already contains all the Feynman diagrams (Fig. 4) relevant for the Peierls mechanism ¹⁴⁾ which, in this case, predicts a resonance at $w = 2w_r$. Clearly the models discussed here cannot support this theory, they have, however, nothing to say on the generalized mechanisms discussed by Kacser and Month ¹⁵⁾, due to the fact that there is only one unstable V present.

One will note that even in the example mentioned in the introduction as well as in the different Lee models, the phase shift actually increases through $\pi/2$ with the same slope as for the input resonance. The fact that in spite of this we do not find a resonance at this point does not contradict Wigner's ¹⁶⁾ causality argument which only implies that a phase shift increasing fast enough through $\pi/2$ is a necessary but not a sufficient condition for a resonance in exactly the same way as we just observed for the condition $\text{Re } D = 0$.

To conclude this note we may ask : how can we make sure we have a resonant zero of $D(s)$? Present day computers allow complex arithmetic and this should be used seriously in future investigations. Another point is to look at the imaginary part of D . In the resonant case it is smooth, in the non-resonant case it shows a peak due to the pole. Also the absolute value squared of $D(w)$ is instructive.

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- 12) The fact that Aitchison used Srivastava's slightly incorrect solution does not affect the point in question. Clearly there can be no resonance in Aitchison model near $w = 2w_r$. This model has so many free parameters that it may be possible to create a peak somewhere else, but this is irrelevant here.
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FIGURE CAPTIONS

Figure 1 : Shows the analytic structure of $D(w)$ in the different Lee models. The arrows indicate the path we take for the analytic continuation and give the position of the point Z_0 .

Figure 2 : a) The complex plane of the integrand for real $w \sim 2w_r$. The pole in the lower half plane is fixed, the one in the upper half plane moves as we change w .
b) The complex plane of the integrand of Eq. (10) after analytic continuation. The moving pole pinches the contour against the fixed one.

Figure 3 : The imaginary part of D_2 peaks just at the zero of real D_2 .

Figure 4 : Shows the lowest order Feynman graphs contributing to the $N2\theta$ amplitude, considering the unstable V as a particle. Each one contains the Peierls diagram (N exchange).

FIG. 1

FIG. 2a

FIG. 2b

FIG.4