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DYNAMICAL SYMMETRIES AS FUNCTION GROUPS ON DYNAMICAL SPACES

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A B S T R A C T

Dynamical symmetries are analyzed from the view point of function groups defined on an n dimensional dynamical space. Limiting the analysis to semi-simple Lie algebras, a number of interesting results are obtained: in particular, the order of the function group (i.e., the number of functionally independent generators employed to obtain an explicit realization of the algebra) is related to the number of independent Casimir operators and the rank of the algebra is shown to be necessarily smaller or equal to n . By applying this method to the recently proposed "invariance" and "non-invariance" dynamical groups, we show that the characteristic features of these groups are independent of any specific Hamiltonian and simply depend on their function group structure. Many general problems related to the explicit realization of "invariance" and "non-invariance" dynamical groups are therefore seen in a completely general frame.

INTRODUCTION

The study of geometrical symmetries of dynamical systems has greatly helped our understanding of stable configurations for classical systems and of energy degeneracy for bound quantum systems. Here, by "geometrical symmetry", we mean the invariance of the Hamiltonian under a set of transformations in configuration space: rotational invariance, space and time translational invariance being responsible for the conservation of the one-valued constants of motion associated with angular momentum, momentum and energy, respectively.

For a number of systems, however, new one-valued constants of motion can be defined. Their existence has been linked to the invariance of the Hamiltonian under a larger transformation group than the geometrical one: for instance, the energy degeneracy of the three-dimensional Coulomb problem (hydrogen atom) has been traced to an invariance under a transformation group isomorphic to $O(4)$ ¹⁾; the energy degeneracy of the n dimensional harmonic oscillator, to an invariance under a transformation group isomorphic to $SU(n)$ ²⁾. For this kind of symmetry of a dynamical system, a symmetry larger than the geometrical one, the expression "dynamical symmetry" (d.s.) has been extensively used³⁾.

However, the very concept of a "dynamical symmetry" has been somewhat extended of late and requires some additional qualification.

On the one hand, it has been shown that the requirement of one-valuedness for the constants of motion can be formally relaxed, so that "formal" dynamical symmetries can be built whose generators, while commuting with the Hamiltonian, are not necessarily one-valued functions of the dynamical variables. Once this extension is accepted, a large number of dynamical systems can be shown to possess, formally, a dynamical symmetry; it is a "local" symmetry, however, unless all generators turn out to be one-valued functions⁴⁾. In what follows we shall not be concerned with the analytic properties of the generators and we shall therefore use the expression "dynamical symmetry" in the extended acceptance defined above.

On the other hand, we have now to deal in the published literature with both "invariance" and "non-invariance" dynamical groups. We can indeed search a dynamical system either for dynamical symmetries whose generators commute all with the Hamiltonian ("invariance" d.s.); or for dynamical symmetries containing at least some generators that do not leave the Hamiltonian invariant ("non-invariance" d.s.). In the latter case, in order to keep some contact with the physical system we want to study, the "non-invariance" d.s. is required to contain as a subgroup the largest "invariance" d.s. that can be built from the constants of motion of the system. It has been claimed that, while "invariance" d.s.'s help to solve the problem of energy degeneracy, "non-invariance" d.s.'s could help to solve the problem of the energy (or mass) spectrum of a system; both "compact" and "non-compact" forms of the "non-invariance" d.s.'s have been extensively discussed ⁵⁾⁻⁸⁾. Some reserves on the role that the "non-invariance" d.s.'s could actually play in understanding the dynamics of a system have been also recently put forward ⁹⁾.

In order to explore the consequences of the existence of a d.s. for a dynamical system with n degrees of freedom, it is necessary to build up its generators explicitly, as functions of the dynamical variables q_i, p_i ($i=1, \dots, n$). This has been done already for many dynamical symmetries, both of the "invariance" and of the "non-invariance" type ¹⁰⁾⁻¹⁵⁾. In any special case studied, a number of quite general and characteristic features have emerged; it is the purpose of the present paper to analyze these common features in detail and to try and understand them in a general frame ¹⁶⁾.

Let us start from an "invariance" d.s.; when its generators are built in terms of all the $2n-1$ functionally independent constants of motion of the system, it is found that not every irreducible representation of the symmetry group can be realized by the Hamiltonian. For instance, in the quantum cases, it is only a "family" of i.r.'s that will be realized by the energy degeneracy of the bound states. In other words, the Casimir operators of our group in the chosen realization of the generators are not independent of each other; and it turns out that

only one of the Casimir operators is functionally independent, irrespectively of the specific Hamiltonian under study and of the dimensionality of the space. It is also found that no d.s. locally described by a semi-simple Lie algebra (of Poisson brackets in the classical case; of commutators in the quantum case) of rank r can be realized in terms of the dynamical variables if r is larger than n ^{13),14)}.

As for a "non-invariance" d.s., when the d.s. is required to contain as a subgroup the "invariance" d.s. discussed above, it is found that its Casimir operators become numbers. In this case the realization chosen for the generators (containing, as it does, arbitrary constants giving rise to inequivalent irreducible representations) contains already information on the i.r. that can be realized by the system; for non-compact d.s.'s, on the infinite dimensional unitary i.r. that will describe all energy levels of the system. Once again, the rank r of the corresponding semi-simple Lie algebra cannot be larger than n ^{12),15)}. We shall show in what follows that these rather peculiar features, presented by both "invariance" and "non-invariance" dynamical symmetries, can be readily and completely understood in a general frame where the specific structure of the Hamiltonian does not play any role. The two quantities that will play a major role in the following analysis will be the number of degrees of freedom of the system, n ; and the number of functionally independent generators employed in building up our algebra, s . These two quantities, together with the rank r of the algebra, will in turn determine the number of independent Casimir operators in both the "invariance" and the "non-invariance" case. The demonstration that r can in no case exceed n will derive as a by-product from our analysis.

The mathematical technique that we shall mainly use will be that connected with the properties of "function groups" realized on an n dimensional dynamical space, $D(n)$ [$\bar{D}(n)$ is defined as a set of $2n$ variables in two complementary sets q_i and p_i ($i=1,\dots,n$), on which a Poisson bracket $\{ \}$ is defined: $\{q_i, q_j\} = \{p_i, p_j\} = 0$; $\{q_i, p_j\} = \delta_{ij}$]. These properties, together with a number of useful theorems, can be found in ¹⁷⁾ and are briefly summarized in Appendix A.

It is clear that henceforth our analysis refers only to classical system; we believe, however, that our results could be extended to the quantum case. We want also to note that a number of papers on the realization of Lie algebras contain points that are somewhat related to our analysis¹⁸⁾⁻²⁰⁾.

In Part I we shall clarify our notation relative to semi-simple Lie algebras in a canonical form; in Part II, we shall deal with simple Lie algebras viewed as function groups on $D(n)$ and derive a number of relevant Lemmas and theorems; in Part III we shall apply our results to the analysis of "invariance" and "non-invariance" dynamical symmetries and show that they allow a general understanding of the peculiar features already observed. Appendix A lists the necessary mathematical apparatus; Appendix B gives some examples of realizations of simple Lie algebras.

I. CANONICAL FORM FOR SEMI-SIMPLE LIE ALGEBRAS

We shall mainly use the notations of Racah ²¹⁾; when necessary, we shall refer to the sign and normalization conventions as used in ²²⁾.

The Poisson brackets between any two of the generators Y_j of a semi-simple Lie algebra ($j=1, \dots, d$; d : dimension of the algebra) in a canonical form, can be expressed by:

$$\begin{aligned} \{H_i, H_j\} &= 0 \\ \{H_i, E_\alpha\} &= \alpha_i E_\alpha \\ \{E_\alpha, E_{-\alpha}\} &= \alpha^i H_i \\ \{E_\alpha, E_\beta\} &= N_{\alpha\beta} E_{\alpha+\beta} \quad (\alpha \neq -\beta) \\ &\quad (\alpha+\beta \text{ a non-zero root}) \end{aligned} \tag{1}$$

where $i=1, \dots, r$ (r : rank of the algebra); $\alpha=1, \dots, e$ ($d=r+e$). The generators H_i span an r dimensional (Cartan) space; the generators E_α correspond to the non-zero, non-degenerate roots $\underline{\alpha}$ in the Cartan space.

It is always possible to find r roots $(\mu, \nu, \dots)$ defining an r dimensional reference system in Cartan space and therefore linearly independent of each other; the remaining $e-r$ generators corresponding to non-zero roots will be then related to roots which are linearly dependent on the set (μ) .

As Cartan has shown that every semi-simple algebra is a direct sum of simple algebras, we shall be concerned only with the root diagrams of simple algebras, apart from some interest for the semi-simple structure of D_2 . We shall use the standard classification for simple Lie algebras, subdivided into the four families A_r, B_r, C_r and D_r and the exceptional algebras $G_2, F_4, E_{6,7,8}$.

II. LIE ALGEBRAS AS FUNCTION GROUPS

We shall now derive a number of necessary conditions to be satisfied by the generators Y_j of a simple algebra of rank r , a_r , when they are explicitly expressed as functions of the dynamical variables q_i, p_i of an n dimensional dynamical space $D(n)$.

a) The set (Y_j) contains a suitable basis for a function group (f.g.) $[Y]$ of order $s \leq \min(d, 2n)$.

It is clear from (1) that the Y_j 's satisfy the requirements of a f.g. (see Appendix A); the dimension of a_r being d , $s \leq d$; $D(n)$ having $2n$ independent variables, $s \leq 2n$.

b) The set (H_i) contains a suitable basis for a commutative f.g. $[H]$ of order $s' \leq \min(r, n)$, $[H]$ being a subgroup of $[Y]$.

A commutative f.g. cannot have order greater than n (A-Corol.1), therefore $s' \leq n$.

Lemma 1: The order of $[H]$ equals the rank of a_r : $s' = r$.

In other words, the H_i 's are functionally independent. If they were not, we could write $H_i = H_i(H_{j \neq i})$ and $\{H_i, E_\alpha\} = \alpha_i E_\alpha$ would lead to $\alpha_i = C_{ij} \alpha_j$ ($j \neq i$) for every α , against the hypothesis that the roots span an r dimensional Cartan space.

Corollary 1: The rank of a_r cannot be greater than n : $r \leq n$.

Lemma 2: There are at least r functionally independent generators among the E_α 's.

Let us select a basis of r non-zero roots (μ) spanning the r dimensional Cartan space; let their corresponding generators be E_μ .

These r E_μ 's are functionally independent. If they were not, we could write $E_\mu = E_\mu(E_\nu, \nu \neq \mu)$ and $\{H_i, E_\mu\} = \mu_i E_\mu$ would lead to $\mu = C_{\mu\nu} \nu$, contrary to the hypothesis.

Lemma 3: Once the set (E_μ) is chosen, the $2r$ functions (H_i, E_μ) are functionally independent.

This can be shown in a way analogous to that used for Lemma 1 and Lemma 2.

Corollary 2: The order of $[Y]$ cannot be smaller than $2r$:
 $2r \leq s \leq \min(d, 2n)$.

As, for all simple Lie algebras, $d > 2r$, we can state:

Corollary 3: If $r=n$, then $s=2n$.

In order to realize an algebra of rank n on $D(n)$ we have to make use, necessarily, of all the $2n$ functionally independent functions that can be defined on $D(n)$.

We can now state the following theorems:

Theorem 1: If a_r is realized with $s=2r$, its r Casimir operators become multiples of the identity.

If we try to realize a_r by as few as $2r$ independent generators, we have to use them for building the set (H_i, E_μ) (Lemma 3). All other generators E_α will be functionally dependent on this set, which represents now a suitable basis for a f.g. $[H, E]$ of order $2r$. The Casimir operators are therefore equivalent to the indicial functions of this f.g. $[H, E] \equiv [X]$; and we know that the number of functionally independent indicial functions is given by $m=2r-t$, where t is the rank of the matrix $\|\{X_j, X_k\}\|$ (A-Theor.3). In order to determine the value of t in our case, we note that it is always possible to choose our set (E_μ) in such a way that $\{E_\mu, E_\nu\} = 0$. This being the case, it can be shown that the linear independence of

8.

the roots μ leads to $t=2r$ and therefore $m=0$. As there are no functions on $D(n)$ indicial with respect to $[X]$, the Casimir operators can only be expressed by numbers.

Theorem 2: If a_r is realized with $s=2n$, its r Casimir operators become multiples of the identity.

It is a direct consequence of (A-Corol. 5).

Theorem 3: If a_r is realized with $s=2n-1$, its r Casimir operators are all functionally dependent on one of them.

It is again a consequence of (A-Corol. 5), according to which for $s=2n-1$ there is only one independent indicial function.

Theorem 4: If a_r contains a sub-algebra b_r of the same rank r , no $s=2r$ realization of a_r is possible.

Let $s=2r$. Because of Corollary 2, the order of the f.g. associated with b_r cannot be smaller than $2r$; but it cannot be larger than $2r$, as $s=2r$; it is therefore equal to $2r$. Theorem 1 forces the Casimir operators of b_r to be numbers; they will therefore commute with all generators of a_r . But some of them (not belonging to b_r) have well defined transformation properties under b_r and cannot therefore commute with all of its Casimir operators; no realization is therefore possible²³).

Corollary 4: If a_n is realized on $D(n)$, its n Casimir operators are multiples of the identity.

It is a direct consequence of Corollary 3 and Theorem 2.

III. APPLICATION TO "INVARIANCE" AND "NON-INVARIANCE" DYNAMICAL SYMMETRIES

We shall now apply the results of Part II to the problem of realizing "invariance" and "non-invariance" dynamical symmetries on the dynamical space $D(n)$. The following conclusions can be readily reached:

- a) If a_n is realized on $D(n)$, then it represents necessarily a "non-invariance" symmetry for any dynamical system.

Any n dimensional dynamical system presents $2n-1$ functionally independent constants of motion (not necessarily uniform); but Corollary 3 requires $s=2n$. As we need $2n$ functionally independent generators, at least one of them cannot commute with the Hamiltonian.

For instance, this implies that no three dimensional dynamical system can present an "invariance" d.s. corresponding to A_3 [$\sim SU(4)$ and to its non-compact forms].

- b) a_{n+1} can be neither an "invariance" nor a "non-invariance" d.s. for any n dimensional dynamical system; the same holds for any a_m with $m > n$.

This is clear from Corollary 1.

- c) If a_r is realized in terms of all and only the $2n-1$ constants of motion of an n dimensional dynamical system ("maximal invariance d.s."), only one of its r Casimir operators will be independent and the Hamiltonian will be a function of this Casimir operator only.

It is enough to refer to Theorem 3 and to remember that the Casimir operators are now the indicial functions of the f.g. corresponding to a_r . As the Hamiltonian is itself a constant of motion, but cannot coincide with any generator, it has to be a function of the Casimir operators of a_r and therefore of the only Casimir operator functionally independent.

This point clarifies the fact, already pointed out in the Introduction, that the explicit realization of the generators of a d.s. for a number of special dynamical systems (hydrogen atom, harmonic oscillator, rigid rotator and so on) has always led to Casimir operators functionally dependent on only one of them; or, in other words, the fact that only special families of irreducible representations (those depending on only one parameter, as the "symmetric tensorial" ones) can be realized by an Hamiltonian possessing a d.s. of the "maximal invariance" type.

- d) If a_r is a "non-invariance" d.s. containing a "maximal invariance" sub-algebra b_r , (realized therefore in terms of the $2n-1$ functionally independent constants of motion), its r Casimir operators are multiples of the identity.

It is clear that a_r , being a "non-invariance" d.s., contains at least one generator which is not a constant of motion; on the other hand, $s \leq 2n$. Therefore $s=2n$ and we can now apply Theorem 2.

It is now clear why all "non-invariance" d.s.'s studied so far have led to Casimir operators that were numbers ¹²⁾⁻¹⁴⁾.

CONCLUSIONS

We have shown that Lie algebras associated with dynamical symmetries realized on a dynamical space $D(n)$ can be analyzed from the view-point of their properties as "function groups" on $D(n)$.

For an n dimensional dynamical system the following results are of interest:

- 1) No semi-simple Lie algebra of rank greater than n can be realized on $D(n)$.
- 2) The realization of a semi-simple Lie algebra of rank r requires at least $2r$ functionally independent generators.

- 3) If we use only $2r$ independent generators, the r Casimir operators of the algebra become multiples of the identity.
- 4) If we make use of all the $2n$ independent functions that can be defined on $D(n)$, the r Casimir operators of the algebra become multiples of the identity.
- 5) If a semi-simple Lie algebra of rank n is realized on $D(n)$, its n Casimir operators become multiples of the identity.
- 6) A semi-simple Lie algebra of rank n is necessarily a "non-invariance" dynamical symmetry.
- 7) A "maximal invariance" dynamical symmetry, making use of all and only the $2n-1$ constants of motion, has only one of its r Casimir operator independent.
- 8) A "non-invariance" d.s. containing a "maximal invariance" symmetry has Casimir operators that are multiples of the identity.

It is clear that our results are strictly limited to the classical case, where the presence of Poisson brackets among generators allows the use of the theory of function groups, as developed in the Treatises on Partial Differential Equations. It is however probable that they are valid also in the quantum case, all quantum analysis of dynamical symmetries presenting the same peculiar features as the classical analyses. In the quantum case, there will be a number of additional difficulties, however; in particular, the presence among the generators of unbounded operators corresponding (at least in the "invariance" case) to observable physical quantities ²⁴⁾.

APPENDIX ATHEOREMS ON FUNCTION GROUPS 17)

Definition: A set of $2n$ variables in two complementary sets (q_i) and (p_i) ($i=1, \dots, n$), on which a Poisson bracket $\{ \}$ is defined:

$$\{q_i, q_j\} = \{p_i, p_j\} = 0 ; \quad \{q_i, p_j\} = \delta_{ij} ,$$

will be called an n dimensional dynamical space, $D(n)$.

Definition: A set of s functionally independent functions u_i on $D(n)$, $[u]$, whose Poisson brackets can be expressed as functions of the u_i 's, defines a function group (f.g.) of order s ; every function of the u_i 's belongs to the f.g. $[u]$.

Definition: A function f on $D(n)$ is in involution with $[u]$ if $\{f, u_i\} = 0$ for every i .

Definition: A f.g. $[u]$ for which $\{u_i, u_j\} = 0$ for every i and j is called a system in involution or a commutative f.g.

(Note: a f.g. can have an unlimited number of forms, depending on the choice of the bases, u_i 's).

a) General Theorems on Function Groups:

Theorem 1: A f.g. on $D(n)$ is of order not greater than $2n$.

Theorem 2: A f.g. of order s on $D(n)$ determines another f.g. of order $2n-s$ (the reciprocal f.g.); every function of each group being in involution with the other group.

Corollary 1: The order of a commutative f.g. cannot be greater than n .

b) Indicial Functions of a Function Group

Definition: A function f belonging to $[u]$ is an indicial function of $[u]$ if $f, u_i = 0$ for every i .

Theorem 3: If the rank of the matrix $\| u_i, u_j \|$ equals $s-m$, the f.g. u has m functionally independent indicial functions.

Corollary 2: Every f.g. of odd order possesses at least one indicial function.

Theorem 4: The number of indicial functions of a f.g. is independent of the form of the f.g. (i.e., of the choice of its basis).

Theorem 5: Two reciprocal f.g.'s have the same indicial functions; if a function belongs to two reciprocal f.g.'s, it is an indicial function of both.

Corollary 3: A non-commutative f.g. of order s cannot possess more than $s-2$ indicial functions.

c) Canonical Form of a Function Group:

Theorem 6: The order of a non-commutative f.g. exceeds the number of its indicial functions by an even integer; and a f.g. of order $s=2q+m$ (with m indicial functions) can be transformed into a f.g. with basis $(Q_1, \dots, Q_{q+m}; P_1, \dots, P_q)$, where:

(canonical form of the f.g.)

Corollary 4: As the set $(Q_1, \dots, Q_{q+m})$ constitutes a commutative f.g., $q+m \equiv n$.

Theorem 7: When a f.g. of order $s=2q+m$ (with m indicial functions) is given in canonical form, it can be amplified by the

association of $2n-(m+2q)$ appropriately determined functions, so that the resulting $2n$ functions are a f.g. of order $2n$ in canonical form.

Corollary 5: A f.g. of order $n+k$ ($k \geq 0$) cannot possess more than $n-k$ indicial functions; therefore, a f.g. of order $2n$ cannot possess any indicial function.

APPENDIX B

SOME EXAMPLES OF REALIZATIONS OF SIMPLE LIE ALGEBRAS

We want here to give a few examples of realizations of simple Lie algebras, on which the general results obtained can be most readily visualized.

We shall be interested in the $s=2r$ realization, to which Theorem 1 and Theorem 4 apply.

In our $s=2r$ realization, we shall take as functionally independent the generators H_i and E_μ ($\mu=1, \dots, r$); all the other generators E_α ($\alpha=1, \dots, d-2r$) will now be functions of the set (H_i, E_μ) .

The E_μ dependence is trivial; given a root $\underline{\alpha} = C_{\alpha\mu} \underline{\mu}$, the relation $\{H_i, E_\alpha\} = \underline{\alpha} E_\alpha$ leads to

$$E_\alpha = C_\alpha(H_i) \prod_\mu E_\mu^{C_{\alpha\mu}}$$

In order to obtain the H_i dependence, one has to solve for every special case the differential equations corresponding to (1); it is easy to see that they can be resolved into differential equations for the $C_\alpha(H_i)$, whose solutions (when they exist) will in general contain a number of arbitrary constants, K_α .

Let us discuss first the simplest case: $A_1 [\sim SU(2)]$. We are looking for a realization with $s=2$ and will therefore take H and E_1 as functionally independent. It is easy to see that:

$$E_{-1} = -\frac{1}{2} (H^2 + \beta) E_1^{-1}$$

(where β is an arbitrary constant). The Casimir operator

$$C = H^2 + 2 E_1 E_{-1} = -\beta$$

is (Theorem 1 !) a number.

We go now to $A_2 [\sim SU(3)]$. We need four independent generators and we shall choose (H_1, H_2, E_1, E_2) , where the notation of ²²⁾ has been followed.

The system of differential equations corresponding to the commutation relations of A_2 gives the following solution:

$$E_3 = \left[\frac{1}{\sqrt{2}} \left(-H_1 + \frac{H_2}{\sqrt{3}} \right) + K \right] E_1^{-1} E_2$$

$$E_{-1} = \left[\frac{1}{2} \left(-H_1^2 + \frac{H_2^2}{3} \right) + \sqrt{\frac{2}{3}} K H_2 + K^2 \right] E_1^{-1}$$

$$E_{-2} = \left[-\frac{1}{\sqrt{3}} H_2 \left(H_1 + \frac{H_2}{\sqrt{3}} \right) + \frac{K}{\sqrt{2}} \left(H_1 - \frac{H_2}{\sqrt{3}} \right) + K^2 \right] E_2^{-1}$$

$$E_{-3} = \left[-\sqrt{\frac{2}{3}} H_2 + K \right] E_1 E_2^{-1}$$

(where K is an arbitrary constant). Of the two Casimir operators, the quadratic one is well known:

$$C_2 = H_1^2 + H_2^2 + 2 E_\alpha E_{-\alpha} = 6 K^2$$

In order to find an expression for the cubic Casimir operator, one can use the general expression given in ²⁵⁾ in terms of the "self-representation" of A_2 (for which we take the λ representation given by Gell-Mann ²⁶⁾) and transform it by going to the canonical representation for the generators. One obtains:

$$\begin{aligned} C_3 &= -H_2^3 + 3 H_1^2 H_2 + 6 H_2 E_1 E_{-1} + 3(\sqrt{3} H_1 - H_2) E_2 E_{-2} + \\ &\quad - 3(\sqrt{3} H_1 + H_2) E_3 E_{-3} + 3\sqrt{6} (E_2 E_{-1} E_{-3} + E_1 E_{-2} E_3) = \\ &= 6 \sqrt{6} K^3 \end{aligned}$$

Once again (Theorem 1!) the two Casimir operators are just numbers.

These generators, when suitably transformed into a non-canonical representation, coincide with those recently proposed by Rosen ¹³⁾.

One could now try and realize B_2 and G_2 with $s=4$, as we have done with A_2 . But B_2 contains as a sub-algebra $D_2=A_1 \oplus A_1$; and G_2 contains as a sub-algebra A_2 . Theorem 4 therefore holds and assures us that no such realization is possible.

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