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UNITARY PADE APPROXIMANTS IN STRONG COUPLING FIELD THEORY
AND APPLICATION TO THE CALCULATION OF THE ρ AND f_0
MESON REGGE TRAJECTORIES

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A B S T R A C T

We show that the unitary Padé approximants, which are well suited for absorbing the essential singularities at $g=0$ in Lagrangian field theory, have in the case of a Lagrangian $\mathcal{L}_I = -g/4(\phi_\alpha \phi^\alpha)^2$, where ϕ_α is the π field, the following nice features :

- 1 - they are built up from the perturbative S matrix expansion, coinciding with it up to fourth order in our case ;
- 2 - they are rigorously unitary (elastic) and contain also inelastic unitarity (4π) ; (the modulus of the diagonalized S matrix is smaller than 1 all over the inelastic cut) ;
- 3 - they have correct analytic properties in the energy variable ;
- 4 - there are no difficulties to extend them to complex values of the angular momentum ;
- 5 - they satisfy the right requirements between the number of inelastic channels open and the precision required on the crossing symmetry ;
- 6 - the complete equivalence between unitary Padé approximants and the approximations derived from the Lippmann-Schwinger variational principle is proved at all orders (using the Cini-Fubini "Ansatz").

We have computed the complete fourth order renormalized, including 2π and 4π contributions both in the direct and crossed channels.

The main results we obtain are the mass of the ρ and f_0 mesons and their Regge trajectories within 15% in agreement with experiment, in terms of only one parameter, the value of g (~ 6). The imaginary parts of Regge trajectories are found instable, so the widths are sensitive to the 6π and $K\bar{K}$ forces : in our model, the ρ and f_0 mesons appear as narrow objects.

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1. INTRODUCTION

Renormalizable Lagrangian field theory has long been regarded as unable to provide a calculational method for strong coupling physics. One relevant argument is the impossibility of taking seriously the formal solution of the field equations (that is, the S matrix expansion), because of the high value of the coupling constant.

The reason why the solution is said to be formal is that it is generally assumed that an essential singularity is present at the origin in the complex plane of the coupling constant. This feature is clearly exhibited by one-dimensional relativistic models, by the $g\phi^3$ theory¹⁾ in four dimensions, and by the Bethe-Salpeter equation²⁾.

For electrodynamics, Dyson³⁾ has given some argument for a singularity at $e=0$, whereas Baker and Chisholm⁴⁾ have shown this to be the case for the Peres model.

However, once the formal solution of a functional set of equations, in particular field equations, is given, there are various mathematical techniques which may be used to try the reconstruction of the actual true solution⁵⁾. Among these, the method of Padé "approximants" seems to be particularly suited for the physical problems of S matrix theory for the following reasons :

- a) it is known⁶⁾ from the theory of functions that Padé approximants can handle a class of functions with various types of singularities, including essential ones, still providing correct uniform convergence; we think that this point is of fundamental importance;
- b) for small coupling, the deviation between the series expansion of the S matrix and the one of the N th approximant starts at the $2N$ th coefficient of the series (and therefore it fulfils crossing to the same orders), while the N th approximant itself is easily expressed in terms of the first $2N$ coefficients of the formal series;

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- c) it has been proved ⁷⁾ that in potential theory, for a certain class of potentials, the Padé approximants do converge to the correct solution;
- d) in the scattering theory, and in particular in field theory, among all Padé approximants, it is possible to choose in a unique way a sequence which is rigorously unitary and at the same time has the best possible analytic properties in the energy variable, compatible with the requirements of field theory.

Let us say a few words on this point. We are giving a formal series

$$S = S_0 + g S_1 + g^2 S_2 + \dots \quad (1.1)$$

and we therefore can construct in various ways Padé approximants from it

$$S^{[N,P]}(g) = \frac{P(g)}{N(g)} \quad (1.2)$$

where $P(g)$ is a polynomial of degree P in g and $N(g)$ a polynomial of degree N , such that the $N+P$ first coefficients of the formal series coincide with the first $N+P$ coefficients of the expansion of (1.2). A first arbitrariness lies in the fact that we can choose many N 's and P 's having the same sum. Another more important arbitrariness comes from the fact that we can construct approximations for either the amplitude or the S matrix element or the phase shifts, and so on. All these procedures would provide us with definitely different results at a given order. Among all these possibilities, we show (Section 2) that there exists a unique Padé prescription which fulfils identically unitarity at all orders of approximation ⁸⁾. This prescription applies only to the S matrix and allows us to define the relativistic analogue of the Jost function introduced in potential scattering.

It is important to notice that unitarity and maximum analyticity (in the energy plane) are, for our approximants, deeply connected; the kind of spurious singularities introduced (if any) in the complex energy plane being at most poles. In the first attempts ⁹⁾ to apply the Padé technique to $\pi - \pi$ relativistic $g\rho^4$ theory, the method was used directly on the phase shifts: in this way unitarity is fulfilled, but very unpleasant essential singularities in the energy plane enter

into the S matrix approximant from the existence of poles in the phase shifts formula, thus spoiling to a large extent the convergence advantages of the Padé approach. Such spurious singularities are automatically avoided in our case.

- e) we show in the Note at the end of Section 2 that the set of unitary Padé approximants are identical with the set of the approximations derived by Cini and Fubini from the variational principle of Lippmann and Schwinger.

In this work we consider the pion-pion interaction at low and intermediate energies (up to 1-1.5 GeV) and introduce a Lagrangian density of the type $1/4g(\phi\phi^*)^2$.

The calculational procedure consists of building a unitary Padé approximant for the S matrix starting from the (renormalized) perturbation terms up to fourth order, including inelastic 4π contributions both in the direct and crossed channels.

We now come to the physical results predicted by this model: after having adjusted the value of the renormalized coupling constant g , in such a way as to reproduce the ρ meson mass, we have computed the Regge trajectory of the ρ meson. Very good agreement with experiment is found, both for the intercept and the slope at zero transfer. The slope turns out to be $1.05 (\text{GeV})^{-2}$ and is completely insensitive to small variation on the mass of the ρ meson. The intercept is 0.68 for a ρ mass of 770 MeV and decreases when increasing the mass.

The value of g is 6.02 for a ρ meson mass of 770 MeV and 6.10 for 750 MeV.

Going now to the $T=0$ channel, we find also a resonance in the d wave. If we adjust the value of g such that the resonance is identified with the f_0 meson, we find $g=7.16$, which is not far from the previous value. However, we do not pretend with such a simple model

to get the mass of the f_0 correctly : inelastic $K\bar{K}$ effects should be relevant at such energies. Nevertheless, the Regge trajectory of the f_0 meson can be computed with accuracy because for low energies and unphysical energies in which we are interested, the inelastic effects (6π and $K\bar{K}$) are small.

We finally discuss briefly the question of the widths of the resonances and imaginary parts of Regge trajectories. We find complete instability : very small variations of few percent in the perturbative fourth order term change the width by a factor two. This situation is typical of narrow resonances : in our model the ρ and f_0 mesons appear as narrow objects. A much more refined calculation which uses higher order approximants and takes into account six pions, $K\bar{K}$ pairs, seems necessary to get stability for the widths.

We conclude these introductory notes by mentioning the impossibility to use the method as it stands for s waves, because of uncontrolled subtraction parameters related to the absence of centrifugal barrier.

2.1 FORMAL DEFINITION OF RELATIVISTIC JOST FUNCTIONS

We consider a general elastic scattering process for which we have a formal development in the renormalized coupling constant g for the S matrix. By projection into partial waves, one can write

$$S(\ell, k, g) = 1 + g S_1(\ell, k) + S_2(\ell, k) g^2 + \dots + S_n(\ell, k) g^n + \dots \quad (2.1)$$

where k is the centre-of-mass momentum, and ℓ is the angular momentum index. Equation (2.1) can, of course, be extended to complex values of ℓ , k and g .

From now on, we shall suppose ℓ , k , and g complex. The unitarity condition reads :

$$S(\ell, k, g) S^*(\ell^*, k^*, g^*) = 1 \quad (2.2)$$

with an appropriate extension of $S(\ell, k, g)$ into complex values of ℓ which will be discussed later on. In the following we shall not write explicitly the ℓ and k dependence of S for convenience.

In this approach we want to approximate the expansion (2.1) by rational fractions in g . To be specific, we write for the N^{th} approximation :

$$S^{[N, N]}(g) = \frac{P_N(g)}{Q_N(g)} \quad (2.3)$$

where $S^{[N, N]}(g)$ is the N^{th} approximation of $S(g)$ and $P_N(g)$, $Q_N(g)$ are two polynomials of degree N in g . To determine P_N and Q_N we ask, as a first condition, that the expansion of $S^{[N, N]}(g)$ in powers of g has the same $2N$ first coefficients as the one of $S(g)$.

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To be specific we write :

$$S(g) = \Pi_{2N}(g) + g^{2N+1} R(g) \quad (2.4)$$

with

$$\Pi_{2N}(g) = 1 + S_1 g + \dots + S_{2N} g^{2N} \quad (2.5)$$

and

$$S^{[N,N]}(g) = \frac{P_N(g)}{Q_N(g)} = \Pi_{2N}(g) + g^{2N+1} \bar{R}(g) \quad (2.6)$$

We shall show that it is possible to do this in a unique way, and determine explicitly the polynomials. In fact, Eq. (2.6) is the definition of the $[N,N]$ Padé approximant to $S(g)$ ⁶⁾.

To determine the polynomials P_N and Q_N , we rewrite (2.6) in the form

$$P_N(g) = Q_N(g) \Pi_{2N}(g) + g^{2N+1} \bar{R}(g) Q_N(g) \quad (2.7)$$

By identification of the coefficient of g^k for $k = N+1, N+2, \dots, 2N$, we obtain a linear set of equations for the coefficients of $Q_N(g)$ which reads :

$$\begin{aligned} q_0 S_{N+1} + q_1 S_N + \dots + q_N S_1 &= 0 \\ q_0 S_{N+2} + q_1 S_{N+1} + \dots + q_N S_2 &= 0 \\ \dots &\dots \\ q_0 S_{2N} + q_1 S_{2N-1} + \dots + q_N S_N &= 0 \\ q_0 + q_1 g + \dots + q_N g^N &= Q_N(g) \end{aligned} \quad (2.8)$$

The solution of (2.8) can be written in a compact way :

$$\frac{Q_N(q)}{Q_N(0)} = \frac{\begin{vmatrix} S_1 & S_2 & \dots & S_N & q^N \\ S_2 & S_3 & \dots & S_{N+1} & q^{N-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ S_{N+1} & S_{N+2} & \dots & S_{2N} & 1 \end{vmatrix}}{\begin{vmatrix} S_1 & S_2 & \dots & S_N \\ \vdots & \vdots & \ddots & \vdots \\ S_{N+1} & S_{N+2} & \dots & S_{2N} \end{vmatrix}} \quad (2.9)$$

Once $Q_N(q)$ is known by (2.9), it is clear that $P_N(q)$ is uniquely determined by (2.7), in terms of the $2N$ first coefficients of the $S(q)$ expansion. We want to point out that the $[N, N]$ approximation involves only the knowledge of the $2N$ first perturbative coefficients of S .

For the time being, we have not yet considered the unitarity condition (2.2). We shall show now that the unitarity implies a very important feature for the $[N, N]$ approximant : the $[N, N]$ approximant satisfies rigorously the unitarity⁸⁾, that is :

$$S^{[N, N]}(q) \cdot S^{[N, N]*}(q^*) \equiv 1 \quad (2.10)$$

We report here an elegant proof due to Martin¹⁰⁾. From (2.6) we deduce :

$$S^{[N, N]}(q) S^{[N, N]*}(q^*) = \Pi_{2N}(q) \Pi_{2N}^*(q^*) + q^{2N+1} \bar{R}_1(q) \quad (2.11)$$

and from (2.2)

$$1 \equiv \Pi_{2N}(q) \Pi_{2N}^*(q^*) + q^{2N+1} R_1(q) \quad (2.12)$$

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Combining (2.11) and (2.12), we get

$$\frac{P_N(g) P_N^*(g^*)}{Q_N(g) Q_N^*(g^*)} = 1 + g^{2N+1} [\bar{R}_1(g) - R_1(g)] \quad (2.13)$$

or

$$P_N(g) P_N^*(g^*) - Q_N(g) Q_N^*(g^*) = g^{2N+1} Q_N(g) Q_N^*(g^*) [\bar{R}_1(g) - R_1(g)] \quad (2.14)$$

The left-hand side of (2.14) is a polynomial of degree $2N$ at most. However, the right-hand side goes to zero with g as g^{2N+1} at least; so, to be consistent, Eq. (2.14) implies that both sides are identically zero. We therefore deduce Eq. (2.10).

It is now easy to get a simple expression for $P_N(g)$. From

$$P_N(g) P_N^*(g^*) \equiv Q_N(g) Q_N^*(g^*) \quad (2.15)$$

we see that $P_N(g)$ is proportional to $Q_N^*(g^*)$.

If g_α is a root of $P_N(g)$, it is also a root of $Q_N(g) Q_N^*(g^*)$ by (2.15). But we suppose that the ratio $(P_N(g)/Q_N(g))$ is already written in irreducible form, so P_N and Q_N have no common roots. It then follows that g_α is a root of $Q_N^*(g^*)$. Therefore, because $P_N(g)$ and $Q_N^*(g^*)$ are both polynomials of the same degree having the same roots, they are proportional :

$$P_N(g) = A Q_N^*(g^*) \quad (2.16)$$

where A is independent of g . It is clear, using (2.15), that $|A| = 1$, and so

$$A = e^{i\alpha} \quad (2.17)$$

The $[N, N]$ approximant of the S matrix can now be written

$$S^{[N, N]}(g) = e^{i\alpha} \frac{Q_N^*(g^*)}{Q_N(g)} \quad (2.18)$$

The phase factor is fixed by the condition $S^{[N, N]}(0) = 1$ which follows by putting $g=0$ in Eq. (2.6).

Finally, Eq. (2.18) can be written :

$$S^{[N, N]}(g) = \frac{\{Q_N^*(g^*)/Q_N^*(0)\}}{\{Q_N(g)/Q_N(0)\}} \quad (2.19)$$

Formula (2.19) allows us to identify $(Q_N(g)/Q_N(0))$ with the N th approximation of the Jost function. In case we let $N \rightarrow \infty$, we obtain a formal expansion for the Jost function. We shall show that this expansion has all the usual properties expected of a Jost function ^{*}).

2.2 ANALYTIC PROPERTIES OF THE $[N, N]$ APPROXIMANT

One can show by unitarity that

$$\begin{vmatrix} S_1 & \dots & S_N \\ S_2 & \dots & S_{N+1} \\ \dots & \dots & \dots \\ S_{N+1} & \dots & S_{2N} \end{vmatrix} = (-1)^N \begin{vmatrix} S_1^* & \dots & S_N^* \\ S_2^* & \dots & S_{N+1}^* \\ \dots & \dots & \dots \\ S_{N+1}^* & \dots & S_{2N}^* \end{vmatrix} \quad (2.20)$$

^{*}) Our denomination of the Jost function for $(Q_N(g)/Q_N(0))$ is conventional. However, if one computes in potential scattering for a potential such as a δ function for which the calculation is easy, one finds precisely that $(Q_N(g)/Q_N(0))$ is the Jost function.

by S_m we mean $S_m(\ell, k)$ and by S_m^* we mean of course $S_m^*(\ell^*, k^*)$.
The $[N, N]$ approximation appears as the ratio of two determinants

$$S^{[N, N]}(q) = (-)^N \frac{\begin{vmatrix} S_1^* & \cdots & S_N^* & q^N \\ \vdots & & \vdots & \\ S_{N+1}^* & \cdots & S_{2N}^* & 1 \end{vmatrix}}{\begin{vmatrix} S_1 & \cdots & S_N & q^N \\ \vdots & & \vdots & \\ S_{N+1} & \cdots & S_{2N} & 1 \end{vmatrix}} \quad (2.21)$$

The numerator and the denominator of $S^{[N, N]}$ are clearly analytic functions of ℓ, k because they are multilinear functions of $S_1, S_2, \dots, S_{2N}$. In particular, the numerator and the denominator have the correct analytic properties required by the general field theory.

Thus our approximation has the following nice features :

- i) it coincides with the perturbation theory up to order $2N$.
[In particular, crossing symmetry is correct to the same order.];
- ii) it is rigorously unitary;
- iii) it has the required analytic properties.

However, we must point out that we can have zeros of the denominator in Eq. (2.21) which may be either physical bound states and resonances or unwanted ghosts, depending on their position in the k plane. This will be discussed later in more detail.

Before turning to the question of convergence, we want to point out an interesting property of our Jost functions.

2.3 A SYMMETRY PROPERTY OF THE $J^N(g)$

We know that due to time reversal, the amplitude is a real analytic function of the invariant s and t . This implies a symmetry for the $S_n(p, k)$, that is

$$S_n(p, k) = S_n^*(p^*, -k^*) e^{2i\pi p} \quad (2.22)$$

(2.22) is obvious for integer p , we shall discuss what happens in the case of complex p , later.

From this condition, and Eq. (2.9) which defines $J^N(g)$, we obtain

$$J^N(p, k, g) = J^{N*}(p^*, -k^*, g^*) \quad (2.23)$$

This relation is well known in potential scattering. With such a relation we can write the N^{th} approximation of our $S(p, k, g)$:

$$S^{[N, N]}(p, k, g) = \frac{J^N(p, -k, g)}{J^N(p, k, g)} \quad (2.24)$$

2.4 CONVERGENCE CONSIDERATIONS

Up to now, all our derivatives were purely formal. What is expected is that presumably in potential scattering the limit when $N \rightarrow \infty$ of $J^N(g)$ is the Jost function. What is already known is that, in potential scattering, for sufficiently regular potentials, the approximations $S^{[N, N]}(g)$ tend to the true $S(g)$ ⁷⁾. However, the existence of a limit for $S^{[N, N]}(g)$ does not imply of course a limit for $J^N(g)$. This problem will be discussed elsewhere.

In field theory it is very likely that point $g=0$ is an essential singularity : one-dimensional models exhibit this feature, and the four-dimensional $g\phi^3$ theory also has it ¹⁾. We shall suppose that this is the case for renormalized field theoretical models.

The series

$$S(g) = 1 + S_1 g + \dots + S_n g^n + \dots \quad (2.25)$$

must be thought as being an asymptotic series.

We remind the reader of some properties of asymptotic series :

- i) the series is divergent;
- ii) for a given value of $g=g_0$ the difference between $S(g_0)$ and

$$\sum_N (g_0) = 1 + S_1 g_0 + \dots + S_n g_0^n \quad (2.26)$$

$\Delta(g_0) = S(g_0) - \sum_N (g_0)$ passes by a minimum when N increases, as shown by Fig. 1, and then increases without limit. The minimal error $\Delta_m(g_0)$ depends on g_0 but can be very small in some cases. In general the minimum error $\Delta_m(g_0)$ increases with g_0 . It may then happen that if g_0 is sufficiently small, one can compute numerically with an extremely good approximation the value of $S(g)$ still using a divergent series !

A practical criterion to see up to which value of g one can trust the asymptotic expansion is to compare the order of magnitude of two subsequent terms.

We shall see that it happens that for our model of $\pi - \pi$ scattering ($g\phi^4$ model), with $g \approx 6$, the terms in (2.25) do not decrease at all and increase slightly instead. This shows clearly that (2.25) is useless.

The Jost function itself has an essential singularity at $g=0$, because otherwise the S matrix would not have it. What is very important to note is that the successive approximations for $J(\ell, k, g)$:

$$J^{(0)}(\ell, k, g) = 1 \quad (2.27)$$

$$J^{(1)}(\ell, k, g) = 1 + J_1^{(1)}(\ell, k) g \quad (2.28)$$

$$J^{(2)}(\ell, k, g) = 1 + J_1^{(2)}(\ell, k) g + J_2^{(2)}(\ell, k) g^2 \quad (2.29)$$

have their coefficients changing with the order of approximation.

This fact is of fundamental importance. It is well known in the theory of functions that the class of functions one can approach by a series of polynomials :

$$J^{(n)}(g) = J_0 + J_1 g + \dots + J_n g^n \quad (2.30)$$

where the coefficients J_i do not change with n , is very small. It is the class of analytic functions in a circle of radius r . On the other hand, the Weierstrass theorem on the approximation tells us that any continuous function can be approximated uniformly in a domain by a sequence of polynomials : however, in this case the coefficients of the polynomials change at each approximation, and as a consequence, the class of functions one can represent is infinitely much larger. In our case, it is precisely the fact that the Padé approximants have their coefficients changing with the order of approximation, which allows them, e.g., for Stieljes functions ⁶⁾, to converge uniformly in a complex neighbourhood of an essential singularity.

Before leaving this subject, we would like to discuss the problem of ghosts.

We are not sure that some zeros of the approximated Jost function $J^N(k, g)$ will not appear in the upper half k plane and give rise to a violation of causality.

At this point, we recall that a good approximation of an analytic function in some region does not need to have the same analytical structure as the original one. A good example is furnished by the Euler $\Gamma(z)$ function which is purely meromorphic and still extremely well approximated for large z by the Stirling function which has a different analytical structure.

Thus, in our case, what is important is to be sure that ghosts, if they exist, are sufficiently far away from the region in which we are interested.

We shall now apply the general ideas developed here to the concrete case of $\pi-\pi$ scattering in the $g\phi^4$ model.

N O T E

On the equivalence between the unitary Padé approximants and the approximations derived from the Lippmann-Schwinger variational principle ^{10')}

Cini and Fubini ^{10'')} have developed a set of equations which express in a simple form the approximations of the variational principle of Lippmann and Schwinger. We here want to point out the complete identity of this approach and the one of the unitary Padé approximants. The practical advantage of the latter method and of the exposition followed in this paper remains, however, in the fact that the renormalization programme is already taken into account in the formal S matrix expansion and one has to deal directly with renormalized quantities.

We here give the proof, for the case of complete diagonalization of the S matrix.

We consider Eqs. (4-A), (5-A) and (11) of Ref. ^{10'')} and we get after trivial steps:

$$S^{(n)} = \frac{2 - D^{(n)}}{D^{(n)}} \quad (\text{M.1})$$

where

$$D^{(n)} = 1 + \frac{1}{2} K^{(n)} \quad (\text{M.2})$$

$D^{(n)}$ fulfils the set of equations

$$D^{(n)} \left[1 + \frac{1}{2} \sum_{\kappa} S_{\kappa} \right] + \sum_{\alpha} S_{i+\alpha} \bar{J}_{\alpha} \quad (\text{M.3})$$

$0 \leq i \leq n$

where $\bar{J}_\alpha = J_n + \alpha'$ $1 \leq \alpha \leq n$ J_K being defined by Eq. (4-A) of Ref. 10").

After introducing the determinants

$$\Delta = \begin{vmatrix} s_1 & s_2 & \dots & s_n & 1 \\ s_2 & \dots & \dots & s_{n+1} & 1 \\ \dots & \dots & \dots & \dots & 1 \\ s_{n+1} & \dots & \dots & s_{2n} & 1 \end{vmatrix} \quad (\text{M.4})$$

$$\Delta' = \begin{vmatrix} s_1 & \dots & \dots & s_n & 1 \\ s_2 & \dots & \dots & s_{n+1} & 1 + \frac{s_1}{2} \\ \dots & \dots & \dots & \dots & \dots \\ s_{n+1} & \dots & \dots & s_{2n} & 1 + \frac{1}{2} \sum_{k=1}^n s_k \end{vmatrix} \quad (\text{M.5})$$

(in this note we take the coupling constant $g=1$), we obtain from (M.1)

$$S^{(n)} = \frac{2\Delta' - \Delta}{\Delta} \quad (\text{M.6})$$

We know, however, that our Padé approximant $S^{[N, \bar{N}]}$ can be written as

$$S^{[N, \bar{N}]} = (-1)^N \frac{\Delta^*}{\Delta}$$

[see Eq. (2.21)]. To show the identity of $S^{[N, \bar{N}]}$ with $s^{(n)}$ it is enough to show that

$$2\Delta' - \Delta = (-1)^N \Delta^* \quad (\text{M.7})$$

We know that $S^{(n)}$ fulfils unitarity

$$(2\Delta' - \Delta)(2\Delta' - \Delta)^* = \Delta\Delta^* \quad (\text{M.8})$$

Furthermore, $2\Delta' - \Delta$ and Δ are polynomials of the variables S_i and are supposed to be irreducible. Δ^* , Δ'^* being polynomials of S_i^* can be re-expressed through perturbative unitarity as polynomials of S_i . It then follows that $2\Delta' - \Delta = e^{i\alpha} \Delta^*$ where α is real and independent of S_i .

To determine α we can take a special set of S_i :

$$S_1 = S_2 = \dots = S_n = 0 \quad S_{n+1} = \dots = S_{2N} \neq 0, \text{ any value.}$$

It follows immediately that $e^{i\alpha} = (-1)^N$. This completes the proof.

3.1 GENERAL CONSIDERATIONS ON THE $\pi - \pi$ SCATTERING IN THE $g\phi^4$ MODEL

In order to apply the previous ideas to a physical problem, we use the $g\phi^4$ theory, including isotopic spin, because it is the simplest renormalized model for direct $\pi - \pi$ interaction.

We expect that a Lagrangian theory, with only one coupling constant, is correct for not too high energies. What limits the validity of such a model is the possibility of production. In our case the $K\bar{K}$ and $N\bar{N}$ channels are forgotten, the multiple (4π) channels being included in the model. However, the $K\bar{K}$ and $N\bar{N}$ production seems to be unimportant up to 1 GeV. We therefore think that above 1.5 GeV the model loses physical meaning. This number fixes the range of application of the model.

Another limitation of the model comes from the fact that it cannot be applied to s waves because the heavy intermediate states play certainly an important role to fix the dynamical parameters : for instance, the subtractions.

We are therefore left with only two possibilities : either the $T=1$, p wave amplitude, or the $T=0$ and two d wave amplitudes. Most of the discussion will be devoted to the $T=1$ amplitude.

3.2 THE JOST FUNCTION APPROXIMATION FOR THE $T=1$ p WAVE

In the special case of the $T=1$ channel, the Bose statistics forbids even angular momentum states. It then follows that the S (renormalized) matrix in $T=1$ has no term in g in its power series expansion. In this case formula (2.1) reads :

$$S_{T=1}(p, k, g) = 1 + S_2(p, k) g^2 + S_3(p, k) g^3 + S_4(p, k) g^4 + \dots \quad (3.1)$$

with $S_1(p, k) \equiv 0$.

The first diagonal Padé approximant is identical to one, as it can be checked directly. We are then obliged to go to the second Padé approximant: that is, to terms of fourth order in perturbation renormalized field theory. This is unavoidable if we want to have unitarity fulfilled.

By applying formula (2.21) for $S^{[2,2]}$ we see that :

$$S^{[2,2]}(p, k, g) = \frac{1 - \frac{S_3(p, k)}{S_2(p, k)} g + \left[\left(\frac{S_3(p, k)}{S_2(p, k)} \right)^2 - \frac{S_4(p, k)}{S_2(p, k)} + S_2(p, k) \right] g^2}{1 - \frac{S_3(p, k)}{S_2(p, k)} g + \left[\left(\frac{S_3(p, k)}{S_2(p, k)} \right)^2 - \frac{S_4(p, k)}{S_2(p, k)} \right] g^2} \quad (3.2)$$

In this case, the approximated Jost function is

$$J^{(2)}(p, k, g) = 1 - \frac{S_3(p, k)}{S_2(p, k)} g + \left[\left(\frac{S_3(p, k)}{S_2(p, k)} \right)^2 - \frac{S_4(p, k)}{S_2(p, k)} \right] g^2 \quad (3.3)$$

To obtain the numerator of (3.2), we have used the perturbative unitarity which tells us :

$$S_2(p, k) + S_2^*(p^*, k^*) = 0$$

$$S_3(p, k) + S_3^*(p^*, k^*) = 0 \quad (3.4)$$

$$S_4(p, k) + S_4^*(p^*, k^*) + S_2(p, k) S_2^*(p^*, k^*) = 0$$

It is convenient to introduce the functions

$$\Delta(\ell, k) = \frac{S_3(\ell, k)}{S_2(\ell, k)} \quad (3.5)$$

and

$$\tilde{\Delta}(\ell, k) = \frac{S_4(\ell, k)}{S_3(\ell, k)} \quad (3.6)$$

From (3.4) it is seen that $\Delta(\ell, k)$ is a real function, i.e.,

$$\Delta(\ell, k) = \Delta^*(\ell^*, k^*) \quad (3.7)$$

With these notations the Jost function $J^{[2]}$ reads :

$$J^{(2)}(\ell, k, g) = 1 + \Delta(\ell, k)g + [\Delta^2(\ell, k) - \tilde{\Delta}(\ell, k)]g^2 \quad (3.8)$$

For $\ell = 1$, we can use (3.8) to find the resonances. A resonance is obtained when the second degree polynomial in g has a real positive root : this gives us a relation between $\text{Im}k$ and $\text{Re}k$. By fixing the value of $\text{Re}k$ by its experimental number (given by the mass of the ρ meson), we deduce the $\text{Im}k$, i.e., the width of the ρ meson, and at the same time the value of g . With this value of g , we can now use (3.8) to compute the Regge trajectory of the ρ meson by going to complex ℓ , still looking for the zeros of (3.8) at fixed given k .

We have therefore a method for computing an approximation to the Regge trajectory of the ρ meson.

Before plunging into the details of calculation of $S_2(\ell, k)$, $S_3(\ell, k)$ and $S_4(\ell, k)$, we shall discuss the point of complex ℓ which has been left aside.

3.3 GENERAL FORMALISM INCLUDING THE CASE OF ℓ COMPLEX

We choose the π mass as a unit of mass; in this case the c.m. momentum is simply related to the s invariant as follows :

$$s = 4 + 4k^2 \quad (3.9)$$

The amplitude, with a given isotopic spin T , for the s channel is :

$$f^T(k, \cos \theta_s) = \sum_{\ell} (2\ell+1) \frac{[S^T(\ell, k) - 1]}{2ik} P_{\ell}(\cos \theta_s) \quad (3.10)$$

where the summation runs only over even (odd) ℓ if T is even (odd).

The invariant amplitude will be defined as :

$$\gamma^T(s, t, u) = -16\pi\sqrt{s} f^T(k, \cos \theta_s) \quad (3.11)$$

Introducing the functions :

$$\gamma^T(\ell, k) = \frac{1}{2} \int_{-1}^{+1} \gamma^T(s, t, u) P_{\ell}(\cos \theta_s) d \cos \theta_s \quad (3.12)$$

we see that

$$S^T(\ell, k) = 1 - \frac{2ik}{16\pi\sqrt{s}} \gamma^T(\ell, k) \quad (3.13)$$

Transforming (3.12) by the Froissart-Gribov formula, and taking into account the symmetries of the $\pi - \pi$ interaction, we can rewrite (3.12) as :

$$\gamma^T(\ell, k) = \frac{1}{\pi k^2} \int_4^{\infty} dt' \text{Abs}_t \gamma^T(s, t') Q_{\ell} \left(1 + \frac{t'}{2k^2} \right) \quad (3.14)$$

This formula must be interpreted in such a way that for even T it gives the even waves (the odd ones being identically zero), and for $T=1$, it gives the odd waves (the even ones being then identically zero). Formula (3.14) makes sense only for $\text{Re } \rho > N$, where N is the number of subtractions.

We shall now discuss the extension to complex ρ of (3.14). In this special case, when T is even (odd), the Carlsonian interpolation of odd (even) waves is trivial : it is the zero function. The interpolation of even (odd) waves is defined precisely by (3.14). However, this Carlsonian interpolation has the following features ¹¹⁾. The reality condition reads :

$$\gamma^T(\rho, s) = e^{2i\pi\rho} \gamma^{*T}(\rho^*, s^*, q^*) \quad (3.15)$$

The unitarity condition, however, reads

$$S^T(\rho, k, q) S^{*T}(\rho^*, k^*, q^*) = 1 \quad (3.16)$$

Equation (3.15) has, as a consequence, Eq. (2.22), which has been used previously, as one can derive immediately.

4. SYMMETRY PROPERTIES OF THE PION-PION INTERACTION (CROSSING RELATIONS)

We define the relativistic S matrix element between an initial two-pion state $|\alpha p_1; \beta p_2\rangle$ (α, β isospin indices, p_1 and p_2 four momenta) and the final $|\gamma p'_1; \delta p'_2\rangle$ as follows :

$$S_{fi} = \delta_{fi} - i (2\pi)^{-2} \frac{1}{\sqrt{16\omega_1\omega_2\omega'_1\omega'_2}} \delta(P_f - P_i) \left[A(s, t, u) \delta_{\alpha\beta} \delta_{\gamma\delta} + \right. \\ \left. + B(s, t, u) \delta_{\alpha\gamma} \delta_{\beta\delta} + C(s, t, u) \delta_{\alpha\delta} \delta_{\beta\gamma} \right] \quad (4.1)$$

where δ_{fi} is the generalized Kroneker symbol and

$$s = (p_1 + p_2)^2 \quad u = (p_4 - p_1)^2 \quad t = (p_3 - p_1)^2 \quad p_1 + p_2 = p_3 + p_4 \quad (4.2) \\ s + t + u = 4$$

The invariant amplitudes verify simple crossing symmetry relations, due to the combination of Bose statistics, analyticity and substitution law :

$$\left. \begin{array}{l} A \rightarrow A \\ B \rightarrow C \end{array} \right\} t \leftrightarrow u \quad s \rightarrow s \quad \left. \begin{array}{l} A \rightarrow B \\ C \rightarrow C \end{array} \right\} s \leftrightarrow t \\ u \rightarrow u \quad (4.3) \\ \left. \begin{array}{l} A \rightarrow C \\ B \rightarrow B \end{array} \right\} s \leftrightarrow u \\ t \rightarrow t$$

We now consider the isospin amplitudes $\chi^T(s, t, u)$ (where s is the energy squared). They are simply related with A, B, C by introducing the isospin projection operators P_T (normalized to one) :

$$P_0 = \frac{1}{3} \delta_{\alpha\beta} \delta_{\gamma\delta} \quad P_1 = \frac{1}{2} (\delta_{\alpha\gamma} \delta_{\beta\delta} - \delta_{\alpha\delta} \delta_{\beta\gamma}) \quad (4.4)$$

$$P_2 = \frac{1}{2} (\delta_{\alpha\gamma} \delta_{\beta\delta} + \delta_{\alpha\delta} \delta_{\beta\gamma}) - \frac{1}{3} \delta_{\alpha\beta} \delta_{\gamma\delta}$$

The crossing relations satisfied by the γ^T 's are easily expressed in terms of the crossing matrix $\alpha^{TT'}$:

$$\alpha^{TT'} = \begin{vmatrix} \frac{1}{3} & 1 & \frac{5}{3} \\ \frac{1}{3} & \frac{1}{2} & -\frac{5}{6} \\ \frac{1}{3} & -\frac{1}{2} & \frac{1}{6} \end{vmatrix} \quad (4.5)$$

$$\begin{aligned} \gamma(s, t, u) &= \sum_{T'} \alpha^{TT'} \gamma^{T'}(t, s, u) = \\ &= \sum_{T'} \alpha^{TT'} \gamma^{T'}(u, t, s) \end{aligned} \quad (4.6)$$

5. RENORMALIZATION AND EXPLICIT EVALUATION OF THE PION-PION AMPLITUDES UP TO THIRD ORDER

We here assume that the direct $\pi - \pi$ interaction is essentially given by the Lagrangian density $\mathcal{L}_I = -(g_0/4)(\phi^\alpha \phi_\alpha)^2$, $\alpha = 1, 2, 3$, where g_0 is the bare coupling constant.

In our approximate Jost function, the quantities S_2, S_3, S_4 must be computed from perturbation theory starting from this Lagrangian and performing the programme of renormalization up to the fourth order. We notice, however, that the Froissart-Gribov formula (3.14) requires only knowledge of the absorptive parts of the invariant amplitudes $\gamma^T(s, t, u)$; this compels us to evaluate the whole γ^T renormalized up to third order, whereas the absorptive parts of γ^T at the fourth order will be derived from unitarity in terms of the previous orders, (noting that it is only for the renormalized S matrix that unitarity applies). We recall now ¹²⁾ that the pion-pion renormalized amplitudes $\gamma^T(s, t, u)$ have the property of being a Clebsch-Gordan factor times the renormalized coupling constant g at the symmetry point $s = t = u = 4/3$. To be more specific, we have :

$$\gamma^T\left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right) = g \cdot \begin{vmatrix} 1 & 0 \\ 0 & 0 \\ 4 & 0 \end{vmatrix} \begin{matrix} T=0 \\ T=1 \\ T=2 \end{matrix} \quad (5.1)$$

The relation between $\gamma^T(s, t, u)$ and the S matrix being ^{*)} :

$$S_{fi} = \delta_{fi} - i \frac{(2\pi)^4}{(2\pi)^6 (2\omega)^2} \delta^{(4)}(P_i - P_f) \sum_0^2 \gamma^T(s, t, u) P^T \quad (5.2)$$

*) Our conventions coincide with those in Drell-Bjorken's book :
 "Relativistic quantum fields", pp. 269-274.

All higher order terms of the γ^T expansion in powers of the renormalized coupling g :

$$\gamma^T = g \gamma^{(1)T} + g^2 \gamma^{(2)T} + g^3 \gamma^{(3)T} + \dots \quad (5.3)$$

must therefore vanish identically at the symmetry point. This condition, together with the fact that up to third order, we do not deal with the renormalization of the pion propagator (internal lines in Feynman graphs), allows us to identify (up to third order) the expansion (5.3) with that of the non-renormalized γ_o^T amplitudes times the $Z^{-\frac{1}{2}}$ constant which renormalizes the external wave functions, provided g_o and $Z^{-\frac{1}{2}}$ are given in power series of g and the bare mass is replaced by the physical one in the bare terms $\gamma_o^{(1)T}$, $\gamma_o^{(2)T}$, $\gamma_o^{(3)T}$:

$$\left. \gamma^T(s, t, u) \right|_{\text{up to third order}} = Z^{-1/2} \{ g_o \gamma_o^{(1)T} + g_o^2 \gamma_o^{(2)T} + g_o^3 \gamma_o^{(3)T} + \dots \} \quad (5.4)$$

with

$$Z^{-1/2} = 1 + \beta_1 g^2 + \beta_2 g^4 + \dots \quad (5.5)$$

and

$$g_o = g [1 + \alpha_1 g + \alpha_2 g^2 + \dots] \quad (5.6)$$

An important feature of the β 's and α 's is that they are independent of T . $\beta_1, \beta_2, \dots, \alpha_1, \alpha_2$ are (infinite) constants that can be easily expressed in terms of a cut-off, as will be shown later.

By putting expressions (5.5) and (5.6) into Eq. (5.4), we have :

$$\gamma^T \underset{\text{up to third order}}{=} g \gamma^{(1)T} + g^2 [\alpha_1 \gamma_o^{(1)T} + \gamma_o^{(2)T}] + g^3 [(\alpha_2 + \beta_1) \gamma_o^{(1)T} + 2\alpha_1 \gamma_o^{(2)T} + \gamma_o^{(3)T}] + \dots \quad (5.7)$$

Finally

$$\gamma^{(1)T} \equiv \gamma_o^{(1)T} = \begin{vmatrix} 10 \\ 0 \\ 4 \end{vmatrix} \quad (5.8)$$

$$\gamma^{(2)T} = \alpha_1 \gamma_o^{(1)T} + \gamma_o^{(2)T} \quad (5.9)$$

$$\gamma^{(3)T} = (\alpha_2 + \beta_1) \gamma_o^{(1)T} + 2\alpha_1 \gamma_o^{(2)T} + \gamma_o^{(3)T} \quad (5.10)$$

The evaluation of $\gamma_o^{(2)T}$, $\gamma_o^{(3)T}$, $\gamma_o^{(1)T}$ is easily performed from the Lagrangian given above. We give here the result for $\gamma_o^{(2)T}$:

$$\gamma_o^{(2)T} = \begin{vmatrix} i [50 I(s) + 30 I(t) + 30 I(u)] \\ i [10 I(t) - 10 I(u)] \\ i [8 I(s) + 18 I(t) + 18 I(u)] \end{vmatrix} \quad (5.11)$$

where the function $I(s)$ is given in Appendix A, as well as the expressions for $\gamma_o^{(3)T}$.

The condition at the symmetry point on the χ^T , $T=0, 2$ (for $T=1$ it is identically verified), looks as follows :

$$\alpha_1 \chi_0^{(1)T} + \chi_0^{(2)T} \Big|_{t=s=u=\frac{4}{3}} = 0 \quad (5.12)$$

$$\eta \chi_0^{(1)T} + 2\alpha_1 \chi_0^{(2)T} + \chi_0^{(3)T} \Big|_{t=s=u=\frac{4}{3}} = 0 \quad (5.13)$$

$$\eta = \alpha_2 + \beta_1$$

From Eqs. (5.12) and (5.13), we obtain

$$\alpha_1 = \frac{11\pi^2}{(2\pi)^4} \left[\Lambda - 2a\left(\frac{4}{3}\right) \right] \quad (5.14)$$

$$\eta = \frac{195}{16} \frac{1}{(2\pi)^4} \left[\Lambda - 2a\left(\frac{4}{3}\right) \right]^2 - \frac{37}{2(2\pi)^4} \left[\frac{1}{4}(1+\Lambda)^2 - (1+\Lambda)a\left(\frac{4}{3}\right) + b\left(\frac{4}{3}\right) + \frac{11\pi^2}{54} - \frac{3}{4} \right] \quad (5.15)$$

We notice that the values of α_1 and η calculated from Eqs. (5.12) and (5.13) with $T=0$ are identical with those obtained by using formulae of the $T=2$ isospin, as expected. We again refer the reader to Appendix A for explicit renormalized formulae for $\chi^{(2)T}$ and $\chi^{(3)T}$.

Finally, we point out that the renormalized $\chi^{T(n)}$ ($n=1,2,3$) verify the crossing relations given in Section 4, as it is easy to check directly.

6. FOURTH ORDER CALCULATION

6.1 GENERALITIES

We are interested in computing the fourth order term of the S matrix expansion : (for convenience we develop the formalism for the $T=1$ channel, very slight variations have to be done for the other cases)

$$S_4^{T=1}(p, k) = \frac{-i}{16\pi^3 k \sqrt{k^2+1}} \int_4^{+\infty} ds' \text{Abs}_t \chi^{T=1}(s, s') Q_p \left(1 + \frac{s'}{2k^2}\right) \quad (6.1)$$

The problem is to compute $\text{Abs}_t \chi^{T=1}(s, s')$. In this respect we use the crossing symmetry relation, (4.6), applied to the absorptive parts :

$$\text{Abs}_t \chi^{T=1}(s, s') = \frac{1}{3} \text{Abs}_s \chi^{T=0}(s', s) + \frac{1}{2} \text{Abs}_s \chi^{T=1}(s', s) - \frac{5}{6} \text{Abs}_s \chi^{T=2}(s', s) \quad (6.2)$$

We are therefore left with the calculation of the absorptive parts in the s channel, which are now computed through unitarity, from the second and third perturbative order. At fourth order, unitarity relates the $\text{Abs}_s \chi^T(s', s)$ to the first, second and third orders. We have to distinguish different kinds of contributions :

- A. the so-called "elastic" unitarity contributions which can be thought as related to the following Feynman graphs (see Fig. 2), which we give for better understanding;
- B. the so-called "inelastic" unitarity contributions which come essentially from the graphs of Fig. 3.

6.2 ELASTIC UNITARITY CONTRIBUTION

We shall first compute the "elastic" part of $\text{Abs}_s \gamma^T(s', s)$ at fourth order (which is the whole contribution for $4 < s' < 16$). We have

$$\text{Abs}_s \gamma^{(el)T}(s', s) = \sum_p (2p+1) \text{Im} \gamma_p^{(el)T}(s') P_p \left(1 + \frac{2s}{s'-4}\right) \quad (6.3)$$

This converges in some region of the s plane. The elastic unitarity reads :

$$\text{Im} \gamma_p^{(el)T}(s') = - \frac{1}{16\pi} \frac{k'}{\sqrt{s'}} |\gamma_p^{(el)T}(s')|^2 \quad (6.4)$$

or, in perturbative fourth order (renormalized) :

$$\begin{aligned} \text{Im} \gamma_p^{(4)T}(s') = - \frac{1}{16\pi} \frac{k'}{\sqrt{s'}} \{ & \gamma_p^{(1)T}(s') \gamma_p^{*(3)T}(s') + \\ & + \gamma_p^{*(1)T}(s') \gamma_p^{(3)T}(s') + \gamma_p^{*(2)T}(s') \gamma_p^{(2)T}(s') \} \end{aligned} \quad (6.5)$$

It is more convenient, which will appear clear in the following, to separate the s wave part (of course only for $T=0$ and 2) from the rest.

6.3 THE EXPLICIT CALCULATION OF THE s WAVE CONTRIBUTION (ELASTIC)

We have, ($T=0$ or 2), from Eq. (5.8)

$$\gamma_{p=0}^{(1)T=0}(s') = 10 \quad (6.6)$$

$$\gamma_{p=0}^{(1)T=2}(s') = 4 \quad (6.7)$$

and from (A.14)

$$\gamma_{\ell=0}^{(el) T=0}(s') = \frac{5}{4\pi^2} \left\{ -11 a\left(\frac{4}{3}\right) + 5 a(s') + 3 \Lambda(\kappa') \right\} \quad \kappa' = \frac{\sqrt{s'-4}}{2} \quad (6.8)$$

where

$$\Lambda(\kappa') = \int_{-1}^{+1} a(t) d\cos\theta_s \quad (6.9)$$

$$\gamma_{\ell=0}^{(el) T=2}(s') = \frac{1}{4\pi^2} \left\{ -22 a\left(\frac{4}{3}\right) + 4 a(s') + 9 \Lambda(\kappa') \right\} \quad (6.10)$$

In the same way we get

$$\gamma_{\ell=0}^{(3) T}(s')$$

(see Appendix A). Finally, we can write for the contributions to

$$Abs \gamma_{(s', s)}^{(4) T=0,2}$$

due to elastic s wave :

$$Im \gamma_{\ell=0}^{T=0,2}(s') = - \frac{1}{32\pi} \sqrt{\frac{s'-4}{s'}} \left[A_{0,2}(s') + B_{0,2}(s') \right] \quad (6.11)$$

where the functions $A_{0,2}(s')$ and $B_{0,2}(s')$ are explicitly given in Appendix B.

6.4 THE EXPLICIT CALCULATION OF NON s WAVE CONTRIBUTIONS (ELASTIC)

We have to compute now

$$Abs_{(s>4) s} \gamma_{\ell=0}^{(el) T}(s', s) - Im \gamma_{\ell=0}^{T(el)}(s') = \sum_{\substack{\ell \neq 0 \\ (even \text{ or } odd)}} (2\ell+1) Im \gamma_{\ell}^{(el) T}(s') P_{\ell}^{(1+\frac{2s}{s'-4})} \quad (6.12)$$

At fourth order :

$$\text{Im } \gamma_p^{(el)(4)T}(s') = -\frac{1}{16\pi} \frac{K'}{\sqrt{s'}} \gamma_p^{*(el)(2)T}(s') \gamma_p^{(el)(2)T}(s') \quad (6.13)$$

for $p \neq 0$. But

$$\gamma_p^{(2)* (el)T}(s') = \frac{1}{\pi K'^2} \int_4^{+\infty} \text{Abs}_t \gamma_p^{(2)T}(s', t') Q_p\left(1 + \frac{t'}{2K'^2}\right) dt' \quad (6.14)$$

by Eq. (3.14).

But following Eq. (A.14) of Appendix A, we have :

$$\text{Abs}_t \gamma_p^{(2)T}(s', t') = \frac{C^T}{4\pi^2} \text{Im } a(t') = -\frac{C^T}{4\pi^2} \frac{\pi}{2} \sqrt{\frac{t'-4}{t'}} \quad (6.15)$$

$$C^T = \begin{pmatrix} 15 \\ 5 \\ 9 \end{pmatrix} \quad \begin{matrix} T=0 \\ T=1 \\ T=2 \end{matrix}$$

So ($p \neq 0$) :

$$\text{Im } \gamma_p^{(el)(4)T}(s') = -\frac{1}{16\pi} \frac{K'}{\sqrt{s'}} \frac{[C^T]^2}{\pi^2 K'^4} \cdot \frac{1}{(8\pi)^2} \int_4^{+\infty} dt' \int_4^{+\infty} dt'' \sqrt{\frac{(t'-4)(t''-4)}{t't''}} \cdot Q_p\left(1 + \frac{t'}{2K'^2}\right) Q_p\left(1 + \frac{t''}{2K'^2}\right) \quad (6.16)$$

which is converging for $p > 0$.

For convenience, we introduce the functions

$$\phi_0(\xi, \eta, z) = \sum_{p \text{ (odd)}}^{+\infty} (2p+1) Q_p(\xi) Q_p(\eta) P_p(z) \quad (6.17)$$

$$\phi_E(\xi, \eta, z) = \sum_{\substack{\ell \geq 2 \\ \text{even}}}^{+\infty} (2\ell+1) Q_\ell(\xi) Q_\ell(\eta) P_\ell(z) \quad (6.18)$$

With these functions we have

$$\text{Abs}_S^{(el)T} \chi^{(4)}(s', s) = \text{Im} \chi_{\ell=0}^{(el)T}(s') - \frac{(C^T)^2}{128\pi^5 \sqrt{s'} (s'-4)^{3/2}} \cdot \int_4^{+\infty} dt' \int_4^{+\infty} dt'' \sqrt{\frac{(t'-4)(t''-4)}{t't''}} \cdot \phi_T(\xi, \eta, z) \quad (6.19)$$

$$\xi = 1 + t'/2k'^2 \quad \eta = 1 + t''/2k'^2 \quad z = 1 + 2s/s'-4$$

where $\phi_T = \phi_O$ for $T=1$, and $\phi_T = \phi_E$ for $T=0$ or 2 . See Appendix B for a compact form for the ϕ functions.

There is no problem of inverting the summation and the double integral because for $0 < s < 4$, for instance, all terms are positive and everything is absolutely convergent. By analytic continuation, formula (6.19) defines the correct $\text{Abs}_S^{(el)T} \chi(s', s)$ when it converges. In particular because we have subtracted the s wave, it is easy to show that it converges everywhere in the s cut plane.

6.5 INELASTIC UNITARITY

The structure of the pion-pion interaction, according to the ϕ^4 theory, correlates the elastic unitarity in the s channel with the four-pion exchange in the crossed channels (t, u) . This is easily seen by looking at spectral functions in Appendix C. We therefore need the absorptive parts of $\chi_T(s, t, u)$ at the fourth order due to the four-pion exchange in the t channel. This calculation can be done in

various ways; we outline here the method based on the unitarity condition, whereas in Appendix C we illustrate alternative derivations of the same formulae. Let us take the S matrix element between a two-pion state $|i\rangle$, or $|f\rangle$ and a four-pion state $|n\rangle$:

$$S_{nf} = -i(2\pi)^4 \delta^{(4)}(P_f - P_n) \frac{\gamma_{nf}}{(2\pi)^9 \sqrt{64 \omega_3 \omega_4 \omega'_1 \omega'_2 \omega'_3 \omega'_4}} \quad (6.20)$$

$$S_{ni} = -i(2\pi)^4 \delta^{(4)}(P_i - P_n) \frac{\gamma_{ni}}{(2\pi)^9 \sqrt{64 \omega_1 \omega_2 \omega'_1 \omega'_2 \omega'_3 \omega'_4}} \quad (6.21)$$

where isospin indices have not been written explicitly for reasons of simplicity. The discontinuity of $\gamma_{if}(s,t,u) \equiv \sum_T \gamma_{if}^T P_T$, due to the four-pion exchange in the s channel, follows from unitarity

$$[\gamma_{if}^{(in)}] = -\frac{1}{4!} \sum_{|n\rangle} (2\pi)^4 \delta^{(4)}(P_i - P_n) \gamma_{ni}^* \gamma_{nf} \quad (6.22)$$

where the symbol $\sum_{|n\rangle}$ stands for summation over all the permutations of the intermediate isospin indices and of p'_1, p'_2, p'_3, p'_4 and also the integration $\int \frac{d^4 p'_1 d^4 p'_2 d^4 p'_3 d^4 p'_4}{(2\pi)^{12}} \delta_+(p_1'^2 - 1) \delta_+(p_2'^2 - 1) \delta_+(p_3'^2 - 1) \delta_+(p_4'^2 - 1) \cdot$

$$\cdot \delta^{(4)}(P_i - p'_1 - p'_2 - p'_3 - p'_4)$$

over the phase space.

Let us consider the fourth order discontinuity $[\gamma_{if}^{(4)(in)}]$ (due to four pions), which follows from combining in all possible ways second order diagrams of the type shown in Fig. 4.

The various contributions $\gamma_A, \gamma_B, \dots$ which we obtain from Eq. (6.22) can be put in correspondence with the graphs of Fig. 5.

We have

$$\frac{1}{2i} [\gamma_{if}^{(4) \text{ in}}] = \text{Abs}_{s' > 16} [\gamma_A^{(\text{in})} + \gamma_B^{(\text{in})} + \gamma_C^{(\text{in})} + \gamma_D^{(\text{in})} + \gamma_E^{(\text{in})} + \gamma_F^{(\text{in})}] \quad (6.23)$$

We give here the final result, details will be found in Appendix C ;

$$\text{Abs}_{s' > 16} \gamma_A^{(\text{in})}(s', t) = - \{ 235 P_0 + 20 P_1 + 76 P_2 \} R(s', t) \quad (6.24)$$

and

$$\text{Abs}_{s' > 16} \gamma_B^{(\text{in})}(s', u) = - \{ 235 P_0 - 20 P_1 + 76 P_2 \} R(s', u) \quad (6.25)$$

$$u = 4 - s' - t$$

and

$$\text{Abs}_{s' > 16} \gamma_D^{(\text{in})} = - \frac{8}{(2\pi)^{12}} \{ 1100 P_0 + 224 P_2 \} \cdot \frac{(2\pi)^4}{2} \cdot \frac{\pi^3}{8 K' \sqrt{s'}} M(s') \quad (6.26)$$

$$\text{Abs}_{s' > 16} \gamma_C^{(\text{in})} = - \frac{8}{3} \frac{1}{(2\pi)^{12}} \{ 375 P_0 + 60 P_2 \} \cdot \frac{(2\pi)^4}{2} \cdot \frac{\pi^3}{s'} N(s') \quad (6.27)$$

$$\text{Abs}_{s' > 16} \gamma_E^{(\text{in})} = - \frac{8}{(2\pi)^{12}} \{ 375 P_0 + 36 P_2 \} \cdot \frac{(2\pi)^4}{2} \cdot \frac{\pi^3}{s'} P(s') \quad (6.28)$$

where the functions $R(s', t)$, $M(s')$, $N(s')$, $P(s')$ are given in Appendix C. We notice here that the terms $\gamma_C^{(\text{in})}$, $\gamma_D^{(\text{in})}$ and $\gamma_E^{(\text{in})}$ depend only on the variable s' , they are related therefore to one-dimensional spectral functions and are present in isospin states $T=0, 2$.

Finally to derive the absorptive inelastic part in the t channel we use (as we did in the elastic case) the crossing relations (4.6), and we get

$$S_4^{(in)}(s) = -\frac{i}{16\pi^2 K \sqrt{K^2+1}} \int_{16}^{+\infty} ds' \text{Abs}_{s'>16}^{(4) T=1} \gamma(s, s') Q_e\left(1 + \frac{s'}{2K^2}\right) \quad (6.29)$$

with

$$\text{Abs}_{s'>16}^{T=1} \gamma(s, s') = \frac{1}{3} \text{Abs}_{s'>16}^{T=0} \gamma(s', s) + \frac{1}{2} \text{Abs}_{s'>16}^{T=1} \gamma(s', s) - \frac{5}{6} \text{Abs}_{s'>16}^{T=2} \gamma(s', s) \quad (6.30)$$

That is

$$\text{Abs}_{s'>16}^{T=1} \gamma(s, s') = -25 R(s', s) - 5 R(s', 4-s'-s) + f(s') \quad (6.31)$$

and

$$f(s') = -\frac{1}{(2\pi)^5} \left\{ \frac{45}{4} \frac{M(s')}{\sqrt{s'} K'} + \frac{75}{6} \frac{N(s')}{s'} + \frac{95}{2} \frac{P(s')}{s'} \right\} \quad (6.32)$$

We are finally left with the contribution γ_F , whose spectral function starts at higher values of s and t ¹³⁾, in regions where six-pion exchange cannot be disregarded.

We therefore neglect the non-coplanar graph F , being aware that we violate slightly the four-pion inelastic unitarity in the $T=0, 2$ amplitudes only. However, we have estimated in numerical calculation some upper bound for this contribution. See the general discussion and Appendix C.

7. NUMERICAL RESULTS AND PHYSICAL CONCLUSIONS

We have used the CDC 6600 computer. The calculations have been made directly with complex numbers by using subroutines for complex log, square roots and Legendre functions of the second kind. The results can be divided as follows:

7.1 THE $T=1, \ell=1$ CASE

For general T and ℓ (excluding the case of s waves), the second order Jost function reads, following formula (2.9)

$$J_T^{(2)}(\ell, k, g) = 1 - \Delta_T(\ell, k) g + [\Delta_T^2(\ell, k) - \tilde{\Delta}_T(\ell, k)] g^2 \quad (7.1)$$

where

$$\Delta_T(\ell, k) = \frac{S_3^{(T)}(\ell, k)}{S_2^{(T)}(\ell, k)} \quad (7.2)$$

$$\tilde{\Delta}_T(\ell, k) = \frac{S_4^{(T)}(\ell, k)}{S_2^{(T)}(\ell, k)} \quad (7.3)$$

In the case of $T=1, \ell=1$, we have computed the two roots g_- and g_+ of $J^{(2)}[1, k, g] = 0$. In general $g_-(k)$ and $g_+(k)$ are complex. We have looked for values of k which make them real. Table I gives the values of $\text{Re } g_{\pm}$, $\text{Im } g_{\pm}$, as functions of $\text{Re } k$ and $\text{Im } k$, in the range $2.40 < \text{Re } k < 2.80$. One sees in that Table two striking facts :

1. only g_- is near to a real positive number, for values of k near the physical ρ meson pole

$$\text{for } M_\rho = 752 \text{ MeV} \quad \begin{aligned} \text{Re } k_\rho &= 2.53 & \text{Im } k_\rho &= -0.23 \\ \Gamma_\rho &= 120 \text{ MeV} \end{aligned}$$

we notice also that this number is of an acceptable order of magnitude $g_- \simeq 6.09$:

2. g_+ is completely out of physical interest, being largely complex. A remarkable fact is that the ratio $|g_-/g_+|$ is of the order of 5%. This means that the g^2 term is very small compared to the g term in (7.1).

Before going to the detailed analysis of the results, we want to discuss the problem of the mathematical accuracy we obtain.

The second order approximation is

$$J^{(2)}(g) = 1 + J_1^{(2)}g + J_2^{(2)}g^2 \quad (7.4)$$

The third order would be

$$J^{(3)}(g) = 1 + J_1^{(3)}g + J_2^{(3)}g^2 + J_3^{(3)}g^3 \quad (7.5)$$

The error could be computed by differentiating between (7.5) and (7.4). However, it is not possible for the moment to compute the third order, or at least some bound on the error, because clearly the $J_1^{(3)}$ terms, e.g., contain already the sixth order perturbation for the S matrix!

We want to emphasize that it would be a mistake to think that the term $J_2^{(2)}g^2$ is only some correction to $J_1^{(2)}g$. The reason is that the second degree polynomial in g is the best of all second degree polynomials for approximating the exact $J(k, l, g)$. For instance, if one neglects the $J_3^{(3)}g^3$ terms in $J^{(3)}$, one gets a polynomial of second degree which will be worse for approximating than $J^{(2)}$.

To have an idea of the order of magnitude of the mathematical error we commit, we refer to the paper of Baker and Chisholm⁴⁾; they compute for the Peres model the error for $g=1, 3$, and 10 . For the $[2,2]$ approximant, the error increases with g , and does not exceed 5% for $g \sim 6$. We think that in our, even more complicated, model the error is of the same order.

We come now to the discussion of the width of the ρ meson. We find a width of the order of 30 MeV. We want to point out that this number is not significant.

The mathematical reason is that a change of factor 2 for $\text{Re } S_4^{(el)}$, changes the width also by a factor 2, as shown by Table I. However, $\text{Re } S_4^{(el)}$ is only 10% of the S_4 contribution, which is nearly purely imaginary. This means that a change of a few percent in the S_4 term changes the width by a factor 2.

Another way to see this point is to note that in the perturbative expansion for $T=1$, $\ell=1$:

$$S(k, g) = 1 + S_2(k) g^2 + S_3(k) g^3 + S_4(k) g^4 \quad (7.6)$$

the functions S_2 and S_3 are purely imaginary, and in S_4 we can also separate the "real" and "imaginary" parts:

$$\text{"Re"} S_4(k) = \frac{1}{2} [S_4(k) + S_4^*(k^*)] \quad (7.7)$$

$$\text{"Im"} S_4(k) = \frac{1}{2i} [S_4(k) - S_4^*(k^*)] \quad (7.8)$$

One can think to write:

$$\text{"Im"} S(k, g) = S_2(k) g^2 + S_3(k) g^3 + i \text{"Im"} S_4(k) g^4 \quad (7.9)$$

$$\text{"Re"} S(k, g) = 1 + \text{"Re"} S_4(k) g^4 \quad (7.10)$$

It is only the presence of $\text{Re } S_4(k)g^4$ which gives rise to unitarity. The other terms do not have the elastic cut. One now understands why it is difficult to get stability in the width: the term $\text{Re } S_4(k)g^4$ is only one tenth of $\text{Im } S_4(k)g^4$, and $\text{Re } S_4(k)g^4$, still being a fourth order contribution with respect to g , is in fact a first order perturbation if one thinks in terms of real and imaginary parts, separately.

The situation can be described, more physically, as follows :

- either the ρ meson is a large resonance, in this case it is not sensitive to the detailed structure of the forces;
- or the ρ meson is a narrow resonance, and then the width is very sensitive to the existence of the 6π and $K\bar{K}$ forces.

In fact, Wanders¹⁴⁾ has found that when one studies the N/D equations, if the resonance is narrow, a variation of a few percent of the left-hand cut discontinuity makes the width completely unstable.

To understand this last fact better, let us go into the structure of partial waves, see Fig. 6.

Suppose we neglect in the direct channel the 4π inelastic cut, the error committed is of the order of 1%, because this error is of order $(\sigma_{\text{tot}}(4\pi)/\sigma_{\text{tot}}(2\pi))$ which, at the energy of the ρ meson, is estimated¹⁵⁾ to be $\sim 1\%$. The error of 1% on the right-hand cut will not affect very much the ρ meson width, because the point in which we compute the function (the ρ meson pole) is not far away from the inelastic cut. Suppose we now make a 1% error on the left-hand cut. For instance, let us neglect the 6π cut, then the error which is 1% on the left, has to be propagated through the complex plane at large distances up to the ρ meson pole; it will generally increase very much and, as shown by Wanders in the case of a narrow resonance, the error can be such that instability occurs. However, the real part of the position of the pole, still being sensitive to such left-hand cut variations, is not very much affected.

In Table I we have given $\text{Re } S_4(\text{el})$, $\text{Re } S_4(\text{inel})$, $\text{Im } S_4(\text{el})$ and $\text{Im } S_4(\text{inel})$ to show the relative importance of these terms and also that it is only $\text{Re } S_4(\text{el})$ which is sensitive to the width change, in agreement with the fact that it is only the presence of this term which makes unitarity to be satisfied.

A point of importance is to check the possible presence of ghosts, i.e., unwanted poles in the upper k plane. We have made the exploration in the domain of interest :

$$0 < \text{Re } k < +10 \\ -1 < \text{Im } k < +1$$

No ghosts were found and only one resonance (which we have identified with the ρ meson) is found. The dependence of g on the ρ mass is summarized in Fig. 7.

7.2 THE REGGE TRAJECTORY OF THE $T=1$ CHANNEL

In Table II we give, for two values of g , the elements to determine the Regge trajectory of the ρ meson.

We have computed the curve between $t=0$ and $t=1.2 (\text{GeV})^2$. The reason why it is difficult to go beyond these limits is the extremely slow convergence of the Froissart-Gribov formulae, which must diverge at $\ell = 0$, because, as is easily seen in the explicit expression for the absorptive parts, they behave like $\log^2 s$. [This does not mean that the partial waves have a pole at $\ell = 0$, because both the numerator and denominator of the Padé fraction have a pole at $\ell = 0$.]

Anyhow the results are quite good, the slope at $t=0$ being insensitive to the value of coupling constant for a reasonable range of values of g around the value 6. The slope is found to be :

$$m = 1.05 (\text{GeV})^{-2}$$

which is in excellent agreement with experiment.

The intercept varies with g and is found to be an increasing function of g . This is very consistent with the principle of saturation of forces by unitarity. If g becomes too high, the intercept will exceed the value 1 and unitarity would be violated through the Froissart bound. We find an intercept which is :

$$\ell = 0.680 \quad \text{for a mass of } 772 \text{ MeV}$$

and

$$\ell = 0.686 \quad \text{for a mass of } 750 \text{ MeV.}$$

These numbers agree with experiment within a few percent. The Regge trajectory is represented on Fig. 8.

NOTE

If instead of choosing the set of unitary Padé approximants, we had chosen a non-unitary set, we would have been in great trouble for calculating Regge trajectories. For instance, we write some of the most significant approximants :

$$[1,2] = \frac{1 - \frac{S_2}{S_3} g}{1 - \frac{S_3}{S_2} g - S_2 g^2} \quad (7.11)$$

$$[2,1] = \frac{1 - \frac{S_3}{S_2} g + S_1 g^2}{1 - \frac{S_3}{S_2} g} \quad (7.12)$$

$$[2,2] = 1 + \frac{S_2 g^2}{1 - \frac{S_3}{S_2} g + \left[\left(\frac{S_3}{S_2} \right)^2 - \frac{S_4}{S_2} \right] g^2} \quad \text{[Unitary]} \quad (7.13)$$

$$[1,3] = \frac{1 - \left[\frac{S_4}{S_3} - \frac{S_2}{S_3} S_2 \right] g}{1 - \left[\frac{S_4}{S_3} - \frac{S_2}{S_3} S_2 \right] g - S_2 g^2 + \left[\frac{S_4}{S_3} S_2 - \frac{S_2}{S_3} S_2^2 - S_3 \right] g^3} \quad (7.14)$$

If the term

$$\left\{ \left(\frac{S_3}{S_2} \right)^2 - \frac{S_4}{S_2} \right\} g^2$$

is very small, which is the case for energies of the order of the ρ mass, then it is easy to check that all these Padé approximants are not very different. However, when the energy is small and in particular not physical, we find great differences. For instance, the Jost approximation :

$$J^{(2,2)} = 1 - \frac{S_3}{S_2} g - S_2 g^2$$

will give much higher values of ℓ for the Regge trajectory, because the term $S_2 g^2$ is extremely small, and for k non-physical, the term in g^2 in (7.13) is not negligible.

7.3 THE $T=0$ CASE

In the $T=0$ case no free parameter is available. We want to point out here that the non-coplanar graph (Fig. 5 F), contrary to the $T=1$ case, gives a contribution. The reason why in the $T=1$ case the contribution of this graph is rigorously zero is that it can be designed as a symmetrical tetrahedron in three dimensions and consequently the corresponding amplitude is completely symmetric under any perturbation of s , t , u . It is therefore clear that it cannot enter into the $T=1$ amplitude.

In this first calculation we have disregarded the non-coplanar graph because, as explained in Appendix C, its evaluation requires an integration of a complex function in eight dimensions. However, we give a majorization of it in one point ($S=0$). We know in what direction the correction should go in this point.

The general features of the $T=1$ case are found in this case also. Table III shows that we find a narrow resonance also in $T=0$, around 1600 for a value of $g \sim 6$ and 1250 for $g \sim 7.15$. No other resonances are found and no unwanted singularities are present in a wide domain of the complex energy plane.

Fig. 9 gives the variation of the mass of this resonance with g . It seems that the introduction of the non-coplanar graph decreases all values of the masses (it raises the Regge trajectory at least near $T=0$). However, the inelastic effects ($K\bar{K}$, 6π) must play a role at such an energy (at least 10%).

The Regge trajectory of the f_0 has been given for values of the transfer t between 0 and 0.6 BeV^2 . The intercept falls around 0.737 and 0.765, the last figure being obtained by including the maximal contribution of the non-coplanar graph. The slope is around $1.6 (\text{BeV})^{-2}$. It seems to us that these figures fall within the acceptable limits given by present parametrization. (see Fig. 10).

7.4 CONCLUSIVE REMARKS

a) Before going on with the conclusions, we would like to point out some of the most important checks, partly analytical and partly numerical, which can be made and have actually been made throughout this work. As far as renormalization is concerned, its isospin invariance has been explicitly verified both at various intermediate steps of the calculation and in the final formulae (see Section 5). Furthermore, crossing relations hold for our perturbative renormalized amplitudes (up to the fourth order). Perturbative unitarity in the partial waves gives other identities which must be fulfilled up to the fourth order.

The double spectral functions can be computed in two ways, either from Feynman diagrams directly or through the absorptive parts of the crossed amplitudes as explained in Appendix C. Many other checks of consistency have been made which are not mentioned here.

b) What has come out of the numerical computation is that, for the physical region of the resonances, both for $T=0$ and $T=1$, most contributions to the forces are negligible except s waves ($A_{0,2}$, $B_{0,2}$ terms in our notations). This fact permits a fast approximate check of our numerical results because with only s wave terms one has to deal with elementary functions.

c) The $T=0$ and $T=2$ channels are in our model similar and we therefore expect attractive forces also in the $T=2$, with the possibility of another resonance. The calculation does exhibit a unique resonance, in fact, in the $T=2$ about the same energy as the $T=0$ (a little higher). The question whether this $T=2$ resonance will survive or will be removed by the introduction of the $K\bar{K}$ system is going to be investigated soon: it is not in fact very difficult to generalize the formalism to the case of the coupled channels πK , $K\bar{K}$. We would like to point out that the introduction of the $K\bar{K}$ system may give more attraction in such a way as to shift the mass (f_0) to lower values and to permit the existence of heavier resonances with the same quantum numbers.

Results about the introduction of the non-coplanar graph ($T=0$), $T=2$) and the calculation of the residues of our Regge poles will also be published very soon.

d) We summarize here the properties of our approximation.

(1) Unitarity is fulfilled rigorously, including the 4π inelastic cut (in particular $|S| < 1$ above the 4π threshold).

NOTE

We would like to show that our $[2,2]$ unitary approximation which defines an S matrix $S(k)$ has the property that

$$|S(k)| = 1 \quad \text{for} \quad 0 < k < \sqrt{s} \quad (\text{N.1})$$

and

$$|S(k)| < 1 \quad \text{for } k > \sqrt{3} \quad (\text{N.2})$$

$\sqrt{3}$ is the beginning of the inelastic cut in the k plane.

The first relation is obtained by construction, the second is equivalent to

$$\operatorname{Re} \frac{1}{S^{[2.2]} - 1} < -\frac{1}{2} \quad (\text{N.3})$$

or

$$\operatorname{Re} \left\{ \frac{1 - \frac{S_3}{S_2} q + \left(\frac{S_3}{S_2} \right)^2 q^2}{S_2 q^2} \right\} - \operatorname{Re} \frac{S_4}{S_2^2} < -\frac{1}{2} \quad (\text{N.4})$$

The functions $S_2(k)$ and $S_3(k)$ have no inelastic cut and are purely imaginary for $k > 0$. We are therefore left with

$$\frac{\operatorname{Re} S_4}{|S_2|^2} < -\frac{1}{2} \quad (\text{N.5})$$

S_4 satisfies the unitarity requirement

$$S_4(k+i\epsilon) + S_4^*(k-i\epsilon) = -|S_2(k)|^2 \quad k > \sqrt{3} \quad (\text{N.6})$$

Introducing

$$\Delta(k) = S_4(k+i\epsilon) - S_4(k-i\epsilon) \quad k > \sqrt{3} \quad (\text{N.7})$$

(N.5) can be rewritten

$$1 - \frac{\Delta(k)}{|S_2|^2} > 1$$

It is clear that $\Delta(k)$ is real; on the other hand, the only contribution to $\Delta(k)$ comes from the box diagram of Fig. 2 (or in the $T=0$ case also from the non-coplanar graph). It is easy to show that those graphs have definite sign spectral functions and consequently $\Delta(k)$ is negative definite.

(2) Correct analytical properties in the energy variable are verified.

(3) Crossing is fulfilled up to the fourth perturbative order.

We would like to emphasize that it is important for crossing to be partly violated in an S matrix approximation having a finite number of inelastic channels.

Indeed, one can show ¹⁶⁾ that an elastic amplitude which is fully crossing invariant is identically zero (axiomatic result). It is very likely ¹⁷⁾ that this can be extended to an amplitude which has only a finite number of inelastic branch points. In this case it is clear that if one tries to improve crossing without opening a sufficient set of inelastic channels, one has a solution which, instead of tending to the correct one, will tend to zero. It is therefore important to keep a certain equilibrium between the precision in which crossing is fulfilled and the number of inelastic channels taken into account. In our method this feature is automatically verified.

We finally point out that our model verifies approximatively the principle of saturation of forces, because the intercepts at $T=0$ of the Regge trajectories are increasing functions of the coupling constant. The Froissart bound certainly applies to our model, and in fact the intercept in $T=0$ (higher than in $T=1$) is below 1.

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A P P E N D I X A

We give here the explicit formulae for the non-renormalized γ_0^T and for the renormalized γ^T amplitudes up to third order, obtained by performing the calculations outlines in Section 5. We recall that our expansions are : $\gamma_0^T = g_0 \gamma_0^{(1)T} + g_0^2 \gamma_0^{(2)T} + g_0^3 \gamma_0^{(3)T} + \dots$

$$\gamma^T = g \gamma^{(1)T} + g^2 \gamma^{(2)T} + g^3 \gamma^{(3)T} + \dots$$

We obtain

$$\gamma_0^{(2)T} = \left[\begin{array}{l} i [50 I(s) + 30 I(t) + 30 I(u)] \\ i [10 I(t) - 10 I(u)] \\ i [8 I(s) + 18 I(t) + 18 I(u)] \end{array} \right] \begin{array}{l} T=0 \\ T=1 \\ T=2 \end{array} \quad (A.1)$$

$$\gamma_0^{(3)T} = \left[\begin{array}{l} i [600 J(s) + 440 J(t) + 440 J(u)] - 250 I^2(s) - 110 I^2(t) - 110 I^2(u) \\ i [80 J(t) - 80 J(u)] - 70 I^2(t) + 70 I^2(u) \\ i [144 J(s) + 224 J(t) + 224 J(u)] - 16 I^2(s) - 86 I^2(t) - 86 I^2(u) \end{array} \right] \quad (A.2)$$

where

$$I(p^2) = (2\pi)^{-4} \int d^4q \frac{1}{[q^2 - 1 + i\epsilon][(p+q)^2 - 1 + i\epsilon]} \quad (A.3)$$

$$J[(p_1+p_2)^2] = i(2\pi)^{-4} \int d^4q \frac{I(q^2)}{[(q-p_1)^2 - 1 + i\epsilon][(q+p_2)^2 - 1 + i\epsilon]} \quad (A.4)$$

with $p_1^2 = p_2^2 = 1$.

The explicit evaluation of $I(p^2)$ and $J[(p_1+p_2)^2]$ requires the introduction of a cut-off Λ which we define by the regularization of the pion propagator :

$$\frac{1}{q^2 - 1} \rightarrow \frac{1}{q^2 - 1} - \frac{1}{q^2 - M^2} \quad (A.5)$$

We then have

$$I(s) = [i\pi^2/(2\pi)^4] [\Lambda - 2a(s)] \quad \Lambda \equiv \log M^2 \quad (A.6)$$

$$J(s) = [-i/8(2\pi)^4] \left[\frac{1}{4}(1+\Lambda)^2 - (1+\Lambda)a(s) + b(s) + \frac{11}{54}\pi^2 - \frac{3}{4} \right] \quad (A.7)$$

where the functions $a(s)$ and $b(s)$ are analytic functions of s in the cut plane from 4 to $+\infty$. For convenience, we give the three different forms on the real axis

$$x \geq 4$$

$$a(x+i\epsilon) = -\frac{i}{2}\pi \sqrt{\frac{x-4}{x}} - \sqrt{\frac{x-4}{x}} \log \frac{\sqrt{x} - \sqrt{x-4}}{2} \quad (A.8)$$

$$0 \leq x \leq 4$$

$$0 \leq \arcsin \frac{\sqrt{x}}{2} \leq \pi/2$$

$$a(x) = \left(\arcsin \frac{\sqrt{x}}{2} \right) \sqrt{\frac{4-x}{x}} \quad (\text{A.9})$$

$$x \leq 0$$

$$a(x) = -\sqrt{\frac{x-4}{x}} \log \frac{-\sqrt{-x} + \sqrt{-x+4}}{2} \quad (\text{A.10})$$

$$x \leq 0$$

$$b(x) = \frac{1}{6} \frac{1}{\sqrt{-x} \sqrt{4-x}} \left[2\pi^2 \log \frac{\sqrt{4-x} - \sqrt{-x}}{2} + 8 \log^3 \frac{\sqrt{4-x} - \sqrt{-x}}{2} \right] + \log^2 \frac{\sqrt{4-x} - \sqrt{-x}}{2} \quad (\text{A.11})$$

$$0 \leq x \leq 4$$

$$b(x) = \frac{1}{6} \frac{1}{(x - \frac{1}{4}x^2)^{1/2}} \left[-\pi^2 \arcsin \frac{\sqrt{x}}{2} + 4 \left(\arcsin \frac{\sqrt{x}}{2} \right)^3 \right] - \left[\arcsin \frac{\sqrt{x}}{2} \right]^2 \quad (\text{A.12})$$

$$x \geq 4$$

$$b(x+i\epsilon) = \frac{i\pi}{2} \left[2 \log \frac{\sqrt{x} - \sqrt{x-4}}{2} - 4 \frac{1}{\sqrt{x} \sqrt{x-4}} \log^2 \frac{\sqrt{x} - \sqrt{x-4}}{2} \right] + \frac{1}{6\sqrt{x} \sqrt{x-4}} \left[4\pi^2 \log \frac{\sqrt{x} - \sqrt{x-4}}{2} - 8 \log^3 \frac{\sqrt{x} - \sqrt{x-4}}{2} \right] + \frac{1}{4} \left[4 \log^2 \frac{\sqrt{x} - \sqrt{x-4}}{2} - \pi^2 \right] \quad (\text{A.13})$$

Collecting all previous results, we have

$$\gamma^{(2)T} = \begin{pmatrix} \frac{5}{4\pi^2} \left\{ -11 a\left(\frac{4}{3}\right) + 5a(s) + 3a(t) + 3a(u) \right\} \\ \frac{5}{4\pi^2} \{ a(t) - a(u) \} \\ \frac{1}{4\pi^2} \left\{ -22 a\left(\frac{4}{3}\right) + 4a(s) + 9a(t) + 9a(u) \right\} \end{pmatrix} \quad (\text{A.14})$$

$$\begin{aligned} \gamma^{(3)T=0} &= \frac{10}{(2\pi)^4} \left\{ \frac{195}{4} a^2\left(\frac{4}{3}\right) + \frac{37}{2} a\left(\frac{4}{3}\right) - \frac{37}{2} b\left(\frac{4}{3}\right) - \frac{11}{4} a\left(\frac{4}{3}\right) [10a(s) + \right. \\ &\quad \left. + 6a(t) + 6a(u)] - \frac{15}{2} a(s) - \frac{11}{2} a(t) - \frac{11}{2} a(u) + \right. \\ &\quad \left. + \frac{15}{2} b(s) + \frac{11}{2} b(t) + \frac{11}{2} b(u) + \frac{35}{4} a^2(s) + \right. \\ &\quad \left. + \frac{11}{4} a^2(t) + \frac{11}{4} a^2(u) \right\} \quad (\text{A.15}) \end{aligned}$$

$$\begin{aligned} \gamma^{(3)T=1} &= \frac{10}{(2\pi)^4} \{ \phi(t) - \phi(u) \} \quad (\text{A.16}) \\ \phi(x) &= b(x) + \frac{7}{4} a^2(x) - \left(1 + \frac{11}{\sqrt{2}} \arcsin \frac{1}{\sqrt{3}}\right) a(x) \end{aligned}$$

$$\begin{aligned} \gamma^{(3)T=2} &= \frac{11}{(2\pi)^4} \left\{ 195 a^2\left(\frac{4}{3}\right) + 74 a\left(\frac{4}{3}\right) - 74 b\left(\frac{4}{3}\right) + \frac{11}{4} a\left(\frac{4}{3}\right) \cdot [-16a(s) \right. \\ &\quad \left. - 36a(t) - 36a(u)] - 18a(s) - 28a(t) - 28a(u) + \right. \\ &\quad \left. + 18b(s) + 28b(t) + 28b(u) + \right. \\ &\quad \left. + 4a^2(s) + \frac{43}{2} a^2(t) + \frac{43}{2} a^2(u) \right\} \quad (\text{A.17}) \end{aligned}$$

We would like to point out that the renormalized $\gamma^{T(n)}$ (for $n=1, 2$ or 3) do verify the crossing relations given in Section 4.

A P P E N D I X B

Elastic unitarity condition at fourth order (two-pion exchange).
s wave contribution.

We complete the set of formulae given in the text, at this point :
 after defining the functions

$$\Lambda(k') = \int_{-1}^{+1} a(t) d\cos\theta_{s1} \quad (\text{B.1})$$

where $t = -2k'^2(1 - \cos\theta_{s1})$

$$\Lambda_1(k') = \int_{-1}^{+1} b(t) d\cos\theta_{s1} \quad \Lambda_2(k') = \int_{-1}^{+1} a^2(t) d\cos\theta_{s1} \quad (\text{B.2})$$

We explicitly give their expressions :

$$\Lambda(k') = \frac{1}{k'^2} \log^2(\sqrt{k'^2+1} - k') - 2 \frac{\sqrt{k'^2+1}}{k'} \log(\sqrt{k'^2+1} - k') - 1 \quad (\text{B.3})$$

$$\begin{aligned} \Lambda_1(k') = & -\frac{1}{3k'^2} \log^4(\sqrt{k'^2+1} - k') + \left[2 + \left(1 - \frac{\pi^2}{6}\right) \frac{1}{k'^2} \right] \log^2(\sqrt{k'^2+1} - k') \\ & + 2 \frac{\sqrt{k'^2+1}}{k'} \log(\sqrt{k'^2+1} - k') + 1 \end{aligned} \quad (\text{B.4})$$

$$\begin{aligned} \Lambda_2(k') = & \frac{2k'^2+1}{k'^2} \log^2(\sqrt{k'^2+1} - k') + \frac{2}{k'} \sqrt{k'^2+1} \log(\sqrt{k'^2+1} - k') + \\ & + 1 - \frac{4}{3k'^2} \log^3(\sqrt{k'^2+1} - k') + \frac{[k' + \sqrt{k'^2+1}]^2}{k'^2} \int_1^{\infty} \frac{dz \log^2 z}{z(z-1)} \end{aligned} \quad (\text{B.5})$$

We obtain, for third order $\ell = 0$:

$$\begin{aligned} \gamma_{\ell=0}^{(el)}(s') &= \frac{10}{(2\pi)^4} \left\{ \frac{195}{4} a^2\left(\frac{4}{3}\right) + \frac{37}{2} a\left(\frac{4}{3}\right) - \frac{37}{2} b\left(\frac{4}{3}\right) - \frac{110}{4} a\left(\frac{4}{3}\right) a(s') \right. \\ &\quad - \frac{66}{4} a\left(\frac{4}{3}\right) \Lambda(k') - \frac{15}{2} a(s') - \frac{11}{2} \Lambda(k') + \frac{15}{2} b(s') + \\ &\quad \left. + \frac{11}{2} \Lambda_1(k') + \frac{25}{4} a^2(s') + \frac{11}{4} \Lambda_2(k') \right\} \end{aligned} \quad (B.6)$$

$$\begin{aligned} \gamma_{\ell=0}^{(el)}(s') &= \frac{1}{(2\pi)^4} \left\{ 195 a^2\left(\frac{4}{3}\right) + 74 a\left(\frac{4}{3}\right) - 74 b\left(\frac{4}{3}\right) - 44 a\left(\frac{4}{3}\right) a(s') \right. \\ &\quad - 99 a\left(\frac{4}{3}\right) \Lambda(k') - 18 a(s') - 28 \Lambda(k') + \\ &\quad \left. + 18 b(s') + 28 \Lambda_1(k') + 4 a^2(s') + \frac{43}{2} \Lambda_2(k') \right\} \end{aligned} \quad (B.7)$$

We again introduce the real parts of $a(s')$, $a^2(s')$, $b(s')$ for $s' > 4$

$$\text{Re } a(s') \equiv \tilde{a}(s') = \frac{-k'}{\sqrt{k'^2+1}} \log(\sqrt{k'^2+1} - k') \quad (B.8)$$

$$\text{Re } a^2(s') \equiv \tilde{a}^2(s') = \frac{k'^2}{k'^2+1} \left\{ \log^2(\sqrt{k'^2+1} - k') - \frac{\pi^2}{4} \right\} \quad (B.9)$$

$$\begin{aligned} \text{Re } b(s') \equiv \tilde{b}(s') &= \frac{1}{6\sqrt{s'(s'-4)}} \left\{ 4\pi^2 \log(\sqrt{k'^2+1} - k') - \right. \\ &\quad \left. - 8 \log^3(\sqrt{k'^2+1} - k') \right\} + \\ &\quad + \log^2(\sqrt{k'^2+1} - k') - \frac{\pi^2}{4} \end{aligned} \quad (B.10)$$

Finally :

$$\begin{aligned} \text{Im}_{s'>4} \gamma_{\ell=0}^{T=0} (s') &= - \frac{1}{32\pi} \sqrt{\frac{s'-4}{s'}} \{ A_0(s') + B_0(s') \} \\ \text{Im}_{s'>4} \gamma_{\ell=0}^{T=2} (s') &= - \frac{1}{32\pi} \sqrt{\frac{s'-4}{s'}} \{ A_2(s') + B_2(s') \} \end{aligned} \quad (\text{B.11})$$

where $A_0(s') =$

$$\begin{aligned} \rightarrow &= \frac{200}{(2\pi)^4} \left\{ \frac{195}{4} a^2\left(\frac{4}{3}\right) + \frac{37}{2} a\left(\frac{4}{3}\right) - \frac{37}{2} b\left(\frac{4}{3}\right) - \frac{110}{4} a\left(\frac{4}{3}\right) \tilde{a}(s') \right. \\ &- \frac{15}{2} \tilde{a}(s') + \frac{15}{2} \tilde{b}(s') + \frac{25}{4} \tilde{a}^2(s') - \frac{33}{2} a\left(\frac{4}{3}\right) \Lambda(\kappa') - \frac{11}{2} \Lambda(\kappa') + \\ &\left. + \frac{11}{2} \Lambda_1(\kappa') + \frac{11}{4} \Lambda_2(\kappa') \right\} \end{aligned} \quad (\text{B.12})$$

$$\begin{aligned} B_0(s') &= \frac{25}{(2\pi)^4} \left\{ 121 a^2\left(\frac{4}{3}\right) - 11 a\left(\frac{4}{3}\right) [10 \tilde{a}(s') + 6 \Lambda(\kappa')] + \right. \\ &+ 25 |a(s')|^2 + 9 \Lambda^2(\kappa') + 30 \Lambda(\kappa') \tilde{a}(s') \left. \right\} \end{aligned} \quad (\text{B.13})$$

$$\begin{aligned} A_2(s') &= \frac{8}{(2\pi)^4} \left\{ 195 a^2\left(\frac{4}{3}\right) + 74 a\left(\frac{4}{3}\right) - 74 b\left(\frac{4}{3}\right) - 11 a\left(\frac{4}{3}\right) [4 \tilde{a}(s') + \right. \\ &+ 9 \Lambda(\kappa')] - 18 \tilde{a}(s') - 28 \Lambda(\kappa') + 18 \tilde{b}(s') \\ &+ 28 \Lambda_1(\kappa') + 4 \tilde{a}^2(s') + \frac{43}{2} \Lambda_2(\kappa') \left. \right\} \end{aligned} \quad (\text{B.14})$$

$$\begin{aligned} B_2(s') &= \frac{1}{(2\pi)^4} \left\{ 484 a^2\left(\frac{4}{3}\right) - 22 a\left(\frac{4}{3}\right) [18 \Lambda(\kappa') + \right. \\ &+ 8 \tilde{a}(s')] + 16 |a(s')|^2 + \\ &+ 81 \Lambda^2(\kappa') + 72 \Lambda(\kappa') \tilde{a}(s') \left. \right\} \end{aligned} \quad (\text{B.15})$$

Non s wave contribution to elastic unitarity (fourth order) :

We give here, for completeness, the representations of $\phi_0(\xi, \eta, z)$ and $\phi_E(\xi, \eta, z)$, where $\xi = 1 + (2t'/s' - 4)$, $\eta = 1 + (2t''/s' - 4)$ and $z = 1 + (2s/s' - 4)$:

$$\phi_0 = \sum_{\ell=1}^{+\infty} P_{\ell}(\text{odd}) (2\ell+1) Q_{\ell}(\xi) Q_{\ell}(\eta) P_{\ell}(z) \quad (\text{B.16})$$

$$\phi_E = \sum_{\ell=2}^{+\infty} P_{\ell}(\text{even}) (2\ell+1) Q_{\ell}(\xi) Q_{\ell}(\eta) P_{\ell}(z) \quad (\text{B.17})$$

$$\phi_0 = \frac{1}{2} [\phi(\xi, \eta, z) - \phi(\xi, \eta, -z)] \quad (\text{B.18})$$

$$\phi_E = -Q_0(\xi) Q_0(\eta) + \frac{1}{2} [\phi(\xi, \eta, z) + \phi(\xi, \eta, -z)] \quad (\text{B.19})$$

$$\phi(\xi, \eta, z) = \frac{z}{\sqrt{z-z_1} \sqrt{z-z_0}} \log \frac{\sqrt{z-z_0} - \sqrt{z-z_1}}{\sqrt{z_1-z_0}} \pm \frac{i\pi}{\sqrt{z-z_0} \sqrt{z-z_1}} \quad (\text{B.20})$$

$$z_0 = \xi \eta - \sqrt{(\xi^2-1)(\eta^2-1)}$$

$$z_1 = \xi \eta + \sqrt{(\xi^2-1)(\eta^2-1)}$$

$$\text{Im } z \geq 0$$

$$\phi(\xi, \eta, z) = \sum_{\ell=0}^{\infty} (2\ell+1) Q_{\ell}(\xi) Q_{\ell}(\eta) P_{\ell}(z) \quad (\text{B.21})$$

A P P E N D I X C

Unitarity condition at fourth order - Inelastic four-pion contribution

We outline here the procedure of calculating the various absorptive parts that contribute to the inelastic unitarity. From the unitarity condition given in Eq. (6.5) we have

$$A \bar{B}_S \gamma_A^{T(4)}(s, t) = - \frac{2}{(2\pi)^{12}} \left\{ 235 P_0 + 20 P_1 + 76 P_2 \right\} \cdot \left\{ \int d^4 p'_1 d^4 p'_2 d^4 p'_3 d^4 p'_4 \frac{\delta^{(4)}(\sigma - \sigma' - p'_1 - p'_2) (2\pi)^4 \prod_i \delta_+(p_i'^2 - 1)}{[(p_1 - \sigma')^2 - 1] [(p_3 - \sigma')^2 - 1]} \right\} \quad (C.1)$$

where $\sigma = p_1 + p_2$, $\sigma' = p'_1 + p'_2$.

By using the formula

$$\int \delta(Q - p'_3 - p'_4) \delta_+(p_3'^2 - 1) \delta_+(p_4'^2 - 1) d^4 p'_3 d^4 p'_4 = \frac{\pi}{2} \theta(Q_0 - 2) \theta(Q^2 - 4) \sqrt{\frac{Q^2 - 4}{Q^2}} \quad (C.2)$$

Equation (C.1) becomes :

$$A \bar{B}_S \gamma_A^{T(4)}(s, t) = - [235 P_0 + 20 P_1 + 76 P_2] R(s, t) \quad (C.3)$$

$$R(s, t) = \frac{1}{2^9 \pi^5} \int_4^{+\infty} dt' \int_4^{+\infty} dt'' \sqrt{\frac{(t'-4)(t''-4)}{t't''}} \frac{\theta(s-t-t'') \theta(A) \sqrt{A} \phi\left(\frac{At}{4A+B}\right)}{4A+B} \quad (C.4)$$

$$A = s^2 + t'^2 + t''^2 - 2st' - 2st'' - 2t't''$$

$$L' = \sigma_0'^2 - \vec{\sigma}'^2$$

$$L'' = s^2 + t'^2 - 2\sigma_0 V s \quad B = 4st't'' \quad (0.5)$$

where

$$\phi(x) = \frac{2}{\sqrt{x} \sqrt{x-1}} \log \{ \sqrt{x} - \sqrt{x-1} \} + \frac{i\pi}{\sqrt{x} \sqrt{x-1}} \quad (0.6)$$

(0.6)

$$\text{Im } x \geq 0$$

$$\phi(x) = \int_1^{+\infty} dx' \frac{1}{(x'-x) \sqrt{x'(x'-1)}} \quad (0.7)$$

$$AB_S \gamma_B^{T(4)}(s, u) = - \{ 235 P_0 - 20 P_1 + 76 P_2 \} R(s, u) \quad (0.8)$$

(0.8)

$$AB_S \gamma_c^{T(4)}(s) = -\frac{4}{3} \frac{1}{(2\pi)^{12}} \{ 375 P_0 + 60 P_2 \}.$$

$$\cdot \left\{ \int d^4 p'_1 d^4 p'_2 d^4 p'_3 d^4 p'_4 \frac{\delta^{(4)}(\sigma - \sigma' - p'_3 - p'_4) (2\pi)^4 \prod_{i=1}^4 \delta_+(p_i^2 - 1)}{[(\sigma - p'_4)^2 - 1]^2} \right\} \quad (0.9)$$

We obtain

$$ABs \underset{s>16}{\gamma}_C^{\tau(4)}(s) = -\frac{1}{6} \frac{1}{(2\pi)^5} \{375 P_0 + 60 P_2\} \frac{1}{s} N(s) \quad (C.10)$$

with

$$N(s) = \int_0^{\frac{1}{2}\sqrt{s-16}} dy \int_{\sqrt{4+y^2}}^{\sqrt{s}-\sqrt{4+y^2}} dx \tilde{\phi}(x,y) \quad (C.11)$$

and

$$\tilde{\phi}(x,y) = \frac{y^2 \sqrt{(x^2-y^2)(x^2-y^2-4)}}{(x^2-y^2)(s-2\sqrt{s}x+x^2-y^2-4)+4x^2} \sqrt{\frac{s-2\sqrt{s}x+x^2-y^2-4}{s-2\sqrt{s}x+x^2-y^2}} \quad (C.12)$$

We notice that for $s > 16$, the square roots in Eq. (C.12) are well defined

$$ABs \underset{s>16}{\gamma}_D^{\tau(4)}(s) = -\frac{4}{(2\pi)^{12}} \{1100 P_0 + 224 P_2\} \cdot \quad (C.13)$$

$$\left\{ \int d^4 p'_1 d^4 p'_2 d^4 p'_3 d^4 p'_4 \frac{\delta^{(4)}(\sigma - \sigma' - p'_3 - p'_4) (2\pi)^4 \prod_i \delta_+(p_i'^2 - 1)}{[(\sigma - p'_1)^2 - 1] [(p'_3 - \sigma')^2 - 1]} \right\}$$

We finally obtain

$$ABs \underset{s>16}{\gamma}_D^{\tau(4)}(s) = -\frac{1}{16 (2\pi)^5} \{1100 P_0 + 224 P_2\} \frac{M(s)}{\kappa \sqrt{s}} \quad (C.14)$$

$$M(s) = \int_0^{\frac{1}{2}\sqrt{s-16}} dy \int_{\sqrt{y^2+4}}^{\sqrt{s}-\sqrt{y^2+4}} dx \phi(x,y). \quad (C.15)$$

$$\phi(x,y) = \sqrt{\frac{s-2\sqrt{s}x+x^2-y^2-4}{s-2\sqrt{s}x+x^2-y^2}} \cdot \log \frac{\sqrt{x^2-y^2}(\sqrt{s}-x) + y\sqrt{x^2-y^2-4}}{\sqrt{x^2-y^2}(\sqrt{s}-x) - y\sqrt{x^2-y^2-4}} \cdot \log \frac{x^2-y^2-x\sqrt{s}+2\kappa y}{x^2-y^2-x\sqrt{s}-2\kappa y} \quad (C.16)$$

In the same way:

$$A_{B-S} \chi_E^{T(4)}(s) = -\frac{4}{(2\pi)^{12}} \{ 375 P_0 + 36 P_2 \} \cdot \left\{ \int d^4 p'_1 d^4 p'_2 d^4 p'_3 d^4 p'_4 \frac{(2\pi)^4 \delta^{(4)}(s - \sigma' - p'_3 - p'_4) \prod_i \delta_+(p_i'^2 - 1)}{[(s - p_1')^2 - 1][(s - p_4')^2 - 1]} \right\} \quad (C.17)$$

We have, after several manipulations :

$$AR_{S>16} \gamma_E^{T(4)}(\lambda) = -\frac{1}{(2\pi)^5} \left\{ 375 P_0 + 36 P_2 \right\} \frac{1}{2} \frac{P(\lambda)}{\lambda} \quad (C.18)$$

$$I(\lambda) = \int_1^{\frac{1}{2}[\sqrt{5}-1-\frac{3}{\sqrt{5}-1}]} dx \int_1^{y_+} dy F(x,y) + \int_1^{\frac{5-8}{2\sqrt{5}}} dx \int_{y_+}^{y_-} dy G(x,y). \quad (C.19)$$

where

$$F(x,y) = \frac{V(x^2-1)(y^2-1)}{(\sqrt{5}-2x)(\sqrt{5}-2y)} \left\{ \sqrt{(\bar{A}+1)(\bar{A}+\alpha^2+1)} - \sqrt{(\bar{A}-1)(\bar{A}+\alpha^2-1)} \right. \\ \left. + \alpha^2 \log \frac{\sqrt{\bar{A}+\alpha^2-1} + \sqrt{\bar{A}-1}}{\sqrt{\bar{A}+\alpha^2+1} + \sqrt{\bar{A}+1}} \right\} \quad (C.20)$$

$$\bar{A} = \frac{\lambda - 2\sqrt{5}(x+y) + 2xy - 2}{2\sqrt{x^2-1}\sqrt{y^2-1}}; \quad \alpha^2 = \frac{2}{\sqrt{(x^2-1)(y^2-1)}}$$

$$G(x,y) = \frac{Vx^2-1}{(\sqrt{5}-2x)} \frac{Vy^2-1}{(\sqrt{5}-2y)} \left\{ \sqrt{(\bar{A}+1)(\bar{A}+\alpha^2+1)} + \alpha^2 \log \frac{\alpha}{\sqrt{\bar{A}+\alpha^2+1} + \sqrt{\bar{A}+1}} \right\} \quad (C.21)$$

with

$$Y_{\pm} = \frac{-2(x-\sqrt{s})(2\sqrt{s}x-s+2) \pm \sqrt{\Delta}}{8\sqrt{s} \left\{ x - \frac{1}{2} \left(\frac{1}{\sqrt{s}} + \sqrt{s} \right) \right\}} \quad (C.22)$$

where

$$\Delta = 4\sqrt{s}(x^2-1)(2x-\sqrt{s})(2\sqrt{s}x-s+18) \quad (C.23)$$

The method of double spectral functions. Inelastic spectral functions

An alternative way of reproducing the results given above for $\text{Abs}_{s>16} \chi_A$ and $\text{Abs}_{s>16} \chi_B$ is the explicit computation of the inelastic double spectral functions from those due to the two-pion exchange by using the crossing symmetry properties ^{12), 18)}. Again we do not consider the spectral functions for the non-coplanar graph F. We start from the Mandelstam representation ^{12), 19)} for the amplitudes $\chi^T(s, t, u)$: our reasoning does not depend on subtractions. We therefore neglect them. In particular, for $T=1$, we do not have subtractions in perturbation theory

$$\begin{aligned} \chi^T(s, t, u) = & \frac{1}{\pi^2} \iint_4^{+\infty} ds' dt' \frac{\rho_{st}^T(s', t')}{(s'-s)(t'-t)} + \frac{1}{\pi^2} \iint_4^{+\infty} ds' du' \frac{\rho_{su}^T(u', s')}{(s'-s)(u'-u)} \\ & + \frac{1}{\pi^2} \iint_4^{+\infty} dt' du' \frac{\rho_{tu}^T(t', u')}{(t'-t)(u'-u)} \quad (C.24) \end{aligned}$$

$$\begin{aligned} \gamma^T(s, t, u) = & \frac{1}{\pi} \int_4^{+\infty} ds' \frac{1}{s' - s} \left[\frac{1}{\pi} \int_4^{+\infty} dt' \frac{e_{st}^T(s', t')}{t' - t} + \frac{1}{\pi} \int_4^{+\infty} du' \frac{e_{su}^T(u', s')}{-t' + t} \right] \\ & + \frac{1}{\pi} \int_4^{+\infty} du' \frac{1}{u' - u} \left[\frac{1}{\pi} \int_4^{+\infty} dt' \frac{e_{tu}^T(t', u')}{t' - t} + \frac{1}{\pi} \int_4^{+\infty} ds' \frac{e_{su}^T(u', s')}{-t' + t} \right] \quad (C.25) \end{aligned}$$

$$\text{Abs}_{s' > 4} \gamma^T = \frac{1}{\pi} \int_4^{+\infty} dt' \frac{e_{st}^T(s', t')}{t' - t} + \frac{1}{\pi} \int_4^{+\infty} du' \frac{e_{su}^T(u', s')}{-t' + t} \quad (C.26)$$

we have

$$e_{st}^0(x, y) = 3e(x, y) + e(y, x) + e_s(x, y)$$

$$e_{st}^1(x, y) = e(y, x) - e_s(x, y) \quad (C.27)$$

$$e_{st}^2(x, y) = e(y, x) + e_s(x, y)$$

and also

$$e_{st}^{0,2}(x, y) = e_{su}^{0,2}(y, x) \quad (C.28)$$

$$e_{st}^1(x, y) = -e_{su}^1(y, x)$$

We split

$$e_{st}^T(x, y) = e_{st}^{T(el)}(x, y) + e_{st}^{T(in)}(x, y) \quad (C.29)$$

and define the strip region $\mathcal{D}_1$ where $4 < x < 16$, $y > 16$, and the spectral functions $e_{st}^T(x, y) \neq 0$. In $\mathcal{D}_1$, $e_{st}^T(x, y)$ coincides with $e_{st}^{T(el)}(x, y)$ and we therefore have the symmetric region $\mathcal{D}_2$ where $e_{st}^T(x, y)$ coincides with $e_{st}^{T(in)}(x, y)$. After introducing the step functions $\theta_1(x, y) = 1$ for $(x, y) \in (1)$, $= 0$ $(x, y) \notin (1)$ and $\theta_2(x, y) = 1$ for $(x, y) \in (2)$ and 0 otherwise, (1) and (2) being the domains for $e_{st}^{T(el)}$ and $e_{st}^{T(in)}$ respectively, we obtain the following representations for $e(x, y)$ and $e_s(x, y)$:

$$e(x, y) = \theta_1(x, y) \alpha(x, y) + \theta_2(x, y) \beta(x, y)$$

$$e(y, x) = \theta_2(x, y) \alpha(y, x) + \theta_1(x, y) \beta(y, x) \quad (C.30)$$

$$e_s(x, y) = e_s(y, x) = \theta_1(x, y) \gamma(x, y) + \theta_2(x, y) \delta(y, x)$$

and in $\mathcal{D}_1$ $e_{st}^T \equiv e_{st}^{T(el)}$ (C.31)

$$\beta(y, x) = \frac{1}{2} [e_{st}^{1(el)}(x, y) + e_{st}^{2(el)}(x, y)] \quad (C.32)$$

$$\gamma(x, y) = \frac{1}{2} [e_{st}^{2(el)}(x, y) - e_{st}^{1(el)}(x, y)] \quad (C.33)$$

$$\alpha(x, y) = \frac{1}{3} [e_{st}^{0(el)}(x, y) - e_{st}^{2(el)}(x, y)] \quad (C.34)$$

We finally obtain

$$e_{st}^{0(in)}(x, y) = \frac{1}{3} \theta_2(x, y) \left[5 e_{st}^{2(el)}(y, x) + e_{st}^{0(el)}(y, x) + 3 e_{st}^{1(el)}(y, x) \right]$$

$$e_{st}^{1(in)}(x, y) = \frac{1}{6} \theta_2(x, y) \left[2 e_{st}^{0(el)}(y, x) - 5 e_{st}^{2(el)}(y, x) + 3 e_{st}^{1(el)}(y, x) \right] \quad (C.35)$$

$$e_{st}^{2(in)}(x, y) = \frac{\theta_2(x, y)}{6} \left[2 e_{st}^{0(el)}(y, x) + e_{st}^{2(el)}(y, x) - 3 e_{st}^{1(el)}(y, x) \right]$$

From the two-pion exchange unitarity condition (neglecting the θ_1 step function):

$$e_{st}^{0(el)}(x, y) = -225 \phi(x, y) \quad (C.36)$$

$$e_{st}^{1(el)}(x, y) = -25 \phi(x, y)$$

$$e_{st}^{2(el)}(x, y) = -81 \phi(x, y)$$

where

$$\phi(x, y) = \frac{\pi/2}{128\pi^5 \sqrt{x} (x-4)^{3/2}} \int_4^{+\infty} dt' \int_4^{+\infty} dt'' \sqrt{\frac{(t'-4)(t''-4)}{t't''}} \frac{\theta(F_+)}{\sqrt{F}} \quad (C.37)$$

where

$$F = z^2 + \xi^2 + \eta^2 - 1 - 2z\xi\eta \quad (C.38)$$

$$z = 1 + \frac{2y}{x-4} \quad \xi = 1 + \frac{2t'}{x-4} \quad \eta = 1 + \frac{2t''}{x-4}$$

$$\theta(F_+) = \theta(F) \theta(z - \xi\eta) \quad (C.39)$$

and then

$$\begin{aligned} \rho_{st}^{0(in)}(x, y) &= -235 \phi(y, x) \\ \rho_{st}^{1(in)}(x, y) &= -20 \phi(y, x) \end{aligned} \quad (C.40)$$

$$\rho_{st}^{2(in)}(x, y) = -76 \phi(y, x)$$

From the equation

$$\text{Abs } \gamma^T(s, t, u) = \frac{1}{\pi} \int_4^{+\infty} dt' \frac{\rho_{st}^T(s, t')}{t' - t} + \frac{1}{\pi} \int_4^{+\infty} du' \frac{\rho_{su}^T(u', s)}{u' - u} \quad (C.41)$$

we finally obtain, for the inelastic contribution (after crossing) :

$$\text{Abs } \gamma^{T=1(in)}(s, s') = -25 R(s', s) - 5 R(s', 4 - s' - s) \quad (C.42)$$

where $R(s', s)$ is the same function defined by Eq. (C.4).

Non-coplanar graph (Figure F)

The absorptive part of the non-coplanar graph is very hard to compute. We know, however, that it is possible to find an upper bound for physical values of s and t , in particular $t=0$. We outline here the procedure :

$$Abs_{s>16} \chi_F^{(4)T}(s,t) = -\frac{8}{(2\pi)^{12}} \{185 P_0 + 74 P_2\} \quad (C.43)$$

$$\cdot \int d^4 p'_1 d^4 p'_2 d^4 p'_3 d^4 p'_4 \delta^{(4)}(\sigma - p'_1 - p'_2 - p'_3 - p'_4) \frac{(2\pi)^4 \prod_{i=1}^4 \delta_+(p_i'^2 - 1)}{[(p_1 - \sigma')^2 - 1][(p_3 - p'_1 - p'_3)^2 - 1]}$$

by comparison with $Abs_{s>16} \chi_A^T(s,t)$ and from the fact that the expressions $1 - (p_1 - \sigma')^2$, $1 - (p_3 - p'_1 - p'_3)^2$ are positive definite for physical s and t , we obtain :

$$\left| Abs_{s>16} \chi_F^{(4)T}(s,0) \right| \leq 4 \left| Abs_{s>16} \chi_A^{(4)T}(s,0) \right| \quad (C.44)$$

To evaluate in general graph F , we remark that the volume element of the phase space :

$$d\Omega_4 = \delta^{(4)}[\sigma - \sum_i p_i'] \prod_{i=1}^4 \delta_+(p_i'^2 - 1) dp_i' \quad (C.45)$$

can be rewritten, using a new system of variables

$$d\Omega_4 = \frac{1}{16} d\omega_1 d\omega_2 d\omega_3 d\varrho d\phi dR^{(20)} \quad (C.46)$$

where $\omega_1, \omega_2, \omega_3$ are the energies of p_1', p_2', p_3' , ϱ is the modulus of $\vec{\sigma}'$, ϕ the angle between the planes $(\vec{p}_1', \vec{p}_2')$ and $(\vec{p}_3', \vec{p}_4')$, and R is the rotation of Euler angles α, β, γ , which brings the trihedron (x,y,z) on the standard trihedron (see Fig.11).

The Monte Carlo method of integration seems the most suitable one with this choice of variables, because the integrand is always finite and smooth. The explicit contribution will be published later on. The upper bound given in (C.44) has been used for the moment for $t=0$.

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Table 1

$T = 1$; $\ell = 1$

Re k	Im k	10^6 Re S_4 (el)	10^5 Im S_4 (el)	10^7 Re S_4 (in)	10^7 Im S_4 (in)	Re g_-	Im g_-	Re g_+	Im g_+	M_ρ MeV	Γ_ρ MeV	g
2.40	-0.20	2.57	1.42	1.03	-1.75	6.231	0.175	98.4	76.15	720	28	6.244
2.40	-0.10	1.23	1.45	1.21	-1.65	6.240	0.055	123.7	63.7			
2.48	-0.30	3.97	1.51	0.92	-1.99	6.123	0.279	75.3	64.7	740	28	6.153
2.48	-0.14	1.85	1.52	1.21	-1.84	6.143	0.097	102.9	53.15			
2.53	-0.30	4.03	1.58	0.961	-2.09	6.067	0.271	73.7	57.3			
2.53	-0.14	1.88	1.59	1.27	-1.93	6.087	0.093	96.6	45.3	752	29	6.09
2.53	-0.07	0.94	1.60	1.40	-1.86	6.090	0.015	105.8	35.5			
2.60	-0.30	4.11	1.675	1.025	-2.23	5.992	0.259	70.8	48.6	770	29	6.019
2.60	-0.14	1.92	1.69	1.34	-2.07	6.010	0.089	88.8	36.8			
2.70	-0.14	1.97	1.83	1.44	-2.27	5.905	0.082	79	28.1	795	~ 30	~ 5.9
2.80	-0.20	2.88	1.965	1.42	-2.545	5.802	0.135	67.7	26.2	820	~ 30	~ 5.8

Table II

Regge trajectory of the ρ meson

$t(\text{GeV})^2$	$t(m_\pi)^2$	g	l
0	0	6.10	0.686
0	0	6.02	0.680
0.095	5	6.10	0.788
0.095	5	6.02	0.780
0.114	6	6.10	0.803
0.114	6	6.02	0.785
0.133	7	6.10	0.817
0.133	7	6.02	0.808
0.152	8	6.10	0.830
0.152	8	6.02	0.822
0.171	9	6.10	0.841
0.171	9	6.02	0.833
0.190	10	6.10	0.852
0.190	10	6.02	0.843
0.285	15	6.10	0.900
0.285	15	6.02	0.889
0.380	20	6.10	0.934
0.380	20	6.02	0.925
0.475	25	6.10	0.964
0.475	25	6.02	0.956
0.565		6.10	1
0.592		6.02	1
0.762	40	6.10	1.054
0.762	40	6.02	1.043
0.857	45	6.10	1.082
0.857	45	6.02	1.070
0.952	50	6.10	1.109
0.952	50	6.02	1.096
1.33	70	6.10	1.202
1.33	70	6.02	1.188
1.71	90	6.10	1.281
1.71	90	6.02	1.266
2.09	110	6.10	1.350
2.09	110	6.02	1.333
2.47	130	6.10	1.411
2.47	130	6.02	1.393

Table III

T = 0 ; $\ell = 2$

Re k	Im k	10^6 Re S_4 (el)	10^6 Im S_4 (el)	10^7 Re S_4 (in)	10^7 Im S_4 (in)	Re g_-	Im g_-	Re g_+	Im g_+	M_{f_0} MeV	g
4.00	-0.30	1.93	8.31	0.842	-1.57	7.635	0.547	-313.8	-337.2	1140	7.64
4.00	-0.10	0.86	8.43	1.096	-1.44	7.637	0.301	-189.8	-240.5		
4.416	-0.217	1.78	10.7	1.18	-2.08	7.165	0.457	+25.3	-286.0	1250	7.16
4.416	-0.10	1.09	10.8	1.35	-2.015	7.160	0.338	-5.2	-244.3		
5.00	-0.10	1.46	14.35	1.71	-2.95	6.647	0.387	65.5	-121.4	1410	6.65
6.00	-0.10	2.15	21.1	2.29	-4.87	5.98	0.439	49.6	-52.2	1680	5.98

Table IV

Regge trajectory of the f_0 meson

$t(\text{GeV})^2$	$t(m_\pi)^2$	g	l	ℓ_{max} including N.C.
0	0	6,02	0,737	0,765
0	0	7,16	0,845	0,877
5	0,095	6,02	0,893	0,944
5	0,095	7,16	1,031	1,088
10	0,190	6,02	0,981	
10	0,190	7,16	1,144	
20	0,380	6,02	1,105	
20	0,380	7,16	1,302	
30	0,570	6,02	1,200	
30	0,570	7,16	1,420

FIG.1 Error for an assymptotic series

FIG.2 Some of the elastic unitarity diagrams

FIG.3 The complete set of the inelastic unitarity diagrams

FIG. 4 Second order 4π production diagrams

FIG. 5 Numeration of the "inelastic" unitarity diagrams

FIG. 6 Partial wave analytical structure

FIG.7 The coupling constant as a function of the ρ Mass

FIG.8a The Regge trajectory of the ρ Meson (Detailed structure)

FIG.8b The Regge trajectory of the ρ Meson

FIG.9 The f_0 Mass as a function of the coupling constant

FIG.10 The Regge trajectory of the f_0 Meson

FIG.11 System of variables for the 4 particles phase space in the case of the non-coplanar diagram