CM-P00057581

FINITENESS OF RADIATIVE CORRECTIONS TO π β -DECAY

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ABSTRACT

We show that the radiative corrections to π β -decay are finite if a sum rule involving weak and electromagnetic transitions for π 's is satisfied.

The sum rule is evaluated in the vector meson dominance hypothesis and found to predict $\Gamma(\omega \rightarrow \pi\gamma)$ and $\Gamma(\pi^0 \rightarrow \gamma\gamma)$ in good agreement with the experimental data.

Some comments are made on the possible general cancellation of the divergences in the radiative corrections to β -decay.

5 June 1967
TH.787

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It has been known for a long time that while the radiative corrections to μ decay are finite ¹⁾, the same is not true, at least up to order α , for the β -decay of a point-like neutron unless $\frac{G_A}{G_V} = 1$ ⁽¹⁾; (the experimental value is ²⁾ $\frac{G_A}{G_V} = 1.18$).

It was hoped that the divergences appearing for point-like hadrons would disappear by taking carefully into account the effects due to the strong interactions. However this hope seemed to be contradicted by a recent theorem by Abers et al. ^{3,4)} ac-

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cording to which, the order α -corrections to an isovector β -decay, in the limit of zero lepton-momentum are given by a universal and infinite constant. For instance the matrix element of π β -decay: $\pi^- \rightarrow \pi^0 e \bar{\nu}$, turns out to be

$$M = \langle \pi^0 e \bar{\nu} | H_W | \pi^- \rangle = \left(1 - \frac{3}{2} \frac{ie^2}{(2\pi)^4} \int \frac{d^4 k}{(k^2 - i\epsilon)^2} \right) G \bar{u}(e) (\not{\pi}^0 + \not{\pi}^-) (1 + \gamma_5) u(\nu) \quad (1)$$

where π_μ^- (π_μ^0) is the momentum of the initial (final) pion.

Although the theorem is certainly correct insofar as the vector part of the weak hamiltonian is concerned, it need not be relevant for the real situation where also an axial part is present.

As we have already remarked the radiative corrections would be finite for the β -decay of a point-like nucleon if its weak interaction with the leptons is of the $V + A$ form. If the π were a loosely bound system of a pair of such nucleons one would expect the radiative corrections to its β -decay to be also finite in contradiction with eq.(1)⁵⁾.

The mechanism by which this is possible is obviously due to the axial part of the hadronic weak-current, whose effect is not included in the derivation of eq.(1).

In view of this let us consider the graph Fig.1, where the $e\bar{\nu}$ pair is emitted through the axial part of the hadronic weak current.

Fig. 1

This contribution, is to be added to eq.(1) and is of the form

$$\Delta M^A = \frac{G}{\sqrt{2}} \frac{ie^2}{(2\pi)^4} \bar{u}(e) \int d^4k D_{\lambda\nu}(k) \gamma_\nu \frac{m - \not{e} - \not{k}}{k^2 + 2(e \cdot k) - i\epsilon} \gamma_\mu T_{\lambda\mu}^A (1 + \gamma_5) u(v) \quad (2)$$

where $D_{\lambda\nu}(k)$ is the photon propagator and $T_{\lambda\mu}^A$ is:

$$T_{\lambda\mu}^A = \frac{1}{\sqrt{3}} \int d^4x e^{iKx} \langle \pi^0 | T(j_\lambda^\beta(x) A_\mu^+(0)) | \pi^- \rangle + S_{\lambda\mu}^A \quad (3)$$

where we have taken into account the fact that only the isoscalar part of the e.m. current can contribute because of G-parity, π is the common momentum of π^0 and π^- , and $S_{\lambda\mu}^A$ is connected with the Schwinger term. $T_{\lambda\mu}^A(K, \pi)$ can be expressed in terms of a single scalar function Φ :

$$T_{\lambda\mu}^A(K, \pi) = \epsilon_{\lambda\mu\nu\rho} K_\nu (\pi^0 \pi^0)_\rho \Phi(K^2, (K\pi)) \quad (4)$$

If we substitute this expression in eq.(2) ⁶⁾, after some standard manipulations we obtain

$$\Delta M^A = G \frac{ie^2}{(2\pi)^4} \frac{3}{2\sqrt{2}} \int \frac{d^4k}{K^2 - i\epsilon} \Phi(K^2, K\pi) \bar{u}(e) (\not{\pi}^0 + \not{\pi}^-) (1 + \gamma_5) u(v) \quad (5)$$

In order for ΔM^A to have a logarithmic divergence which exactly cancels the divergence in eq.(1), the high K^2 behaviour of Φ must be

$$\Phi \underset{K^2 \rightarrow \infty}{\sim} \frac{\sqrt{2}}{K^2} \quad (6)$$

In the following we show that the $\frac{1}{K^2}$ behaviour is a reasonable

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possibility ⁷⁾, and that eq.(6) gives rise to a sum rule of a new kind involving observable amplitudes.

Let us then compute $T_{\lambda\mu}^A - S_{\lambda\mu}^A$ by inserting a complete set of intermediate states:

$$\begin{aligned} T_{\lambda\mu}^A - S_{\lambda\mu}^A = & \frac{i}{\sqrt{3}} \int \frac{dm^2}{2p_0} \sum_n' \frac{\langle \pi^- | j_\lambda^8(0) | n, \vec{p} = \vec{k} \rangle \langle n, \vec{p} = \vec{k} | A_\mu^-(0) | \pi^0 \rangle}{K_0 - p_0 + \pi_0 - i\epsilon} \\ & - \frac{i}{\sqrt{3}} \int \frac{dm^2}{2p_0} \sum_n' \frac{\langle \pi^- | A_\mu^-(0) | n', \vec{p} = -\vec{k} \rangle \langle n', \vec{p} = -\vec{k} | j_\lambda^8(0) | \pi^0 \rangle}{K_0 + p_0 - \pi_0 - i\epsilon} \end{aligned} \quad (7)$$

where we have assumed the π 's to be at rest, and $(\vec{p}, p_0)$ is the four-momentum carried by the intermediate states of invariant mass m^2 .

Lorentz and TP invariance require that

$$\begin{aligned} & \sum_n' \langle \pi^- | j_\lambda^8(0) | n, \vec{p} = \vec{k} \rangle \langle n, \vec{p} = \vec{k} | A_\mu^-(0) | \pi^0 \rangle = \\ & = \epsilon_{\lambda\mu\sigma\rho} K_\sigma(\pi^-\pi^0)_\rho \sigma(m^2, p_\pi, m_\pi^2) \end{aligned} \quad (8)$$

and isospin symmetry implies that the two numerators of eq.(7) are actually equal.

If we substitute this expression in eq.(7) we would obtain for Φ a complicated function of K^2 and (K, π) . Since, however, our aim is to show that the divergence in eq.(1) can indeed be cancelled by ΔM^A , we restrict ourselves to the limit of zero π -mass, i.e. of zero π -momenta as we are computing everything in the π rest frame. This has to be done with care because in this limit both M and ΔM^A have a kinematical zero.

M is itself (see eq.(1)) of the first order in m_π , so that we have to keep the first order term in ΔM^A , i.e. the zero order term in $\bar{\Phi}$.

In this limit $\bar{\Phi}(k^2, k\pi) \rightarrow \bar{\Phi}_0(k^2)$, and $\sigma(m^2, k\pi) \rightarrow \sigma_0(m^2)$, and m_π can be dropped in the denominators of eq.(7). Also in this limit the r.h.s. of eq.(7) becomes a covariant expression

$$T_{\lambda\mu}^A - S_{\lambda\mu}^A = \epsilon_{\lambda\mu\sigma\rho} K_\sigma(\pi^+\pi^0)_\rho \frac{1}{\sqrt{3}} \int \frac{dm^2 \sigma_0(m^2)}{k^2 + m^2 - i\epsilon} \quad (9)$$

If the integral converges, eq.(9) can be taken to represent the complete amplitude, and the Schwinger term can be dropped from the l.h.s.

The condition of eq.(6) which guarantees the convergence of radiative corrections to π β -decay, implies the following sum rule:

$$\frac{1}{\sqrt{3}} \int dm^2 \sigma_0(m^2) = \sqrt{2} \quad (10)$$

We note that should eq.(9) hold and the integral in eq.(10) diverge, the radiative correction term ΔM^A would give rise to a stronger than logarithmic divergence.

As a test of the sum rule eq.(10) we try to saturate it with the lowest single particle intermediate state, i.e. the ρ -meson. This means putting

$$\begin{aligned} \epsilon_{\lambda\mu\sigma\rho} K_\sigma(\pi^+\pi^0)_\rho \sigma_0(m^2) = \\ = \delta(m^2 - m_\rho^2) \sum_{\text{spin}} \langle \pi^- | j_\lambda^8(0) | \rho \rangle \langle \rho^- | A_\mu^-(0) | \pi^0 \rangle \end{aligned} \quad (11)$$

where in the r.h.s. we have to take the term which is of the

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first order in the mass of the pion. It is easily seen that the second factor is finite in this limit so that it can be computed using PCAC and current commutators, while the first factor is related to $\rho \rightarrow \pi\gamma$ decay. As a result we find

$$\sigma_0(m^2) = \delta(m^2 - m_\rho^2) 4\sqrt{3} \frac{m_\rho^2}{f_\pi f_\rho} g_{\rho\pi\gamma} \quad (12)$$

where we have defined

$$\langle \pi^- | j_\lambda^{e.m.}(0) | \rho^- \rangle = g_{\rho\pi\gamma} \epsilon_{\lambda\mu\nu\rho} p_\mu (\pi+k)_\nu (\pi-k)_\rho$$

Requiring now eq.(6) to hold, and using the relation ⁸⁾ $f_\rho f_\pi = m_\rho$, we find

$$g_{\rho\pi\gamma} = \frac{1}{2\sqrt{2} m_\rho} \quad (13)$$

This gives a partial width for $\rho \rightarrow \pi\gamma$

$$\Gamma(\rho \rightarrow \pi\gamma) = .12 \text{ MeV}$$

This process has not yet been observed, however by using SU_3 and the usual ω - ϕ mixing angle eq.(13) implies

$$g_{\omega\pi\gamma} = \frac{3}{2\sqrt{2}} \frac{1}{m_\rho}$$

yielding $\Gamma(\omega \rightarrow \pi\gamma) = 1.1 \text{ MeV}$, to be compared with the experimental value of $\Gamma(\omega \rightarrow \pi\gamma) = (1.15 \pm .2) \text{ MeV}$.

Eq.(13) can also be used to evaluate the $\pi^0 \rightarrow \gamma\gamma$ rate, using the vector meson dominance model, which gives

$$\Gamma(\pi^0 \rightarrow \gamma\gamma) = 8.3 \text{ eV}$$

in very good agreement with the experimental numbers $(7.5 \pm 1.5) \text{eV}^9$.

These results give us confidence in the validity of the sum rule eq.(10), which insures the finiteness of order α radiative corrections in the limit of vanishing π mass.

One is then led to the question of the possible cancellation of ultraviolet divergences in the general case.

The universality of the divergent corrections to the vector part of a weak leptonic transition $A \rightarrow B e \nu$, proved in ref. 3), could be understood by looking at the crossed process $\bar{A} B \rightarrow e \nu$, and by assuming that it is dominated by a vector meson (i.e. the ρ -meson).

The whole amplitude including the ultraviolet part of order α radiative corrections would then be given by the graphs in fig.2.

Fig. 2'

In this spirit the new divergent contribution ΔM^A is also dominated by ρ -meson exchange, (i.e. is given by the graph (b) when the lepton pair is emitted through an axial current).

The cancellation insured by our sum rule in a particular case would now be universal and the radiative corrections would then turn out to be finite in the physical π β -decay as well as in any other vector β -decay.

The extension of this argument to axial transitions is quite straightforward by replacing the ρ -meson with the A_1 -meson.

The task remains of computing the finite part of the radiative corrections to β -decay. However given the absence of an ultraviolet catastrophe, these corrections should be dominated by the soft photon contributions, and the previous estimates ¹⁾ should be quite reliable.

In this paper we have taken the attitude that the radiative corrections should be finite. It will be interesting to see what constraint this requirement poses on the general structure of e.m. and weak hadronic currents.

REFERENCES AND FOOTNOTES

- 1) S.M.Berman, A.Sirlin, Ann. of Phys. 20, 20 (1962). Here the references to earlier literature can be found.
- 2) C.S.Wu, S.A.Moszkowski, Beta Decay. Interscience Publishers. New York, (1966).
- 3) E.S.Abers, R.E.Norton, D.A.Dicus, Phys.Rev.Letters, 18, 676 (1967).
- 4) J.D.Bjorken, Phys.Rev. 148, 1467 (1966).
- 5) Motivated by this remark we have computed the order α radiative corrections to the β -decay of a free, point-like quark: n -quark $\rightarrow$ p -quark + $e^- + \bar{\nu}$, and found that if the zero order amplitude is

$$\frac{G}{\sqrt{2}} \bar{e} \gamma_\mu (1 + \gamma_5) n$$

then the divergent part of the amplitude, to order α is:

$$-\frac{G}{\sqrt{2}} \frac{e^2}{2\pi} \ln\left(\frac{\lambda}{m_Q}\right) \left\{ (1+\rho) \left[\frac{1}{2} Q_n + \frac{1}{2} (Q_p - Q_n)^2 - 2Q_p \right] \bar{e} \gamma_\mu (1 + \gamma_5) n + \right. \\ \left. - (1-\rho) \left[\frac{1}{2} Q_p - \frac{1}{2} (Q_p - Q_n)^2 - 2Q_n \right] \bar{e} \gamma_\mu (1 - \gamma_5) n \right\} \epsilon_\mu$$

where Q (Q_n) is the p -quark (n -quark) electric charge, equal to $\frac{2}{3}(-\frac{1}{3})$. Contrary to the case of the nucleon β -decay, these corrections are never finite, independently of ρ .

In our opinion this fact could raise some doubts on the consistency of the naive, realistic quark model. This difficulty has already been noticed by J.D.Bjorken, in his report at the Fourth Coral Gables Conference on "Symmetry at high Energies", February 1967.

- 6) This term is gauge invariant by itself, so that we are free to use the Feynman gauge: $D_{\lambda\mu}(k) = \delta_{\lambda\mu} \frac{1}{k^2 + i\epsilon}$.
- 7) The term ΔM^A has already been considered by Bjorken ⁴⁾. He suggests that the asymptotic behavior of $\Phi(K^2, K\pi)$ is $\propto \frac{(K\pi)}{K^4}$, which would make ΔM^A convergent. This however conflicts with PT invariance, which requires $T_{\lambda\mu}^A$ to be an odd function of K , and therefore Φ to be even in K .
- 8) K.Kawarabayashi, M.Suzuki, Phys.Rev.Letters 16, 255 (1966) and ibid. (E) 16, 384 (1966).
Riazzuddin, Fayazzuddin, Phys.Rev. 147, 1071 (1966).
- 9) A.H.Rosenfeld et al., Rev.Mod.Phys. 39, 1 (1967).

FIGURE CAPTIONS

Figure 1 : Additional radiative correction to π
 β -decay. The lepton pair is emitted
 through an axial current.

Figure 2 : Graphs for the vector β -decay $A \rightarrow B + e^- + \bar{\nu}$
 a) zero order amplitude;
 b), c), d) α -order radiative corrections.
 In b) the lepton pair can be emitted through
 an axial and vector current.