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LOWER BOUNDS FOR THE ABSORPTIVE PART OF THE SCATTERING
AMPLITUDE AT HIGH ENERGY

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ABSTRACT

Within the framework of relativistic Mandelstam representation, we study the analytic continuation of partial wave amplitude $a(\ell, k)$ into the lower half k plane, where k is the centre-of-mass momentum, for complex values of the angular momentum ℓ . By constructing an envelope of meromorphy for $a(\ell, k)$, we show that for $4\mu^2(1-\delta) \leq s \leq 4\mu^2$ the t channel absorptive part $A_t(t, s)$ cannot decrease faster than $t^{-1-\varepsilon}$ as $t \rightarrow +\infty$, where $\varepsilon = (N+1/\pi) \sqrt{\delta/3}$ for $0 < \delta \ll 1$ and N is the number of subtractions in the Mandelstam representation. It is shown further that $\sigma_{\text{total}}(t)$ has a lower bound $t^{-(N+5)/2}$ for $t \rightarrow +\infty$.

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I. INTRODUCTION

The purpose of this paper is to explore in some detail the analyticity property of partial wave amplitude $a(\ell, k)$ as a simultaneous function of angular momentum variable ℓ and centre-of-mass momentum k . To avoid unnecessary complications, we shall restrict ourselves to the elastic scattering of neutral pseudoscalar particles of equal mass and assume that the scattering amplitude $f(s, t)$, where $s = 4(\mu^2 + k^2)$ and $t = -2k^2(1 - \cos\theta)$, satisfies the Mandelstam representation with a finite number of subtractions. An additional assumption we make is that, for every physical values of ℓ , the partial wave amplitude $a(\ell, k)$ can be continued analytically into the second sheet of the s plane through the elastic cut. As was noted by Martin¹⁾, this assumption does not follow from field theory or even from Mandelstam representation. Thus we have to postulate it explicitly in our considerations.

Analyticity structure of $a(\ell, k)$ for complex ℓ is complicated by the fact that, besides the usual physical and unphysical cuts in the k plane, it has an additional branch point at $k=0$. In the case of non-relativistic potential scattering, Regge²⁾ has shown that one can construct from $a(\ell, k)$ a new function $Z(\ell, k)$ which is free from this extra branch point. We show in Section II that such a function can also be defined in the relativistic case and that its nice analyticity property is in fact a consequence of unitarity, reality, and the Mandelstam representation. In Section III this result is utilized to obtain certain analytic continuation into the lower half k plane for values of ℓ satisfying $\text{Re } \ell < N$, where N is the number of subtractions.

As an application, we show in Section IV that the t channel absorptive part $A(t, s)$ for $4\mu^2(1 - \delta) \leq s \leq 4\mu^2$ cannot decrease faster than $t^{-1-\epsilon}$ as $t \rightarrow +\infty$, where $\epsilon = \sqrt{\delta}(N+1)/\sqrt{3}\pi$ for $0 < \delta \ll 1$. We also show that the absorptive part $A(t, 0)$ of the forward scattering amplitude has a lower bound $t^{-(N+3)/2}$ for $t \rightarrow +\infty$.

II. ANALYTICITY PROPERTY OF $Z(\ell, k)$

As is well known, the Froissart-Gribov formula

$$a(\ell, s) = \frac{4}{\pi(s-4)} \int_4^\infty A(x, s) Q_\ell\left(1 + \frac{2x}{s-4}\right) dx, \quad (\text{II.1})$$

where we have put the mass of the particle μ equal to one and assumed the u-t crossing symmetry for simplicity, defines uniquely the analytic interpolation of the physical partial wave amplitudes³⁾. If we rotate the usual cut $(-\infty, -1)$ of $Q_\ell(z)$ to the new position $(-1, +\infty)$ through the upper half z plane, we find that $a(\ell, s)$ is holomorphic in the product domain $(\text{Re } \ell > N) \otimes (s \text{ plane with the physical and unphysical cuts along the real axis})$ ⁴⁾. It is easy to show that $a(\ell, s)$ thus defined satisfies the reality condition⁴⁾

$$a(\ell, s) = e^{2i\pi\ell} a^*(\ell^*, s^*). \quad (\text{II.2})$$

Since we are interested in the analytic continuation of $a(\ell, s)$ into the second sheet, it is more convenient to discuss the properties of $a(\ell, s)$ in the k plane, where $k = \frac{1}{2}(s-4)^{\frac{1}{2}}$. Then the cut s plane is represented by the upper half of the k plane in Fig. 1, and the reality condition (II.2) becomes

$$a(\ell, k) = e^{2i\pi\ell} a^*(\ell^*, -k^*), \quad \text{Im } k > 0, \quad (\text{II.3})$$

where we use the notation $a(\ell, k)$ instead of $a(\ell, 4(1+k^2))$. This can be easily translated into the reality condition of the S matrix

$$S(\ell, k) - 1 = e^{2i\pi\ell} [S^*(\ell^*, -k^*) - 1], \quad \text{Im } k > 0, \quad (\text{II.4})$$

where

$$S(\ell, k) = 1 + \frac{2ik}{\sqrt{1+k^2}} a(\ell, k). \quad (\text{II.5})$$

We shall now examine how the S matrix $S(\ell, k)$ is continued into the lower half k plane through the "elastic window" $0 < k < \sqrt{3}$. For this purpose, we note that, in the elastic region $0 < k < \sqrt{3}$, $S(\ell, k)$ satisfies the unitarity condition

$$S(\ell, k) S^*(\ell, k) = 1 \quad (\text{II.6})$$

for all physical values of ℓ . According to our basic assumption, $S(\ell, k)$ is analytically continuable to the lower half k plane through the line $0 < k < \sqrt{3}$. Thus, if we consider the function

$$\psi(\ell, k) \equiv S(\ell, k) S^*(\ell, k^*) \quad (\text{II.7})$$

for even integer $\ell > N$, $\psi(\ell, k)$ is analytic in the neighbourhood of the line $0 < k < \sqrt{3}$. Since $\psi(\ell, k) = 1$ on this line, we therefore obtain by the Schwartz reflection principle the equation

$$S(\ell, k) S^*(\ell, k^*) = 1, \quad \text{even integer } \ell > N, \quad (\text{II.8})$$

which is valid everywhere in the cut k plane of Fig. 1. We shall call this domain D .

Next we introduce

$$\phi(\ell, k) \equiv S(\ell, k) S^*(\ell^*, k^*), \quad (\text{II.9})$$

where we shall at first restrict k to a small neighbourhood of the line $0 < k < \sqrt{3}$. It is easily seen that $\phi(\ell, k)$ is holomorphic in the domain $\text{Re } \ell > N$, bounded by $\exp[C(|\text{Im } \ell|)^{1-\varepsilon}]$ for $\text{Re } \ell > N$, $\varepsilon > 0$, (the above restriction on k is needed here), and equal to one for any even integer ℓ greater than N . Thus we have

$$S(\ell, k) S^*(\ell^*, k^*) = 1, \quad \text{for } \operatorname{Re} \ell > N, \quad (\text{II.10})$$

by Carlson's theorem⁵⁾. Now, since $\phi(\ell, k)$ is analytic in k , the restriction on k can be easily removed for $\operatorname{Re} \ell > N$, and (II.10) holds in the cut k plane D , with an additional branch point at $k=0$, as is shown below. The formula (II.10) determines completely the analytic continuation of $S(\ell, k)$ into the lower k plane through the window $0 < k < \sqrt{3}$.

We have thus seen that $S(\ell, k)$ satisfies the relations (II.4) and (II.10). By combining these two, we can derive a useful circuit relation of $S(\ell, k)$ for a path going around the origin $k=0$. Let us first restrict ourselves to real ℓ . Then, (II.4) and (II.10) reduce to

$$S(\ell, k) - 1 = e^{2i\pi\ell} [S^*(\ell, -k^*) - 1], \quad \operatorname{Im} k > 0, \quad (\text{II.11})$$

$$S(\ell, k) S^*(\ell, k^*) = 1. \quad (\text{II.12})$$

If we start from a point k in the second quadrant and move to $-k^*$ along a continuous curve in the cut k plane as is shown in Fig. 2, we find easily from (II.11) that

$$S^*(\ell, -k^*) = 1 + e^{-2i\pi\ell} [S(\ell, k) - 1]. \quad (\text{II.13})$$

Substituting this in

$$S(\ell, k e^{-i\pi}) = [S^*(\ell, -k^*)]^{-1} \quad (\text{II.14})$$

obtained from (II.12), where $-k^*$ and $k e^{-i\pi}$ are again connected by a continuous curve in the cut k plane, we find

$$S(\ell, k e^{-i\pi}) = \frac{e^{2i\pi\ell}}{S(\ell, k) + e^{2i\pi\ell} - 1}. \quad (\text{II.15})$$

Obviously, this relation can be easily extended to complex ℓ satisfying $\text{Re } \ell > N$.

Apply the same operation to the path $ke^{-i\pi} \rightarrow k$ which lies in the cut k plane and goes through the interval $(-\sqrt{3}, 0)$, we now obtain the circuit relation

$$S(\ell, k e^{-2i\pi}) = \frac{S(\ell, k) + 2i e^{i\pi\ell} \sin \pi \ell}{2i e^{-i\pi\ell} \sin \pi \ell S(\ell, k) + 1 - 4 \sin^2 \pi \ell}. \quad (\text{II.16})$$

This formula shows clearly that $k=0$ is a branch point for any non-integer ℓ .

In order to examine the nature of this branch point, it will be more convenient to diagonalize (II.16), namely, to construct from $S(\ell, k)$ a new function $Z_1(\ell, k)$ such that

$$Z_1(\ell, k e^{-2i\pi\ell}) = \lambda(\ell) Z_1(\ell, k). \quad (\text{II.17})$$

It is easily seen that there is only one solution of this diagonalization problem, and the answer is

$$Z_1(\ell, k) = \frac{S(\ell, k) + e^{2i\pi\ell}}{S(\ell, k) - 1} \quad (\text{II.18})$$

6.

with the eigenvalue

$$\lambda(\ell) = e^{4i\pi\ell}. \quad (\text{II.19})$$

This shows that, under the "rotation" $k \rightarrow ke^{-2i\pi}$, $Z_1(\ell, k)$ behaves in exactly the same manner as $k^{-2\ell+n}$, where n is any integer. Thus, if we introduce

$$Z_2(\ell, k) = k^{2\ell+1} Z_1(\ell, k), \quad (\text{II.20})$$

we obtain

$$Z_2(\ell, ke^{-2i\pi}) = Z_2(\ell, k). \quad (\text{II.21})$$

Namely, $Z_2(\ell, k)$ is meromorphic in the neighbourhood of $k=0$ for $\text{Re } \ell > N$.

The function $Z_2(\ell, k)$ can be made real by multiplying with a phase factor independent of k . In this manner, we finally find the function

$$Z(\ell, k) = i e^{-i\pi\ell} k^{2\ell+1} \frac{S(\ell, k) + e^{2i\pi\ell}}{S(\ell, k) - 1}, \quad (\text{II.22})$$

which is real and meromorphic in the domain $(\text{Re } \ell > N) \otimes D$. Furthermore, $Z(\ell, k)$ is even in k , as is readily checked using (II.15). By inverting (II.22), we obtain

$$S(\ell, k) = \frac{Z(\ell, k) + i k^{2\ell+1} e^{i\pi\ell}}{Z(\ell, k) - i k^{2\ell+1} e^{-i\pi\ell}}. \quad (\text{II.23})$$

The function $Z(\ell, k)$ is essentially the same function that was introduced by Regge in his study of non-relativistic potential scattering²⁾. What we have shown here is that this function can also be defined in the relativistic case, and its meromorphy in the domain $(\text{Re } \ell > N) \otimes D$ is a consequence of the Mandelstam analyticity, reality and unitarity. It is interesting to note that for an integer ℓ , $Z(\ell, k)$ reduces to a function familiar in the effective range theory of scattering.

III. ANALYTIC COMPLETION

The partial wave amplitude $a(\ell, s)$ defined by (II.1) is holomorphic in the domain $(\text{Re } \ell > N) \otimes (\text{cut } s \text{ plane})$. In general, however, $a(\ell, s)$ has additional holomorphy or meromorphy in some region outside this domain due to some extra properties satisfied by the scattering amplitude. For instance, the Froissart bound for $A(t, 0)$ in the t channel allows us to obtain such an extension of analyticity domain. We may thus ask to what extent the meromorphy domain of $a(\ell, s)$ can be extended when the scattering amplitude is known to satisfy some additional conditions. In this Section we shall try to answer this question making use of the analyticity property of the function $Z(\ell, k)$ established in Section II.

We are particularly interested in the restriction of the form

$$A(x, s_0) < \frac{C}{x^\gamma}, \quad x \gg x_0, \quad (\text{III.1})$$

where s_0 is a fixed number satisfying $0 \leq s_0 < 4$. The constant γ will depend on the choice of s_0 in general. If we define $d = \frac{1}{2}(4-s_0)^{\frac{1}{2}}$, d satisfies $0 < d \leq 1$.

Under the assumption (III.1), it is easy to see that $a(\ell, k)$ is holomorphic in the half plane $\text{Re } \ell > -\gamma$ for any $k = i\xi$, $\xi > d$. This is obvious for $0 < s < 4(1-d^2)$ since $A(x, s)$ is monotonically increasing in this interval of s by unitarity in the t channel and hence

$$A(x, s) < A(x, 4(1-d^2)) < \frac{C}{x^\gamma} \quad \text{for } 0 < s < 4(1-d^2). \quad (\text{III.2})$$

When $s < 0$, we split the integral (II.1) into two parts as

$$\int_4^{4-s} A(x, s) Q_\ell \left(1 + \frac{2x}{s-4} \right) dx + \int_{4-s}^{\infty} A(x, s) Q_\ell \left(1 + \frac{2x}{s-4} \right) dx. \quad (\text{III.3})$$

Since the first integral is over a finite interval, its domain of analyticity in ℓ is no worse than that of Q_ℓ . Namely, it is meromorphic in ℓ with fixed simple poles at $\ell = -1, -2, \dots$. As for the second integral, we note that $A(x, s)$ is always in the physical scattering region of the t channel and hence satisfies the condition (III.1) by unitarity. Thus it is holomorphic in the domain $\text{Re } \ell > -\gamma$. This is seen immediately from (III.3) if $\text{Re } \ell > -\frac{1}{2}$. When $\text{Re } \ell < -\frac{1}{2}$, it can be proved using the alternative form of the second term of (III.3) obtained by the substitution ⁶⁾

$$Q_\ell(z) = Q_{-\ell-1}(z) + \pi \cot \pi \ell P_{-\ell-1}(z). \quad (\text{III.4})$$

We have thus seen that, under the assumption (III.1), $a(\ell, k)$ and hence $S(\ell, k)$ are meromorphic at least in the union of two domains $(\text{Re } \ell > N) \otimes (\text{the upper half } k \text{ plane with a cut on } k = i\xi, \xi > d)$ and $(\text{Re } \ell > -\gamma) \otimes (k = i\xi, \xi > d)$. If we consider the function

$$\bar{S}(\ell, k) \equiv (\ell+1)(\ell+2) \dots (\ell+n) S(\ell, k), \quad (\text{III.5})$$

where n is the largest positive integer which does not exceed γ , then $\bar{S}(\ell, k)$ is holomorphic in the same domain. Using an adaptation of Malgrange's theorem ⁷⁾, we find that $\bar{S}(\ell, k)$ is holomorphic in the envelope of holomorphy of these two domains. This means that for any ℓ satisfying $\text{Re } \ell > -\gamma + \varepsilon_1$ ($\varepsilon_1 > 0$), there is a complex neighbourhood $D_1(-\gamma + \varepsilon_1)$ of the semi-axis $k = i\xi, \xi > d$, in which $\bar{S}(\ell, k)$ is holomorphic. Obviously $S(\ell, k)$ and $Z(\ell, k)$ are meromorphic in this enlarged domain.

Since $Z(\ell, k)$ is real, the domain of meromorphy is symmetric with respect to the real k axis. Thus $Z(\ell, k)$ is also meromorphic in the domain $(\operatorname{Re} \ell > -\gamma + \varepsilon_1) \otimes D_2(-\gamma + \varepsilon_1)$ where D_2 is the mirror image of D_1 with respect to the real k axis. Furthermore, the equality

$$Z(\ell, k e^{-i\pi}) = Z(\ell, k), \quad (\text{III.6})$$

which was originally proved for $\operatorname{Re} \ell > N$, serves to define $Z(\ell, k)$ in the domain $(\operatorname{Re} \ell > -\gamma + \varepsilon_1) \otimes D_2(-\gamma + \varepsilon_1)$ in terms of $Z(\ell, k)$ in $(\operatorname{Re} \ell > -\gamma + \varepsilon_1) \otimes D_1(-\gamma + \varepsilon_1)$. Of course this does not necessarily mean that $Z(\ell, k)$ in these two domains are analytic continuations of each other when the continuation is made along a path in the cut k plane D for a fixed value of ℓ satisfying $\operatorname{Re} \ell < N$.

In summary $Z(\ell, k)$ is meromorphic in the union of three domains

$$\begin{aligned} & (\operatorname{Re} \ell > N) \otimes D, \\ & (\operatorname{Re} \ell > -\gamma + \varepsilon_1) \otimes D_1(-\gamma + \varepsilon_1), \\ & (\operatorname{Re} \ell > -\gamma + \varepsilon_1) \otimes D_2(-\gamma + \varepsilon_1). \end{aligned}$$

Now we shall determine the envelope of meromorphy of these domains. For this purpose we note that the envelope of meromorphy is identical with the envelope of holomorphy in our problem⁸⁾. Determining the envelope of holomorphy, on the other hand, is equivalent to solving the following boundary value problem: find a harmonic function $v(k)$ in the k plane which takes the value N on the lines $(-\infty, -\sqrt{3})$ and $(\sqrt{3}, +\infty)$ and the value $-\gamma$ on the lines $(i\infty, i\omega)$, $(-i\omega, -i\infty)$.

To solve this problem, it is convenient to regard these four lines as the cuts of the k plane as is shown in Fig. 3 and map this plane onto the upper half w plane of Fig. 4. This mapping can be done making use of the transformation

$$k(w) = -\sqrt{3} \frac{1 + w^2}{\sqrt{1 - (2 + \frac{12}{d}) w^2 + w^4}}, \quad (\text{III.7})$$

which may be easily constructed by the Schwartz-Christoffel method ⁹⁾. Corresponding points in Figs. 3 and 4 are denoted by the same capital letters. Let us put

$$d = \sqrt{3} \tan \varphi, \quad (0 < \varphi \leq \frac{\pi}{6}). \quad (\text{III.8})$$

Then

$$\alpha = \tan \frac{\varphi}{2}, \quad \beta = \cot \frac{\varphi}{2}, \quad (\text{III.9})$$

where α, β are the w co-ordinates of the points F, H , respectively.

In the w plane the boundary condition can be stated as follows :

$$\begin{aligned} v(k(w)) &= N \quad \text{on} \quad \overline{AB}, \overline{DF}, \overline{HA}, \\ &= -\gamma \quad \text{on} \quad \overline{BD}, \overline{FH}. \end{aligned} \quad (\text{III.10})$$

This problem can be readily solved, the result being

$$v(k(w)) = N - \frac{N+\gamma}{\pi} \operatorname{Im} \left[\ln \left(\frac{w+\alpha}{w-\alpha} \frac{w-\beta}{w+\beta} \right) \right]. \quad (\text{III.11})$$

If we express w in terms of k solving the equation (III.7), we obtain

$$v(k) = N - \frac{N+Y}{\pi} \arg \left(\frac{w(k) + \alpha}{w(k) - \alpha} \frac{w(k) - \beta}{w(k) + \beta} \right). \quad (\text{III.12})$$

For any given k in the cut k plane, $Z(\ell, k)$ is meromorphic for all values of ℓ satisfying $\text{Re } \ell > v(k)$. Thus (III.12) determines the envelope of meromorphy of our problem.

IV. APPLICATIONS

If we choose $\gamma = -1$ and $d=1$ in (III.12), we obtain a solution of the problem previously studied by Bardakci¹⁰⁾.

However, we shall be concerned here exclusively with some other aspects of the scattering problem, namely lower bounds of the absorptive part of the scattering amplitude at high energy. For this purpose let us suppose that

$$\gamma > 1, \quad (\text{IV.1})$$

and ask what is the value of $v(k)$ at $k=0$. As is seen from (III.7), $k=0$ corresponds to $w=i$. Thus we obtain from (III.12)

$$v(0) = N - \frac{N+\gamma}{\pi} (\pi - 2\varphi), \quad (\text{IV.2})$$

where φ is given by (III.8). For given values of N and γ , $v(0)$ depends on the value of φ . In particular, if we can choose

$$\varphi < \frac{\pi(\gamma-1)}{2(\gamma+N)} \quad (\text{IV.3})$$

or, equivalently, if

$$d < \sqrt{3} \tan \left(\frac{\pi(\gamma-1)}{2(\gamma+N)} \right), \quad (\text{IV.4})$$

then we find that $v(0) < -1$.

This means that, if $A(x,s)$ satisfies the inequality

$$A(x, s_0) < \frac{C}{x^\gamma}, \quad \gamma > 1, \quad (\text{IV.5})$$

where $s_0 = 4(1-d^2)$ and d satisfies (IV.4), then we can find a positive constant ε such that $-1-\varepsilon > -Y + \varepsilon_1$ and the domains $D_1(-1-\varepsilon)$ and $D_2(-1-\varepsilon)$ have the point $k=0$ in common. Hence $D_1(-1-\varepsilon)$ and $D_2(-1-\varepsilon)$ are connected as is shown in Fig. 5. Thus $Z(\ell, k)$ is meromorphic in the connected domain

$$(\operatorname{Re} \ell > -1-\varepsilon) \otimes (D_1(-1-\varepsilon) + D_2(-1-\varepsilon)), \quad (\text{IV.6})$$

and (III.6) determines the analytic continuation of $Z(\ell, k)$ from $D_1(-1-\varepsilon)$ to $D_2(-1-\varepsilon)$. Consequently, $S(\ell, k)$ and $a(\ell, k)$ are also meromorphic in the domain (IV.6), except that we now have to include a cut starting at $k=0$, which may be conveniently taken along the negative real axis of the k plane.

Now, according to (II.1), when $Y > 1$, $a(\ell, k)$ is meromorphic in the domain $(\operatorname{Re} \ell > -1-\varepsilon) \otimes D_1(-1-\varepsilon)$ and has a fixed pole at $\ell = -1$. As is shown above, however, $a(\ell, k)$ can be continued analytically to the lower half k plane [domain $D_2(-1-\varepsilon)$] for any value of ℓ satisfying $\operatorname{Re} \ell > -1-\varepsilon$, in so far as d satisfies the condition (IV.4). In the domain $D_2(-1-\varepsilon)$, the continuation is given explicitly by

$$S(\ell, k e^{-i\pi}) = \frac{e^{2i\pi\ell}}{S(\ell, k) + e^{2i\pi\ell} - 1}. \quad (\text{IV.7})$$

However, this has obviously no pole at $\ell = -1$ for $\operatorname{Im}[k e^{-i\pi}] < 0$. It follows from the continuity theorem ¹¹⁾ for functions of many complex variables that such a behaviour of the analytic function $S(\ell, k)$ is not possible ¹²⁾. Namely, the assumption (IV.5) cannot be imposed on the scattering amplitude if d is very small, or s_0 is sufficiently close to the elastic threshold $s=4$. In other words, to be consistent with unitarity and Mandelstam analyticity, the t channel absorptive part $A(t, 4-\delta)$ cannot decrease more rapidly than $C/t^{1+\varepsilon}$ for all $t > t_0$, where $\varepsilon = (\sqrt{\delta}(N+1)/\sqrt{3}\pi)$ and t_0 is a sufficiently large constant ¹³⁾.

Twisting the above consideration around slightly, let us now ask what is the lower limit of γ such that $v(0) \geq -1$ for $d=1$. In this case we obtain from (IV.4) the condition

$$\frac{\gamma-1}{\gamma+N} > \frac{1}{3},$$

namely

$$\gamma > \frac{N+3}{2}. \quad (\text{IV.8})$$

Thus, from the requirement that the scattering amplitude satisfies unitarity and Mandelstam analyticity [thus $v(0) \geq -1$], we have obtained a lower bound for the absorptive part of the forward scattering amplitude in the t channel

$$A(t, 0) > C t^{-(N+3)/2}. \quad (\text{IV.9})$$

Correspondingly, the total cross-section has a lower bound

$$\sigma_{\text{total}}(t) > C' t^{-(N+5)/2} \quad (\text{IV.10})$$

as $t \rightarrow +\infty$. Recently, Jin and Martin¹⁴⁾ obtained a lower bound t^{-5} for $A(t, 0)$ from the dispersion relation for the forward scattering amplitude $f(t, 0)$. Although ours is not much of an improvement of their result from the practical viewpoint, it is interesting to note that our bound depends on the number of subtraction N , and that it in fact represents an improvement over the Jin-Martin result for $N < 7$.

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FIGURE CAPTIONS

- Figure 1 : The k plane cut along the lines $(-\infty, -\sqrt{3})$, $(\sqrt{3}, \infty)$, $(-i\infty, -i)$ and $(i, i\infty)$. This plane will be called D .
- Figure 2 : A continuous path in the cut plane D .
- Figure 3 : The k plane with four cuts starting at $k = \sqrt{3}, -\sqrt{3}, +id$, and $-id$.
- Figure 4 : The w plane whose upper half is the image of the cut k plane of Fig. 3.
- Figure 5 : The domain $D_1(-1-\xi) + D_2(-1-\xi)$. Several contour curves for $v(k) = \text{const}$ are also indicated by dashed lines.

FIG.1

FIG.2

FIG.3

FIG.4

FIG.5