ON SINGH'S LEMMA FOR LOW ENERGY COMPTON SCATTERING

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A new derivation will be given of a lemma of Singh \(^1\) which has proved useful in establishing low energy theorems for Compton scattering \(^1,\)\(^2\). The original treatment depends on the validity of a certain infinite expansion over intermediate states. This can be avoided. The result follows essentially from gauge invariance and the notion that the amplitude, after extraction of the pole terms, is a smooth function.

Let

\[ k \equiv (k_x, i\omega) \quad k' \equiv \left( k'_x, i\omega' \right) \quad P_I \equiv (p_{I_x}, i(p_{I}^2 + p_{I}^4)) \quad P_F \equiv (p_{F_x}, i(p_{F}^2 + p_{F}^4)) \]

be initial and final photon and target four-momenta; the photons are not restricted to the mass shell. Let \( T_{\mu\nu} \) be the Compton amplitude for some specified target spin states, where \( \mu \) and \( \nu \) are photon polarization indices. A particular contribution is that of unexcited intermediate states:

\[
U_{\mu\nu} = \sum_N \frac{\langle F | J_{\mu} | N \rangle \langle N | J_{\nu} | I \rangle}{\omega + E_I - E_N} S(\vec{P}_{I} + \vec{p}_{I} - \vec{p}_{N})
\]

\[
+ \left( k \cong k', \mu \cong \nu \right)
\]

(1)

where the currents \( J \) refer to the origin of co-ordinates, the last bracket indicates the crossed term, and the summation is over unexcited target states \( N \) \(^3\). The residual "excited" contribution \( E_{\mu\nu} \) is defined by

\[
T_{\mu\nu} = U_{\mu\nu} + E_{\mu\nu}
\]

(2)

Singh's lemma is that for fixed \( P_I \)

\[
E_{00} = k'_m k_n \Lambda_{mn} (k', k)
\]

(3)
where $\Lambda_{mn}$ is free from singularities for small values of the variables (just "F-S" hereafter), and has the crossing property

$$\Lambda_{mn}(k', k) = \Lambda_{nm}(-k, -k') \quad (4)$$

Latin indices take values $1, 2, 3$. In obtaining this result, Singh represented $E_{oo}$ by an infinite expansion over intermediate states. However, it will be seen a certain property of the finite summation defining $V_{oo}$ is sufficient.

Note first that as independent variables we can take $\vec{k}, \vec{k}'$, and $\omega + \omega'$; $\omega - \omega'$ is determined $^4$ by the mass shell condition

$$\Delta = (\vec{p}_I + \omega - \omega')^2 - (\vec{p}_I + \vec{k} - \vec{k}')^2 - M^2 = 0 \quad (5)$$

The relevant property of $U_{oo}$ is that it vanishes in general (i.e., avoiding singularities) when either $\vec{k}$ or $\vec{k}'$ approaches zero, the other variables being held fixed. This follows from the construction $^1$ and

$$\sum N \delta(\vec{p}_F - \vec{p}_N) \langle N | J_o | N \rangle = Z \langle f | \quad (6)$$

$$\sum N \delta(\vec{p}_I - \vec{p}_N) \langle N | J_o | I \rangle = Z \langle I |$$

which are consequences of the commutation of total charge with spin, $Z$ being the target charge. That $T_{oo}$ has the same property follows from

$$\omega' \omega T_{oo} = k'_m k_n T_{mn} \quad (7)$$

which is a consequence $^5$ of the gauge invariance conditions $k'_\mu T_{\mu\nu} = = T_{\mu\nu} k_\nu = 0$. Then $E_{oo} (= T_{oo} - U_{oo})$ must also vanish in general for $k=0$ or $k'=0$; since it is an F-S function ($U_{oo}$ supposedly accounting entirely for the singularities of $T_{oo}$ for small values of the variables)

$$E_{oo} = k'_m W_{mn} k_n \quad (8)$$

where $W_{mn}$ is F-S.
In the first instance (8) is subject to the mass shell restriction (5). However we can continue $E_{oo}$ and $W_{mn}$, for example by specifying that they do not depend on $\omega' - \omega$, in such a way that (8) remains true. Then identifying $W_{mn}$ with $\Lambda_{mn}$ we have the result (3). The crossing property (4) follows from the crossing properties of $T_{\mu\nu}$ and $U_{\mu\nu}$.

The infinite expansion employed by Singh implies the validity of certain unsubtracted dispersion relations in $\omega$. With the present approach, on the contrary, no implication about high energy behaviour is made. Proof of the low energy theorems then depends only on low energy assumptions.

I am much indebted to Professor A. Pais in connection with this question.
REFERENCES

1) V. Singh, Rockefeller preprint (1967).
2) A. Pais, Brookhaven preprint, and CERN preprint TH.816 (1967).
3) The states $N$ are real, on-the-mass-shell, states.
4) It is assumed that $P_{\omega'0} = P_{\omega'0}^2 - \omega'$ is positive.