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FAMILIES OF SUM RULES FROM CURRENT ALGEBRA ^{*)}

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A B S T R A C T

By making use of a simple method we obtain different families of sum rules from current algebra. The method allows to understand the relation of the reference frame in which equal time commutators are considered with the form of the sum rules obtained and dispersion relations. The different contributions to the sum rules are then analyzed and it is recognized that besides the contributions which are generally considered, there are others which come from disconnected intermediate states, which play an important role. By discussing in detail the $P \rightarrow \infty$ sum rule we put in evidence the physical content of these extra terms and how they are responsible for the automatic cancellation of the singularities in the variables which, due to the locality of equal time commutators, do not appear in the right-hand side of the sum rules. The origin and value of these terms is purely dynamical, they have nothing to do with Schwinger and their presence is not related to convergence of the sum rule integrals. It is shown that the assumption that they vanish (so to recover the Fubini sum rule for the $P \rightarrow \infty$ family) implies superconvergence so that it appears clearly that superconvergence is a purely dynamical assumption and is not related to current algebra or locality of equal time commutators. The structure of other families of sum rules is also discussed and, in particular, that corresponding to the " $\Delta \rightarrow \infty$ " family which appears to be of particular interest.

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In this paper we discuss the general structure of the sum rules that can be obtained from current algebra. As was already recognized, the dependence of the equal time commutator on the reference frame allows to obtain different families of sum rules ¹⁾. Of these different families, two which have particularly simple properties are investigated in more detail ; they are the ones corresponding to the $P \rightarrow \omega$ and $\Delta \rightarrow \omega$ frames of reference. The content of these sum rules is then discussed and it is recognized that some contributions which were generally discarded (being "infinite mass" contributions in the language of Ref. ²⁾ or subtraction terms in that of Ref. ³⁾) play a very natural and important role. In particular, it is shown that they automatically guarantee the absence of singularities in other variables (k_1^2 and k_2^2 for the $P \rightarrow \omega$ sum rules). If these "infinite mass" contributions are ad hoc supposed to vanish - assumption which, being purely dynamical, has no relation with current algebra or locality of current commutators - then one is automatically assuming that all singularities of these contributions also vanish and this - as shown - amounts to assume the superconvergence relations ⁴⁾.

If the superconvergence relations are assumed then it is shown that the Fubini-Dashen-Gell-Mann (FDG) sum rules ^{2,3)} are recovered for the $P \rightarrow \omega$ frame. Similarly, for the $\Delta \rightarrow \omega$ frame the sum rules proposed by us before ⁵⁾ are obtained if superconvergence of form factors is assumed.

However, perfectly defined sum rules are obtained even if superconvergence does not hold. For the $P \rightarrow \omega$ frame, these sum rules differ from the FDG ones by presenting extra terms which are related to the asymptotic behaviour of hadron production amplitudes. These are just the amplitudes which are supposed to vanish asymptotically in a superconvergent theory.

In Section 2 we shall describe a simple method in order to obtain sum rules from current algebra. This method exhibits explicitly the dependence of the form of the sum rules on the reference frame.

In Section 3, the possible contributions of the sum rules are discussed in terms of physical intermediate states.

In Sections 4 and 5, the $P \rightarrow \infty$ frame sum rules are discussed in more detail. In Section 4 the different contributions are evaluated in terms of physical amplitudes (or - actually - absorptive parts of physical amplitudes) and in Section 5 it is shown that the sum of all contributions is automatically free of singularities in the variables $(k_1^2$ and $k_2^2)$ which - due to locality - do not appear in the matrix element of equal time commutators. The t singularities of both sides of the sum rule are also discussed.

In Section 6, the same procedure is applied to the $\Delta \rightarrow \infty$ frame sum rules.

Finally, in Section 7, we show the absolute analogy between the method developed here and the dispersive approach. It is shown that the disconnected intermediate states that give rise to the "infinite mass contributions" (in the language of Ref. ²⁾) in the $P \rightarrow \infty$ frame appear as subtraction terms in a dispersive approach. More exactly, they appear as contributions in which the dispersive variable is not the energy ν but is related to the square of the momenta transferred to the currents.

2. -

THE SUM RULES

We start from the equal time commutation relations between vector and/or axial currents of the form

$$\left[j_\sigma^\alpha\left(\frac{x}{2}\right), j_\sigma^\beta\left(-\frac{x}{2}\right) \right] \delta(x_0) = \left\{ f^{\alpha\beta\gamma} j_\sigma^\gamma(0) + i \sum^{\alpha\beta} (0) \delta_{\sigma i} \frac{\partial}{\partial x_\sigma} \right\} \delta^4(x) \quad (2.1)$$

where α, β, γ are isospin or SU_3 indices and $\sum^{\alpha\beta}(x)$ is the (eventual) operator occurring in the Schwinger term. We use the same symbols for V and A currents, with the understanding that if both currents are V (or A) in the left-hand side of Eq. (2.1) then j_σ^γ in the right-hand side is V, while if one is V and one A in the left j_σ^γ is A in the right.

Let us consider the matrix element of the commutator of two currents between states with momenta p_1 and p_2 respectively and let us define its Fourier transform, i.e.,

$$t_{p\sigma}^{\alpha\beta}(K, K_0) = \int d^4x e^{iKx} \langle 2 | [j_\sigma^\alpha\left(\frac{x}{2}\right), j_\sigma^\beta\left(-\frac{x}{2}\right)] | 1 \rangle \quad (2.2)$$

For convenience of further use, we introduce the following kinematical notations :

$$\begin{aligned} P &= \frac{1}{2}(p_1 + p_2), \quad \Delta = \frac{1}{2}(p_1 - p_2), \quad k_1 = K - \Delta, \quad k_2 = K + \Delta, \\ p_1 + k_1 &= p_2 + k_2, \\ s &= (P + K)^2, \quad u = (P - K)^2, \quad t = 4\Delta^2, \\ v &= 2PK = \frac{1}{2}(s - u), \quad v = 2\Delta K = \frac{1}{2}(k_2^2 - k_1^2) \\ \zeta_1 &= k_{1x}^2 + k_{1y}^2, \quad \zeta_2 = k_{2x}^2 + k_{2y}^2, \quad \bar{s} = (p_x + k_x)^2 + (p_y + k_y)^2, \\ \bar{u} &= (p_x - k_x)^2 + (p_y - k_y)^2. \end{aligned} \quad (2.3)$$

Following a method proposed by Bollini and Giambiagi ⁶⁾, we consider the matrix element of Eq. (2.1) between $\langle 2|$ and $|1\rangle$, and then we take the Fourier transform of both sides. Remembering that the Fourier transform of the product of two distributions is the convolution of the Fourier transform of the two factors, one obtains ⁶⁾

$$\frac{1}{2\pi} \int t_{\rho\sigma}^{\alpha\beta}(\vec{k}, k_0) dk_0 = \int_0^{\alpha\beta\gamma} \langle 2| j_{\sigma}^{\gamma}(0) |1\rangle + \langle 2| \sum^{\alpha\beta} (0) |1\rangle K_{\sigma} \delta_{\sigma i} \quad (2.4)$$

In Eq. (2.4), the integral runs over k_0 for every $\vec{k}$ and the result must be independent of $\vec{k}$, except for the trivial linear dependence on $\vec{k}$ of the right-hand side. Higher order Schwinger terms [i.e., higher order derivatives of $\vec{k}$ in the right-hand side of Eq. (2.1)] would simply give rise to corresponding higher powers of $\vec{k}$ in the right-hand side of Eq. (2.4). Equation (2.4) must hold in any frame of reference. As it involves not manifestly covariant quantities, different frames are expected to give different information. For the sake of definiteness we take in Eq. (2.1) j_{ρ}^{α} , j_{σ}^{β} and so j_{σ}^{γ} , to be of vector type, and the states $|1\rangle$, $|2\rangle$ in Eq. (2.2) to be spinless particles of equal mass, for instance pions.

We can then define

$$\langle 2| j_{\sigma}^{\gamma}(0) |1\rangle = F_1^{\gamma}(t) P_{\sigma} + F_2^{\gamma}(t) \Delta_{\sigma} \quad (2.5)$$

(actually $F_2^{\gamma}(t) = 0$, due to current conservation), and

$$\langle 2| \sum^{\alpha\beta} (0) |1\rangle = S^{\alpha\beta}(t)$$

As $t_{\rho\sigma}$ is a tensor under Lorentz transformations for spinless particles, we have the decomposition

$$\begin{aligned} t_{\rho\sigma} = & a P_{\rho} P_{\sigma} + \frac{1}{2} a_1 (P_{\rho} K_{\sigma} + P_{\sigma} K_{\rho}) + \frac{1}{2} a_2 (P_{\rho} \Delta_{\sigma} + P_{\sigma} \Delta_{\rho}) \\ & + \frac{1}{2} a_3 (P_{\rho} K_{\sigma} - P_{\sigma} K_{\rho}) + \frac{1}{2} a_4 (P_{\rho} \Delta_{\sigma} - P_{\sigma} \Delta_{\rho}) \\ & + b \Delta_{\rho} \Delta_{\sigma} + \frac{1}{2} b_1 (\Delta_{\rho} K_{\sigma} + \Delta_{\sigma} K_{\rho}) + \frac{1}{2} b_2 (\Delta_{\rho} K_{\sigma} - \Delta_{\sigma} K_{\rho}) \\ & + c K_{\rho} K_{\sigma} + d \delta_{\rho\sigma} \end{aligned} \quad (2.6)$$

where a, a_1 , etc., are invariant quantities. According to the definitions of Eqs. (2.3), they are in general function of the four scalars t, v and k_1^2, k_2^2 or t, v, s, u :

$$a = a(t, v, k_1^2, k_2^2) = a(t, v, s, u) \quad \text{etc.} \quad (2.7)$$

The left-hand side of the sum rule Eq. (2.4) involves a linear combination of those invariant functions, for values of the arguments v, k_1^2, k_2^2 (or v, s, u) which depend on K_0 and $\vec{K}$, the former being the integration variable and the latter an arbitrary but fixed vector, as indicated in Eq. (3.3). Changing the reference frame changes the dependence of the variables k_1^2, k_2^2 or s, u on the integration parameter K_0 , and therefore different contributions are picked up in different frames during the integration process. The explicit form of those contributions shall be the content of the next Section.

The four vector P_μ (see Eq. (1.3)) can be timelike or spacelike.

In the first case we can define a Breit system as the one in which $\vec{P} = 0$, and therefore $\Delta_0 = 0$ and $P_0 = \sqrt{-P^2}$. Any other system can be obtained from the Breit one by a Lorentz transformation characterized by a relative velocity β and a direction which we define to be the z -direction. If, in the Breit system,

$$P = (\sqrt{-P^2}, 0, 0, 0),$$

$$\Delta = (0, \Delta_x, \Delta_y, \Delta_z),$$

in the general one

$$P = (\gamma\sqrt{-P^2}, 0, 0, -\beta\gamma\sqrt{-P^2}), \quad (2.8)$$

$$\Delta = (-\beta\gamma\Delta_z, \Delta_x, \Delta_y, \gamma\Delta_z),$$

where, as usual, $\gamma = (1 - \beta^2)^{-\frac{1}{2}}$. K_0 and $\vec{K}$, which appear in Eq. (2.4), are of course the components of K in the system in question. From Eq. (2.3), we have

$$K_0 = -\frac{v}{2\gamma\sqrt{-P^2}} - \beta K_z \quad (2.9)$$

and correspondingly

$$\begin{aligned}
k_1^2 &= \zeta_1 + (K_z - \gamma \Delta_z)^2 - \left(\frac{\nu}{2\gamma\sqrt{-p^2}} + \beta K_z \right)^2, \\
k_2^2 &= \zeta_2 + (K_z + \gamma \Delta_z)^2 - \left(\frac{\nu}{2\gamma\sqrt{-p^2}} + \beta K_z \right)^2.
\end{aligned}
\tag{2.10}$$

The integration in Eq. (2.4) is done at fixed values of $\vec{K}$ (and ζ_1, ζ_2) ; it is therefore clear from Eqs. (2.3), (2.10) that k_1^2, k_2^2 vary with ν along the integration.

We define the $P_0 \rightarrow \infty$ system, which proves to be particularly interesting, by

$$\Delta_z = 0, \quad \beta \rightarrow 1 \tag{2.11}$$

In this system

$$P_0 \rightarrow \infty, \quad P_z \rightarrow -P_0, \quad \Delta_0 = \Delta_z = 0$$

and, as far as finite values of ν are concerned, $K_0 \rightarrow -K_z$, so that Eqs. (2.10) reduce to

$$\begin{aligned}
k_1^2 &= \zeta_1 \\
k_2^2 &= \zeta_2
\end{aligned}
\tag{2.12}$$

In other words, in the $P_0 \rightarrow \infty$ system contributions from finite values of ν enter in the sum rule with k_1^2 and k_2^2 fixed at the values ζ_1 and ζ_2 , independent of ν *).

*) The same discussion applies even if $|1\rangle$ and $|2\rangle$ have different masses, provided that Δ is spacelike and the "Breit system" is defined as the one in which

$$\begin{aligned}
P &= (P_0, P\Delta/\Delta_x, 0, 0) \\
\Delta &= (0, \Delta_x, \Delta_y, \Delta_z).
\end{aligned}$$

We have studied up to now the kinematical situation in which P is timelike (Δ spacelike). The other physical situation is the one for which Δ is timelike, and P spacelike. For this case, we can repeat the preceding discussion, with the substitution $P_\mu, k_1^2, k_2^2 \rightarrow \Delta_\mu, s, u$. Define a "Breit" frame by

$$P = (0, P_x, P_y, P_z)$$

$$\Delta = (\sqrt{-\Delta^2}, 0, 0, 0) .$$

Given any system, obtained through a Lorentz transformation with relative velocity β in the z direction, we have

$$K_0 = -\frac{v}{2\gamma\sqrt{-\Delta^2}} - \beta K_z$$

Then, analogously to Eqs. (2.10)

$$s = \bar{s} + (K_z + \gamma P_z)^2 - \left(\frac{v}{2\gamma\sqrt{-\Delta^2}} + \beta K_z\right)^2 ,$$

$$u = \bar{u} + (K_z - \gamma P_z)^2 - \left(\frac{v}{2\gamma\sqrt{-\Delta^2}} + \beta K_z\right)^2 ,$$

i.e., s and u vary with v along the integration. We define the $\Delta_0 \rightarrow \infty$ system by

$$\beta \rightarrow 1, \quad P_0 = P_z = 0, \quad \Delta_z \rightarrow -\Delta_0. \quad (2.13)$$

In this system, as far as finite values of v are concerned

$$\begin{aligned} s &= \bar{s} , \\ u &= \bar{u} , \end{aligned} \quad (2.14)$$

i.e., s and u become independent of v and fixed at the values $\bar{s}, \bar{u}$.

It must be noted that the $P_0 \rightarrow \omega$ and $\Delta_0 \rightarrow \omega$ systems just discussed are nothing but two particular cases of reference frames in which sum rules can be analyzed. Any other system could in principle be considered on the same footing. The $P_0 \rightarrow \omega$ and $\Delta_0 \rightarrow \omega$ systems are, in our opinion, the most easy to handle, as in those systems there are wide classes of contributions to the sum rules for fixed values of k_1^2 and k_2^2 or s and u respectively.

3. - CONTRIBUTIONS TO THE SUM RULES

Let us now discuss the values of the integration variable K_0 for which there are contributions to the sum rules.

By inserting a complete set of states $|m\rangle$ between the two currents of Eq. (2.2), we have

$$t_{\rho\sigma}^{\alpha\beta}(\vec{K}, K_0) = \sum_m (2\pi)^4 \delta^4(P+K-P_m) \langle 2 | j_\rho^\alpha(0) | m \rangle \langle m | j_\sigma^\beta(0) | 1 \rangle - \\ - \sum_{m'} (2\pi)^4 \delta^4(P-K-P_{m'}) \langle 2 | j_\sigma^\beta(0) | m' \rangle \langle m' | j_\rho^\alpha(0) | 1 \rangle \quad (3.1)$$

and the discussion of the contribution to the sum rules can be better carried out in terms of intermediate states. We will have in general contributions from any physical state with the appropriate quantum numbers (according to the quantum numbers associated to the currents and to the external states). The space components of the δ function in Eq. (3.1) impose to the intermediate state a momentum $\vec{P} + \vec{K}$ for the first term in the right-hand side and $\vec{P} - \vec{K}$ for the second. The structure of the intermediate state determines its energy in terms of its space momentum, and the fourth component of the δ function specifies which is the corresponding value of K_0 .

Let us visualize the structure of the intermediate states by graphs. In these graphs straight lines will represent physical states, sliced intermediate lines the sum over polarization, a wiggled line attached to a "bubble" the matrix element of a current between incoming and outgoing lines. For simplicity, we will take $K_z = 0$.

To begin with, we have the graphs

Fig. 1a

Fig. 1b

Graphs (1a), (1b) correspond to the first and second term in Eq. (3.1) respectively. In (1a), the intermediate state consists of the state n_1 , with four-momentum $P + K$ and $P_0 + K_0 > 0$. If m_s is its mass, it follows

$$K_0 = \sqrt{(\vec{P} + \vec{K})^2 + m_s^2} - P_0, \quad s = -m_s^2, \quad dK_0 = \frac{dm_s^2}{2E_s} \quad (3.2)$$

$$\text{where } E_s = \sqrt{(\vec{P} + \vec{K})^2 + m_s^2}.$$

If $P_0 \rightarrow \infty$:

$$K_0 \rightarrow 0, \quad E_s \rightarrow P_0, \quad v = -m_s^2 - \vec{K}^2, \quad k_1^2 = \xi_1, \quad k_2^2 = \xi_2. \quad (3.3)$$

If $\Delta_0 \rightarrow \infty$:

$$K_0 = E_s = \sqrt{\vec{S} + m_s^2}, \quad v = -2\Delta_0 \sqrt{\vec{S} + m_s^2} \rightarrow -\infty, \quad u = \bar{u} - \vec{S} - m_s^2 \quad (3.4)$$

In (1b) the intermediate state is the state n_2 , with four-momentum $P - K$, $P_0 - K_0 > 0$. If m_u is its mass, it follows

$$K_0 = -\sqrt{(\vec{P} - \vec{K})^2 + m_u^2} + P_0, \quad u = -m_u^2, \quad dK_0 = \frac{dm_u^2}{2E_u} \quad (3.5)$$

$$\text{where } E_u = \sqrt{(\vec{P} - \vec{K})^2 + m_u^2}.$$

If $P_0 \rightarrow \infty$:

$$K_0 \rightarrow 0, \quad E_u \rightarrow P_0, \quad v = m_u^2 + \vec{K}^2, \quad k_1^2 = \xi_1, \quad k_2^2 = \xi_2. \quad (3.6)$$

If $\Delta_0 \rightarrow \infty$

$$-K_0 = E_u = \sqrt{\bar{u} + m_u^2}, \quad v = 2\Delta_0 \sqrt{\bar{u} + m_u^2} \rightarrow \infty, \quad s = \bar{s} - \bar{u} - m_u^2 \quad (3.7)$$

Graphs (1a), (1b) give in the $P_0 \rightarrow \infty$ system the contributions at finite v which were referred to in the previous Section.

A second class of graphs is given by

Fig. 2a

Fig. 2b

In (2a), the intermediate state corresponding to $|m\rangle$ of Eq. (3.1) is disconnected. It consists of the final particle together with the state h_2 , i.e., $|m\rangle = |h_2\rangle|2\rangle$ and therefore

$$\langle 2 | j_P^\alpha(0) | m \rangle = \langle 2 | 2 \rangle \langle 0 | j_P^\alpha(0) | h_2 \rangle$$

while $\langle m | j_\sigma^\beta(0) | 1 \rangle$ is the amplitude for the production of the hadronic state $|h_2\rangle$. The total momentum of the intermediate state $|m\rangle$ being $P + K$, the momentum of $|h_2\rangle$ is $k_2 = K + \Delta$, with $K_0 + \Delta_0 > 0$. If μ_2 is the mass of $|h_2\rangle$, it follows

$$K_0 = \sqrt{(\vec{\Delta} + \vec{K})^2 + \mu_2^2} - \Delta_0, \quad k_1^2 = -\mu_2^2, \quad dK_0 = \frac{d\mu_2^2}{2E_2}$$

where

$$E_2 = \sqrt{(\vec{\Delta} + \vec{K})^2 + \mu_2^2}. \quad (3.8)$$

If $P_0 \rightarrow \infty$:

$$K_0 = E_2 = \sqrt{\xi_2 + \mu_2^2}, \quad v = -2P_0 \sqrt{\xi_2 + \mu_2^2} \rightarrow -\infty, \quad k_1^2 = \xi_1 - \xi_2 - \mu_2^2 \quad (3.9)$$

Further the invariant mass of the intermediate state is $-(P+K)^2 = -P^2 + 2P_0 \sqrt{\xi_2 + \mu_2^2} - (\xi_2 + \mu_2^2) \rightarrow \infty$, i.e., in the $P_0 \rightarrow \infty$ limit the invariant mass of the intermediate state $|m\rangle$ as a whole goes to infinity.

If $\Delta_0 \rightarrow \infty$:

$$K_0 \rightarrow 0, \quad E_2 \rightarrow \Delta_0, \quad v = -\mu_2^2 - K^2, \quad s = \bar{s}, \quad u = \bar{u}. \quad (3.10)$$

The structure of the graph (2b) is very similar to the previous one. The intermediate state corresponding to $|m'\rangle$ of Eq. (3.1) is disconnected, and consists of the final particle together with the state $|h_1\rangle$. The total momentum of the intermediate state being $P - K$ the momentum of $|h_1\rangle$ is $-k_1 = \Delta - K$, with $\Delta_0 - K_0 > 0$. If μ_1 is the mass of $|h_1\rangle$, it follows :

$$K_0 = -\sqrt{(\vec{\Delta} - \vec{K})^2 + \mu_1^2} + \Delta_0, \quad k_1^2 = -\mu_1^2, \quad dK_0 = \frac{d\mu_1^2}{2E_1},$$

$$E_1 = \sqrt{(\vec{\Delta} - \vec{K})^2 + \mu_1^2}. \quad (3.11)$$

If $P_0 \rightarrow \infty$:

$$-K_0 = E_1 = \sqrt{\xi_1 + \mu_1^2}, \quad v = 2P_0 \sqrt{\xi_1 + \mu_1^2} \rightarrow \infty, \quad k_2^2 = \xi_2 - \xi_1 - \mu_1^2 \quad (3.12)$$

and the invariant mass of the intermediate state as a whole $-(P-K)^2$ becomes infinite.

If $\Delta_0 \rightarrow \infty$:

$$K_0 \rightarrow 0, E_1 \rightarrow \Delta_0, v = \mu_1^2 + \vec{K}^2, s = \bar{s}, u = \bar{u}. \quad (3.13)$$

Graphs (2a), (2b) give in the $\Delta_0 \rightarrow \infty$ system the contributions at finite v which we referred to in the previous Section.

In correspondence with (1b) and (1a) there is a third class of graphs

Fig. 3a

Fig. 3b

where $\bar{n}_1, \bar{n}_2$ are the antiparticle states of n_1, n_2 . The intermediate states in this case consist of the two external particles together with the states $|\bar{n}_1\rangle, |\bar{n}_2\rangle$ respectively. In graph (3a), $\bar{n}_2$ has momentum $K - P$, so that $K_0 - P_0 > 0$ and

$$K_0 = \sqrt{(\vec{P} - \vec{K})^2 + m_u^2} + P_0, u = -m_u^2, \int dK_0 = \int \frac{dm_u^2}{2E_u} \quad (3.14)$$

where E_u is the same as in (1b).

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If $P_0 \rightarrow \infty$:

$$K_0 \rightarrow 2E_u = 2P_0, \nu \rightarrow -4P_0^2 \rightarrow -\infty, k_1^2 \rightarrow -4P_0^2, k_2^2 \rightarrow -4P_0^2 \quad (3.15)$$

and the invariant mass $-(P+K)^2$ of the whole intermediate state becomes infinite.

If $\Delta_0 \rightarrow \infty$:

$$K_0 = E_u = \sqrt{\bar{u} + m_u^2}, \nu = -2\Delta_0 \sqrt{\bar{u} + m_u^2} \rightarrow -\infty, s = \bar{s} - \bar{u} - m_u^2 \quad (3.16)$$

In graph (3b), $\bar{n}_1$ has momentum $-(P+K)$, so that $-(P_0+K_0) > 0$ and

$$K_0 = -\sqrt{(\vec{P}+\vec{K})^2 + m_s^2} - P_0, s = -m_s^2, \int dK_0 = \int \frac{dm_s^2}{2E_s} \quad (3.17)$$

where E_s is the same as in (1a).

If $P_0 \rightarrow \infty$:

$$K_0 \rightarrow -2E_s = -2P_0, \nu \rightarrow 4P_0^2 \rightarrow \infty, k_1^2 \rightarrow -4P_0^2, k_2^2 \rightarrow -4P_0^2 \quad (3.18)$$

and the invariant mass $-(P-K)^2$ of the intermediate state becomes infinite.

If $\Delta_0 \rightarrow \infty$:

$$-K_0 = E_s = \sqrt{\bar{s} + m_s^2}, \nu = 2\Delta_0 \sqrt{\bar{s} + m_s^2} \rightarrow \infty, u = \bar{u} - \bar{s} - m_s^2 \quad (3.19)$$

Finally, in correspondence with (2b), (2a), we have

Fig. 4a

Fig. 4b

where $\bar{h}_1, \bar{h}_2$ are the antiparticle states of h_1, h_2 . The intermediate states in this case consist of the initial particle together with the states $|\bar{h}_1\rangle, |\bar{h}_2\rangle$ respectively. In graph (4a), $\bar{h}_1$ has momentum $K - \Delta$, so that $K_0 - \Delta_0 > 0$ and

$$K_0 = \sqrt{(\vec{\Delta} - \vec{K})^2 + \mu_1^2} + \Delta_0, \quad k_1^2 = -\mu_1^2, \quad dK_0 = \frac{d\mu_1^2}{2E_1} \quad (3.20)$$

where E_1 is the same as in (2b).

If $P_0 \rightarrow \infty$:

$$K_0 = E_1 = \sqrt{\xi_1 + \mu_1^2}, \quad \nu = -2P_0 \sqrt{\xi_1 + \mu_1^2} \rightarrow -\infty, \quad k_2^2 = \xi_2 - \xi_1 - \mu_1^2 \quad (3.21)$$

and the invariant mass $-(P+K)^2$ of the intermediate state becomes infinite.

If $\Delta_0 \rightarrow \infty$:

$$K_0 \rightarrow 2E_1 = 2\Delta_0, \quad \nu \rightarrow -4\Delta_0^2 \rightarrow -\infty, \quad s \rightarrow -4\Delta_0^2, \quad u \rightarrow -4\Delta_0^2 \quad (3.22)$$

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In graph (4b), $\bar{h}_2$ has momentum $-(K+\Delta)$, so that $-(K_0+\Delta_0) > 0$ and

$$K_0 = -\sqrt{(\vec{\Delta}+\vec{K})^2 + \mu_2^2} - \Delta_0, \quad k_2^2 = -\mu_2^2, \quad dK_0 = \frac{d\mu_2^2}{2E_2} \quad (3.23)$$

where E_2 is the same as in (2a).

If $P_0 \rightarrow \infty$:

$$-K_0 = E_2 = \sqrt{\xi_2^2 + \mu_2^2}, \quad \nu = 2P_0 \sqrt{\xi_2^2 + \mu_2^2} \rightarrow \infty, \quad k_1^2 = \xi_1^2 - \xi_2^2 - \mu_2^2 \quad (3.24)$$

and the invariant mass of the intermediate state becomes infinite.

If $\Delta_0 \rightarrow \infty$:

$$K_0 \rightarrow -2E_2 \rightarrow -2\Delta_0, \quad \nu \rightarrow 4\Delta_0^2 \rightarrow \infty, \quad s \rightarrow -4\Delta_0^2, \quad u \rightarrow -4\Delta_0^2 \quad (3.25)$$

4. - CONTENT OF THE $P_0 \rightarrow \omega$ SUM RULE

In this Section we study the content of the sum rule Eq. (2.4) in the $P_0 \rightarrow \omega$ system. For simplicity, we will consider only $G = 0$ in Eq. (2.4) (i.e., the commutator of two fourth components of currents) avoiding therefore the effects of eventual Schwinger terms, and we will drop the $SU(3)$ indices. Furthermore, we call $C(1,a)$, $C(1,b)$, etc., the contributions to the sum rule of the intermediate states whose structure is represented by the graphs (1a), (1b), etc., writing therefore

$$\frac{1}{2\pi} \int t_{\infty}(\vec{k}, k_0) dK_0 = \sum_{v=1}^4 [C(v,a) - C(v,b)] . \quad (4.1)$$

In order to study the contributions from the various classes of graphs, it is convenient to introduce the amplitude

$$T_{\rho\sigma}(\vec{k}, k_0) = \int d^4x e^{ikx} \theta(x_0) \langle 2 | [j_\rho(\frac{x}{2}), j_\sigma(-\frac{x}{2})] | 1 \rangle , \quad (4.2)$$

which can be graphically visualized by

Fig. 5

Let us write the usual decomposition

$$\begin{aligned}
 T_{p\sigma} = & A P_p P_\sigma + \frac{1}{2} A_1 (P_p K_\sigma + P_\sigma K_p) + \frac{1}{2} A_2 (P_p \Delta_\sigma + P_\sigma \Delta_p) \\
 & + \frac{1}{2} A_3 (P_p K_\sigma - P_\sigma K_p) + \frac{1}{2} A_4 (P_p \Delta_\sigma - P_\sigma \Delta_p) \\
 & + B \Delta_p \Delta_\sigma + \frac{1}{2} B_1 (\Delta_p K_\sigma + \Delta_\sigma K_p) + \frac{1}{2} B_2 (\Delta_p K_\sigma - \Delta_\sigma K_p) \\
 & + C K_p K_\sigma + D \delta_{p\sigma}
 \end{aligned}
 \tag{4.3}$$

The invariants A, A_1 , etc., depend on the scalars t, v, k_1^2, k_2^2 : $A = A(t, v, k_1^2, k_2^2)$, etc. It is easily seen that the various graphs of Section 3 are the singularities of T . The graphs (1a), (1b) and (3a), (3b) give the singularities in s and u (i.e., in v), graphs (2a), (2b) and (4a), (4b) the singularities in k_1^2, k_2^2 (i.e., in v).

Let us indicate the discontinuity of the function $f(x, y, \dots)$ with respect to the variable x at the point x_0 by

$$D_x f(x_0, y, \dots) \equiv \frac{i}{2\pi} [f(x_0 + i\epsilon, y, \dots) - f(x_0 - i\epsilon, y, \dots)]$$

So equipped, we can examine the $P_0 \rightarrow \infty$ sum rules. From Eq. (3.3) and Eq. (3.6), we write

$$\begin{aligned}
 C(1a) - C(1b) &= \int \frac{dm_s^2}{2E_s} (D_s A) P_0^2 - \int \frac{dm_u^2}{2E_u} (D_u A) P_0^2 \\
 &= \frac{P_0}{2} \int dv' D_v A(t, v', \zeta_1, \zeta_2)
 \end{aligned}$$

Assuming that the integral converge, and, defining

$$\mathcal{A}(t, k_1^2, k_2^2) \equiv \int dv' D_v A(t, v', k_1^2, k_2^2) \tag{4.4}$$

we have for large ν the behaviour ^{*})

$$A(t, \nu, k_1^2, k_2^2) \rightarrow \frac{1}{\nu} \mathcal{A}(t, k_1^2, k_2^2) \quad (4.5)$$

Therefore

$$C(1, a) - C(1, b) = \frac{P_0}{2} \mathcal{A}(t, \zeta_1, \zeta_2) \quad (4.6)$$

Similarly, making use of Eqs. (3.9) and (3.12), graph (2a) gives in the $P_0 \rightarrow \infty$ limit

$$C(2, a) = \frac{P_0}{2} \lim_{\nu \rightarrow -2P_0 \sqrt{\zeta_2 + \mu_2^2}} \left(\frac{d\mu_2^2}{\sqrt{\zeta_2 + \mu_2^2}} \left[P_0 D_{k_2^2} A + \sqrt{\zeta_2 + \mu_2^2} D_{k_2^2} A_1 \right] \right)$$

where the functions $D_{k_2^2} A$, etc., are evaluated at $k_1^2 = \zeta_1 - \zeta_2 - \mu_2^2$, $k_2^2 = -\mu_2^2$.

Analogously, graphs (2b), (4a), (4b) give in the $P_0 \rightarrow \infty$ system

$$C(2, b) = \frac{P_0}{2} \lim_{\nu \rightarrow 2P_0 \sqrt{\zeta_1 + \mu_1^2}} \left(\frac{d\mu_1^2}{\sqrt{\zeta_1 + \mu_1^2}} \left[P_0 D_{k_1^2} A - \sqrt{\zeta_1 + \mu_1^2} D_{k_1^2} A_1 \right] \right)$$

^{*}) In general, for large ν $A(\nu) \rightarrow A(\infty) + \frac{\mathcal{A}}{\nu}$. $A(\infty)$ and $\mathcal{A}$ cannot be both different from zero if $A(\nu)$ has definite crossing properties for $\nu \rightarrow -\nu$. It is easy to see that the sum rule picks up only the part of A odd under $\nu \rightarrow -\nu$, and a possible constant $A(\infty)$ would play no role in the following.

where $D_{k_1^2} A$, etc., are evaluated at $k_1^2 = -\mu_1^2$, $k_2^2 = \zeta_2 - \zeta_1 - \mu_1^2$,

$$C(4,a) = \frac{P_0}{2} \lim_{\nu \rightarrow -2P_0\sqrt{\zeta_1+\mu_1^2}} \int \frac{d\mu_1^2}{\sqrt{\zeta_1+\mu_1^2}} \left[P_0 D_{k_1^2} A + \sqrt{\zeta_1+\mu_1^2} D_{k_1^2} A_1 \right]$$

where $D_{k_2^2} A$, etc., are evaluated at $k_1^2 = -\mu_1^2$, $k_2^2 = \zeta_2 - \zeta_1 - \mu_1^2$,
and finally

$$C(4,b) = \frac{P_0}{2} \lim_{\nu \rightarrow 2P_0\sqrt{\zeta_2+\mu_2^2}} \int \frac{d\mu_2^2}{\sqrt{\zeta_2+\mu_2^2}} \left[P_0 D_{k_2^2} A - \sqrt{\zeta_2+\mu_2^2} D_{k_2^2} A_1 \right]$$

where $D_{k_2^2} A$, etc., are evaluated at $k_1^2 = \zeta_1 - \zeta_2 - \mu_2^2$, $k_2^2 = -\mu_2^2$.

Recalling Eq. (4.5) and putting ^{*})

$$\lim_{\nu \rightarrow \pm\infty} A_1 = \mathcal{A}_1$$

we have

$$\begin{aligned} C(2,a) - C(2,b) + C(4,a) - C(4,b) = \\ = -\frac{P_0}{2} \int \frac{d\mu_1^2}{\zeta_1+\mu_1^2} D_{k_1^2} \mathcal{A}(t, -\mu_1^2, \zeta_2-\zeta_1-\mu_1^2) \\ - \frac{P_0}{2} \int \frac{d\mu_2^2}{\zeta_2+\mu_2^2} D_{k_2^2} \mathcal{A}(t, \zeta_1-\zeta_2-\mu_2^2, -\mu_2^2) \\ + P_0 \int d\mu_1^2 D_{k_1^2} \mathcal{A}_1(t, -\mu_1^2, \zeta_2-\zeta_1-\mu_1^2) \\ + P_0 \int d\mu_2^2 D_{k_2^2} \mathcal{A}_1(t, \zeta_1-\zeta_2-\mu_2^2, -\mu_2^2) \end{aligned} \quad (4.7)$$

^{*}) Note that the crossing properties of A and A_1 are opposite ³⁾.
The sum rule indeed picks up the part of A_1 symmetric for $\nu \rightarrow -\nu$.

Finally, the graphs (3a), (3b) (the so-called "zeta" graphs), as shown by Eq. (3.15) and Eq. (3.18), give for $P_0 \rightarrow \infty$ a contribution involving form factors evaluated at infinite values of the momentum transfers k_1^2, k_2^2 . We will indicate it as $\frac{1}{2}P_0 Z(t, -\infty, -\infty)$. It can be neglected if forms factors satisfy unsubtracted dispersion relations: anyhow, it does not depend on ζ_1, ζ_2 .

The sum rule (2.4) for $\sigma = 0$ and $P_0 \rightarrow \infty$ can now be written as

$$\begin{aligned} \mathcal{R}(t, \zeta_1, \zeta_2) - \int \frac{d\mu_1^2}{\zeta_1 + \mu_1^2} D_{k_1^2} \mathcal{R}(t, -\mu_1^2, \zeta_2 - \zeta_1 - \mu_1^2) - \int \frac{d\mu_2^2}{\zeta_2 + \mu_2^2} D_{k_2^2} \mathcal{R}(t, \zeta_1 - \zeta_2 - \mu_2^2, -\mu_2^2) \\ + 2 \int d\mu_1^2 D_{k_1^2} \mathcal{R}_1(t, -\mu_1^2, \zeta_2 - \zeta_1 - \mu_1^2) + 2 \int d\mu_2^2 D_{k_2^2} \mathcal{R}_1(t, \zeta_1 - \zeta_2 - \mu_2^2, -\mu_2^2) \\ + Z(t, -\infty, -\infty) = 4\pi F_1(t). \end{aligned} \quad (4.8)$$

It should be noted that ζ_1, ζ_2 and t are the squared norms of spacelike vectors, and $\sqrt{\zeta_1}, \sqrt{\zeta_2}, \sqrt{t}$ satisfy triangular relations, i.e., anyone of the three must be smaller (or equal) than the sum of the other two and larger (or equal) than the modulus of the difference.

However, once the sum rule Eq. (4.8) has been obtained in this region, it holds in all regions which can be reached by analytic continuation. The starting region as well as the analytic continuation procedure is the same which one encounters in the demonstration of dispersion relations ⁷⁾.

5. - SINGULARITIES OF THE $P_0 \rightarrow \infty$ SUM RULES

From the general form of the sum rule Eq. (4.8), we see that the right-hand side (i.e., the part containing the information of the equal time commutation relations) does not depend on ζ_1, ζ_2 - in particular, it has no singularities in those variables. On the other hand, the various terms on the left-hand side once analytically continued in ζ_1, ζ_2 , do exhibit singularities in ζ_1, ζ_2 .

We will show, however, that the discontinuities of the various terms cancel exactly each other, so that the sum of all the contributions to the left-hand side of the sum rule is automatically free of singularities in ζ_1 and ζ_2 . As contributions from (3a), (3b) are independent of ζ_1, ζ_2 they can be safely disregarded in the discussion of the singularities.

For what follows, it is convenient to recall two formulae :

- i) if $D_x f(x_0)$ is the discontinuity of $f(x)$ with respect to x at the point x_0 , the discontinuity of $f(x = \pm \xi + c)$ with respect to ξ at the point ξ_0 is

$$D_{\xi} f(x = \pm \xi_0 + c) = \pm \int dx_0 \delta(\pm \xi_0 + c - x_0) D_x f(x_0); \quad (5.1)$$

- ii) the discontinuity of the product of two functions $f(x), g(x)$ is

$$D_x [f(x_0) g(x_0)] = [D_x f(x_0)] g(x_0 + i\varepsilon) - f(x_0 - i\varepsilon) [D_x g(x_0)] \quad (5.2)$$

Let us start looking at the discontinuity in ζ_1 of the term containing $\mathcal{A}_1$ in Eq. (4.8). From Eq. (5.1), the discontinuity at $\zeta_1 = \bar{\varepsilon}_1$ is seen to be

$$2 \int d\mu_1^2 d\mu_2^2 \left[D_{k_1^2} D_{k_2^2} \mathcal{R}_1(t, -\mu_1^2, -\mu_2^2) \delta(\bar{\zeta}_1 - \zeta_2 - \mu_2^2 + \mu_1^2) \right. \\ \left. - D_{k_1^2} D_{k_2^2} \mathcal{R}_1(t, -\mu_1^2, \mu_2^2) \delta(\zeta_2 - \bar{\zeta}_1 - \mu_1^2 + \mu_2^2) \right] = 0.$$

The same cancellation occurs for the discontinuity with respect to ζ_2 .

The remaining discontinuities are those of $\mathcal{R}$, coming from the contributions of Eq. (4.6) and Eq. (4.7).

Collecting all the terms in $\mathcal{R}$ from Eq. (4.7), recalling Eq. (5.2) and $D_x \frac{1}{x} \Big|_{x=x_0} = \delta(x_0)$, their total discontinuity in ζ_1 at $\bar{\zeta}_1$ is seen to be

$$\begin{aligned} & - \int d\mu_1^2 \delta(\bar{\zeta}_1 + \mu_1^2) D_{k_1^2} \mathcal{R}(t, -\mu_1^2, \zeta_2 - \bar{\zeta}_1 - \mu_1^2) \\ & + \int d\mu_1^2 d\mu_2^2 \delta(\zeta_2 - \bar{\zeta}_1 - \mu_1^2 + \mu_2^2) \frac{1}{\bar{\zeta}_1 + \mu_1^2} D_{k_1^2} D_{k_2^2} \mathcal{R}(t, -\mu_1^2, -\mu_2^2) \\ & - \int d\mu_1^2 d\mu_2^2 \delta(\bar{\zeta}_1 - \zeta_2 - \mu_2^2 + \mu_1^2) \frac{1}{\zeta_2 + \mu_1^2} D_{k_1^2} D_{k_2^2} \mathcal{R}(t, -\mu_1^2, -\mu_2^2) \\ & = - D_{k_1^2} \mathcal{R}(t, \bar{\zeta}_1, \zeta_2). \end{aligned} \tag{5.3}$$

On the other hand, the discontinuity at $\zeta_1 = \bar{\zeta}_1$ of Eq. (4.6) is clearly $D_{k_2^2} \mathcal{R}(t, \bar{\zeta}_1, \zeta_2)$ and therefore cancels exactly the contribution of all the other terms as given by Eq. (5.3).

The same discussion applies to the ζ_2 discontinuities. The ζ_1 and ζ_2 discontinuities are therefore automatically absent in the left-hand side of the sum rule Eq. (4.8), their cancellation being tautological.

What about the t singularities in Eq. (4.8)?

The right-hand side of the sum rule is expected to have singularities at values of t corresponding to square masses of states which are connected to the vacuum by the current j_σ^γ of Eq. (2.1). From the very inspection of the graphs of Section 2, we see that the "large bubble" in (2a), (2b) and (4a), (4b) can account very naturally for t singularities of the left-hand side. Therefore, the sum rule at some t singularity can be graphically visualized by

where the dotted vertical line represents the state giving rise to the t singularity. By "factorizing out" from each term the "residuum"

one is left with the matrix elements of the equal time commutator Eq. (1.1) between the vacuum and a one-particle state :

$$\langle 0 | \left[j_0^\omega\left(\frac{x}{2}\right), j_0^\beta\left(-\frac{x}{2}\right) \right] | t \rangle \delta(x_0) = \int_0^{\beta} \langle 0 | j_0^\gamma(0) | t \rangle \delta^4(x)$$

which can be treated according to the general methods described in the preceding Sections (of course, in dealing with just one particle, the distinction between $P_0 \rightarrow \infty$ and $\Delta_0 \rightarrow \infty$ systems makes no longer sense).

To sum up, in the $P_0 \rightarrow \infty$ sum rule Eq. (4.8), we can distinguish two parts: a "low mass contribution" - essentially a sum of coupling constants of form factors at finite momentum transfer, coming from graphs (1a), (1b) - and an "infinite mass contribution", involving asymptotic values of hadron production amplitudes. The explicit form of the second part depends of course on dynamics⁸⁾. It could be that those "infinite mass" terms vanish by themselves. This pattern is referred to as superconvergence, and happens actually in some Regge pole models.

Under the purely dynamical assumption that those terms do vanish, the sum rule Eq. (4.8) becomes

$$\int dv' D_\nu A(t, \nu'; \zeta_1, \zeta_2) = 4\pi F(t) \quad (5.4)$$

which is the Fubini - Dashen - Gell-Mann (F.D.G.M.) sum rule. The assumption that the other contributions in Eq. (4.8) vanish, implies that their ζ_1 and ζ_2 discontinuities also vanish. We have seen, however, that their discontinuities are nothing but the ζ_1 and ζ_2 discontinuities of the left-hand side of the F.D.G.M. sum rule. Therefore, the dynamical assumption done in order to write the sum rule in the F.D.G.M. form of Eq. (5.4) implies directly superconvergence.

It should be pointed out that superconvergence is not implied by local commutativity or current algebra alone, and, besides, it is not related to the convergence of the integrals appearing in the sum rules. Superconvergence means that a certain integral vanishes, while for writing the sum rule it is sufficient that the same integral exists. Even if the concerned amplitudes are not superconvergent, sum rules still exist - their form is given in Eq. (4.8). Anyhow, the sum rule is free from singularities in the variables which do not appear in the right-hand side, as required by local commutativity. As we have shown this is achieved by means of cancellations between different terms, without imposing any real constraint on the physical structure of the singularities.

6. - THE $\Delta_0 \rightarrow \infty$ SUM RULES

The $\Delta_0 \rightarrow \infty$ sum rules can be discussed following a pattern similar to the preceding one, with Δ , v , s , u playing the role that P , v , k_1^2 , k_2^2 respectively had in the $P_0 \rightarrow \infty$ case. Graphs (2a), (2b) give the "finite v contributions", to be written as $\int dv' D_v B(t, v', \bar{s}, \bar{u})$, where B is the coefficient of $\Delta_p \Delta_\sigma$ in the expansion Eq. (4.3). It corresponds to a hadron production amplitude at fixed, finite (and unphysical) values of s and u . The other graphs give "infinite v contributions". Graphs (4a), (4b) correspond to the zeta graph (3a), (3b) in the $P_0 \rightarrow \infty$: they involve the same kind of discontinuities as (2a), (2b), but at infinite values of s , u , and anyhow with no dependence on $\bar{s}$, $\bar{u}$ [see Eqs. (3.22), and (3.25)]. Graphs (1a), (1b), (3a), (3b) give in the $\Delta_0 \rightarrow \infty$ asymptotic values of form factors, for the transitions between any of the external particles and the sliced intermediate states [see Eqs. (3.4), (3.7), (3.16) and (3.19)]. In the $\Delta_0 \rightarrow \infty$ sum rules, $\bar{s}$, $\bar{u}$ and $4P^2$ are the squared norms of spacelike vectors, and $\sqrt{\bar{s}}$, $\sqrt{\bar{u}}$, $\sqrt{4P^2}$ satisfy triangular relations as $\sqrt{\xi_1}$, $\sqrt{\xi_2}$ and $\sqrt{t}$ do in the $P_0 \rightarrow \infty$ case. Once the sum rules have been obtained in this region, they hold in all regions which can be reached by analytic continuation. Local commutativity requires the right-hand side of the sum rule, and therefore also the left-hand side, to be free from singularities in $\bar{s}$, $\bar{u}$. This is actually the case. The $\bar{s}$ and $\bar{u}$ discontinuities of the term coming from graphs (1a), (1b) are cancelled by the corresponding discontinuities of the other graphs (infinite v contributions) as can be seen from an analysis similar to that made before for the $P_0 \rightarrow \infty$ case. Assuming that the "infinite v term" vanish, one gets the $\Delta_0 \rightarrow \infty$ sum rules Eqs. (6) of Ref. 5). As for the F.D.G.M. sum rules, such sum rule turns out to be free from $\bar{s}$ and $\bar{u}$ singularities by the very assumption that it can be written in that way. In Ref. 5), the explicit structure of the discontinuities is given.

7. -

CONNECTION WITH THE DISPERSION RELATIONS

Consider the Fourier transform of the matrix element T of the retarded commutator of two currents, already introduced in Section 4:

$$T_{\rho\sigma}^{\alpha\beta}(\vec{K}, K_0) = \int d^4x e^{iKx} \vartheta(x_0) \langle 2 | \left[j_\rho^\alpha\left(\frac{x}{2}\right), j_\sigma^\beta\left(-\frac{x}{2}\right) \right] | 1 \rangle \quad (7.1)$$

Recalling that the Fourier transform of the step function $\vartheta(x_0)$ is $-i(2\pi)^3 \delta^3(\vec{K})(K_0 - i\varepsilon)^{-1}$, from the convolution theorem we have ⁶⁾

$$T_{\rho\sigma}^{\alpha\beta}(\vec{K}, K_0) = -\frac{i}{2\pi} \int \frac{dK'_0}{K_0 - K'_0 - i\varepsilon} t_{\rho\sigma}^{\alpha\beta}(\vec{K}, K'_0) \quad (7.2)$$

This equation can be regarded as a dispersion relation for $T_{\rho\sigma}$ in the variable K_0 at fixed $\vec{K}$; $t_{\rho\sigma}(\vec{K}, K_0)$, which is the same as in Eq. (2.2), is then seen to be the discontinuity of $T_{\rho\sigma}$ with respect to K_0 at fixed $\vec{K}$.

From Eq. (7.1), with standard manipulation, we have

$$\begin{aligned} iK_0 T_{\rho\sigma}^{\alpha\beta} &= \int d^4x e^{iKx} \delta(x_0) \langle 2 | \left[j_\rho^\alpha\left(\frac{x}{2}\right), j_\sigma^\beta\left(-\frac{x}{2}\right) \right] | 1 \rangle \\ &+ \int d^4x e^{iKx} \vartheta(x_0) \langle 2 | \frac{\partial}{\partial x_0} \left[j_\rho^\alpha\left(\frac{x}{2}\right), j_\sigma^\beta\left(-\frac{x}{2}\right) \right] | 1 \rangle \end{aligned}$$

By using Eq. (7.2) for the left-hand side, Eq. (2.1) for the first term in the right-hand side and again the convolution theorem for the last term, we get

$$\begin{aligned} \frac{1}{2\pi} \int \frac{K_0 dK'_0}{K_0 - K'_0} t_{\rho\sigma}^{\alpha\beta}(\vec{K}, K'_0) &= \int^{\alpha\beta} \langle 2 | j_\sigma^\beta(0) | 1 \rangle + \langle 2 | Z^{\alpha\beta}(0) | 1 \rangle K_\sigma \delta_{\sigma i} \\ &+ \frac{1}{2\pi} \int \frac{K'_0 dK'_0}{K_0 - K'_0} t_{\rho\sigma}^{\alpha\beta}(\vec{K}, K'_0) \end{aligned}$$

i.e.,

$$\frac{1}{2\pi} \int t_{0\sigma}^{\alpha\beta}(\vec{K}, K'_0) dK'_0 = \int^{\alpha\beta\gamma} \langle 21 j_{\sigma}^{\gamma}(0) | 1 \rangle + \langle 21 \sum^{\alpha\beta}(0) | 1 \rangle K_{\sigma} \delta_{\sigma i} \quad (7.3)$$

This is the sum rule Eq. (2.4), which can therefore be derived by means of dispersive techniques, starting from the suitable dispersive representation (7.2), regardless of the reference frame.

Let us see what is the connection between Eq. (7.2) and the usual dispersion relations in ν , at fixed t, k_1^2, k_2^2 .

By using the decomposition Eqs. (2.6) and (4.3), Eq. (6.2) can be recast as dispersion relations for the scalar amplitudes $A, A_1, \dots$, in terms of the discontinuities $a, a_1, \dots$. It turns out that while $A, A_1, \dots$ depend on t, ν, k_1^2, k_2^2 the discontinuities $a, a_1, \dots$ depend on $t, \nu' = 2PK', k_1'^2 = (K' - \Delta)^2, k_2'^2 = (K' + \Delta)^2$ if $K' = (K'_0, \vec{K})$. Therefore $k_1'^2, k_2'^2$ - the arguments of the discontinuities - vary with K'_0 along the integration as in dealing with sum rules; furthermore, they are in general different from k_1^2, k_2^2 .

As in the discussion of the sum rules, we can analyze the contributions to the right-hand side of Eq. (7.2) in terms of intermediate states according to Section 3. In connection with the $P_0 \rightarrow \infty$ sum rules, we will examine the $P_0 \rightarrow \infty$ limit [see Eq. (2.11)] of Eq. (7.2). We will take $K_z = 0$ for simplicity, so that finite values of ν correspond to $K_0 \rightarrow 0$ in the $P_0 \rightarrow \infty$ system, and $k_1^2 = \zeta_1, k_2^2 = \zeta_2$.

In considering the intermediate states corresponding to the graphs (1a), (1b) due to Eqs. (3.3) and (3.6) we can write

$$\int \frac{dK'_0}{K_0 - K'_0} = \int \frac{d\nu'}{\nu - \nu'} \quad (7.4)$$

and the corresponding contributions to the dispersive integral in the right-hand side of Eq. (6.2) are taken at values of $k_1'^2$ and $k_2'^2$ fixed and equal to ζ_1, ζ_2 , i.e., just the values of k_1^2, k_2^2 in the left-hand side.

Contributions from the connected graphs (1a), (1b) have therefore the usual form of unsubtracted dispersion relations in ν , at fixed k_1^2, k_2^2 . However, they do not exhaust the content of Eq. (7.2), as in principle it gets contributions also from all the other disconnected graphs of Section 3.

For graphs (3a), (3b), we can write [see Eqs. (3.15), (3.18)]

$$\int \frac{dK'_0}{K_0 - K'_0} = \int \frac{d\nu'}{\nu \mp 2P_0^2} \quad (7.5)$$

the corresponding contribution being evaluated at infinite values of $\nu', k_1'^2, k_2'^2$. For reasonable asymptotic behaviours in these variables the presence of P_0^2 in the denominator of Eq. (7.5) implies the vanishing of these contributions for $P_0 \rightarrow \infty$.

There are, besides, graphs (2a), (2b), (4a), (4b). Let us consider the first of them, the other being very similar. For finite (i.e., $K_0 \rightarrow 0$ for $P_0 \rightarrow \infty$) from Eq. (3.9) we have

$$\int \frac{dK'_0}{K_0 - K'_0} \rightarrow -\frac{1}{2} \int \frac{d\mu_2^2}{\zeta_2 + \mu_2^2} \quad (7.6)$$

The corresponding contribution is independent of ν , and is taken at $k_1'^2 = \zeta_1 - \zeta_2 - \mu_2^2, k_2'^2 = -\mu_2^2$.

Summing up all the contributions to the dispersive integral, we have therefore:

- a term coming from the connected graphs, which has the form of an unsubtracted dispersion relation in ν at fixed values of the other variables;
- terms coming from the disconnected graphs (2a), (2b), (4a), (4b), of the form of dispersive integral on k_1^2, k_2^2 , independent of ν .

The whole expression can be regarded as a dispersive representation in ν plus a subtraction term.

The dispersion relations, having subtraction terms which possess k_1^2 and k_2^2 singularities, give rise to sum rules whose form shall depend on these terms. This dependence is of course the same as was realized in the preceding Sections and therefore the discussion of their dynamical content is also the same.

As a last comment, we note that, if Δ is timelike from Eq. (7.2), in the $\Delta_0 \rightarrow \infty$ system of Eq. (2.3), one obtains a new kind of dispersion relations in v , the difference of the invariant masses associated to the currents. The intermediate states of graphs (2a), (2b) give rise to a term which has the form of an unsubtracted dispersive integral in v at fixed t, s, u ; on the other hand, the intermediate states of graphs (1a), (1b) and (3a), (3b) give terms of the form of dispersive integrals on the other variables, independent of v , which therefore can be regarded as subtraction terms.

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