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DOUBLE REGGE MODEL AND DENSITY MATRIX ELEMENTS

IN SINGLE PARTICLE PRODUCTION

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A B S T R A C T

A formalism is developed which allows one to take spin fully into account in three-body production. Expressions for the density matrix elements are given. The double Regge model is used to compare the density matrix of a resonance produced in a three-body production process with the corresponding density matrix in a two-body reaction having in common the vertex at which the resonance is produced.

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1. INTRODUCTION

In the last few years the double peripheral model ¹⁾⁻⁶⁾ has had a reasonable success in accounting for the data in single particle production at high energies. In the Reggeized version of Chan et al. ^{3),4)} the distribution of events in the Dalitz plot and the distributions in momentum transfer were compared to experiment. Since these distributions involve averages over the spins of the external particles, the complications due to spin were not considered in ^{3),4)}. However, as data at higher energies and better statistics become available one can think of measuring such quantities as, e.g., density matrix elements of resonances produced in three-body processes. From a theoretical point of view it should be interesting to analyze resonances produced in the upper or lower vertex of a double Regge graph (see the Figure), because through the factorization of the residue functions one can compare them easily to similar vertices in two-body reactions. Also the analysis of resonances produced at the middle vertex of the figure is important since it can give information on the coupling of two Regge poles with one elementary particle which is otherwise unknown. It is clear that in all these analyses the spin of the particles must be fully taken into account. Recently, a formalism using helicity amplitudes (which seem to be very well suited for Reggeization, see e.g., ⁷⁾⁻⁹⁾) has been introduced by Roberts and Fraser ⁵⁾. The idea in ⁵⁾ is to express the full three-body amplitudes in terms of quasi two-body amplitudes (following Wick ¹⁰⁾) and to use crossing relations for these quasi two-body processes ^{11),12),13)}.

In this paper we develop further the formalism of ⁵⁾, especially concerning the high energy region, and give expressions for the density matrix elements (Sections 2a and 2b).

2.

In Sections 3a and 3b we compare, in the framework of the double Regge model, the density matrix elements of a resonance produced in a three-body process with the corresponding density matrix elements in a two-body reaction having in common the vertex at which the resonance is produced. As a consequence of the factorization of the residue functions it is shown that, when the energy is high enough for only one leading trajectory to dominate, both density matrices are equal at the same momentum transfer (except near the forward direction). Concluding remarks are made in Section 3c.

2. a. DENSITY MATRIX ELEMENTS IN TERMS OF CROSSED CHANNEL HELICITY AMPLITUDES

Let us consider the process

$$a + b \rightarrow 1 + 2 + 3 \quad (2.1)$$

p_a , p_b and p_i being the momenta of the incoming and outgoing particles and λ_a , λ_b and λ_i the corresponding helicities. Let the helicity amplitude for reaction (2.1) in its center-of-mass system (c.m.s.) be $G_{\lambda_a \lambda_b}^{\lambda_1 \lambda_2 \lambda_3}$. This amplitude depends on five variables. The following set of variables is usually used ¹⁴⁾

$$\begin{aligned} s &= (p_a + p_b)^2 & s_{12} &= (p_1 + p_2)^2 & s_{23} &= (p_2 + p_3)^2 \\ t_{a1} &= (p_a - p_1)^2 & t_{b3} &= (p_b - p_3)^2 \end{aligned} \quad (2.2)$$

We shall find it useful to use for G instead of (2.1) the following set of independent variables

$$s, s_{23}, t_{a1}, \cos \vartheta_d, \varphi_d \quad (2.3)$$

where ϑ_d is the angle between the three-momenta of particles 3 and 1, and φ_d is the angle between the normals to the planes ab and 31 , both measured in the c.m. frame of particles 2 and 3. The relation between the sets (2.2) and (2.3) is given in Appendix A.

Following Wick¹⁰⁾ it is possible to form out of particles 2 and 3 a state e having an invariant "mass-squared" S_{23} , angular momentum j , and helicity m (see Appendix B). The helicity amplitude for the process $a+b \rightarrow 1+e$ will be denoted by $g_{\lambda_a \lambda_b}^{\lambda_1(jm \lambda_2 \lambda_3)}$. It depends on the three invariants S , t_{a1} and S_{23} ¹⁵⁾.

Let us now consider the crossed processes

$$a + \bar{1} \rightarrow \bar{b} + 2 + 3 \quad (2.4)$$

and

$$a + \bar{1} \rightarrow \bar{b} + e \quad (2.5)$$

and let us call the corresponding helicity amplitudes $F_{\lambda_a \lambda_{\bar{1}}}^{\lambda_{\bar{b}} \lambda_2 \lambda_3}$ and $f_{\lambda_a \lambda_{\bar{1}}}^{\lambda_{\bar{b}}(jm \lambda_2 \lambda_3)}$. For F we shall use instead of (2.2) the following set of independent variables¹⁶⁾

$$S, t_{a1}, S_{23}, \cos \vartheta_c, \varphi_c \quad (2.6)$$

where ϑ_c is the angle between the three-momenta of particles 2 and $\bar{b}$ and φ_c is the azimuthal angle between the normals to the planes $a\bar{1}$ and $3\bar{b}$, both measured in the c.m. frame of particles 2 and 3. For the relation between the two sets (2.2) and (2.6), see Appendix A.

Using the results of¹⁰⁾ and⁵⁾ one has the following relation between G and g

An important simplification occurs if formula (2.11) is integrated over the variables $\cos \vartheta_d$ and φ_d . Indeed, using the orthogonality properties of the rotation matrices ²⁴⁾ we obtain

$$\rho_{nn^+}(s, t_{a1}, s_{23}) = NC^{-2} \sum_{j^m} \sum_{\substack{\lambda_2 \lambda_3 \\ \lambda_a \lambda_b}} f_{\lambda_a \lambda_{\bar{a}}=n}^{\lambda_b(j^m \lambda_2 \lambda_3)} f_{\lambda_a \lambda_{\bar{a}}=n^+}^{*\lambda_b(j^m \lambda_2 \lambda_3)} \quad (2.12)$$

Inserting (2.8) into (2.12) we find the (twice integrated) density matrix elements in terms of the three-body crossed channel amplitude F .

It should be stressed that in deriving (2.11) and (2.12) we have not made any assumption about crossing symmetry for the three-body amplitude. We have instead relied entirely on the crossing symmetry of quasi two-body helicity amplitudes. For this reason, when inserting (2.8) into (2.11) a double integral appears which would not be expected otherwise. However, in going from $\rho_{nn^+}(s, t_{a1}, s_{23}, \cos \vartheta_d, \varphi_d)$ to $\rho_{nn^+}(s, t_{a1}, s_{23})$ no farther integration is introduced.

2. b. THE HIGH ENERGY REGION

As the amplitude F is usually known as function of the five invariants (2.2) we want to perform explicitly the change from the variables $\cos \vartheta_c$ and φ_c to the variables s_{12} and t_{b3} in Eq. (2.8). We shall restrict ourselves to the important case in which the non-integrated variables s , s_{23} and t_{a1} are taken in the high energy region of the direct channel.

Let us assume first $s_{23} \gg m_1^2, |t_{a1}|$. Formula (A.3) of Appendix A can be written within this approximation

$$\cos \vartheta_c \approx 1 + 2 \frac{t_{b3}}{s_{23}} \quad (2.13)$$

which implies for the integration limits of t_{b3}

$$-s_{23} \leq t_{b3} \leq 0 \quad (2.14)$$

These are the approximate maximum and minimum values of t_{b3} , at fixed values of t_{a1} and S_{23} satisfying the condition stated above. Since the values of t_{b3} are physical and S_{23} is large, the most important contribution to the t_{b3} integral comes from small values of $|t_{b3}|$ because F is appreciable only for small $|t_{b3}|$ (forward peak at high energies). Thus, let us limit ourselves to a region where

$$S \gg S_{23} \gg |t_{a1}|, |t_{b3}|, m_a^2, m_b^2, m_i^2 \quad (2.15)$$

holds. The first condition ($S \gg S_{23}$) is assumed in order to allow (from kinematics) S_{12} to become large ^{1),3)}. Under the assumption (2.15), Eq. (A.4) of Appendix A can be written

$$\cos \varphi_c \simeq \frac{S_{12} S_{23} + s(t_{a1} + t_{b3} - m_2^2)}{2s\sqrt{-t_{a1}}\sqrt{-t_{b3}}} \quad (2.16)$$

For the limits of integration of S_{12} we have

$$S_{12}^{(\min)} \simeq \frac{S}{S_{23}} \left[m_2^2 + (\sqrt{-t_{a1}} + \sqrt{-t_{b3}})^2 \right] \quad (2.17)$$

From (2.17) we obtain

$$\frac{S_{12} S_{23}}{s} \leq \left[m_2^2 + (\sqrt{-t_{a1}} + \sqrt{-t_{b3}})^2 \right] \quad (2.18)$$

8.

which leads to the "cornering effect" in the Dalitz plot described in ^{3),4)}. It is known ³⁾ that Eq. (2.18) taken with the equality sign gives, in the high energy limit, the border of the physical region.

From (2.15) and (2.17) we see that $S_{12}^{\min}$ is large. This allows the use of the asymptotic form of the crossed channel amplitude F in the integrand in Eq. (2.8).

To summarize, in the high energy limit, Eq. (2.8) can be written

$$f_{\lambda_2 \lambda_1}^{\lambda_2 \lambda_3}(s, t_{a1}, s_{23}) = C e^{i\pi S_3} N_j \int_{-s_{23}}^0 dt_{b3} \int_{\frac{s}{s_{23}} [m_2^2 + (\sqrt{-t_{b3}} + \sqrt{-t_{a1}})^2]}^{\frac{s}{s_{23}} [m_2^2 + (\sqrt{-t_{b3}} - \sqrt{-t_{a1}})^2]} ds_{12} \cdot$$

$$\cdot \gamma(s, s_{12}, s_{23}, t_{a1}, t_{b3}) e^{-i\varphi_c [m - (\lambda_2 - \lambda_3)]} d_{m, \lambda_2 - \lambda_3}^j(\varphi_c). \quad (2.19)$$

$$\sum_{\nu_2 \nu_3} d_{\nu_2 \lambda_2}^{s_2}(\beta_2^c) d_{\nu_3 \lambda_3}^{s_3}(\beta_3^c) F_{\lambda_a \lambda_1}^{\lambda_2 \nu_1 \nu_2}(s, s_{12}, s_{23}, t_{a1}, t_{b3})$$

with the Jacobian

$$\gamma \simeq -2 \left\{ 4s^2 t_{a1} t_{b3} - [s_{12} s_{23} + s(t_{a1} + t_{b3} - m_2^2)]^2 \right\}^{-1/2}. \quad (2.20)$$

Of course, when the energy is not very high one should use more accurate values for the limits of integration, especially for $t_{b3}^{\min}$. Note that, although $\cos \varphi_c = 1$ if $s_{23} \rightarrow \infty$ [see (2.13)], the amplitude F is not independent on the azimuthal angle φ_c , as it could be thought ⁵⁾. Indeed, when inverting (2.16) we obtain

$$s_{12} \simeq 2 \cos \varphi_c \frac{s}{s_{23}} \sqrt{-t_{a1}} \sqrt{-t_{b3}} - \frac{s}{s_{23}} (t_{a1} + t_{b3} - m_2^2). \quad (2.21)$$

Therefore, the dependence of F on φ_c through s_{12} is rather strong ²⁵⁾.

3. a. CONSEQUENCES OF THE DOUBLE REGGE MODEL FOR THE DENSITY MATRIX ELEMENTS

If one assumes that reaction (2.1) takes place via a double Regge graph (see the Figure) the crossed channel helicity amplitudes have in the high energy region of the direct channel (both S_{12} and S_{23} large) the very simple expression ⁸⁾

$$F_{\lambda_a \lambda_{\bar{a}} \lambda_2 \lambda_3}^{\lambda_b \lambda_{\bar{b}} \lambda_1 \lambda_3}(s, s_{12}, s_{23}, t_{a1}, t_{b3}) = \Delta(t_{a1}, t_{b3}) \gamma_{\lambda_2}(t_{a1}, t_{b3}, \varphi) \cdot N_\lambda \beta_{\lambda_a \lambda_{\bar{a}}}(t_{a1}) N_\mu \beta_{\lambda_b \lambda_{\bar{b}}}(t_{b3}) \cdot S_{12}^{\alpha_A(t_{a1})} S_{23}^{\alpha_B(t_{b3})} \quad (3.1)$$

$$\gamma_{\lambda_2} = \sum_{\mu, \lambda'} \gamma_{\lambda_2 \mu' \lambda'}(t_{a1}, t_{b3}, \varphi) \delta_{\lambda', \mu' - \lambda_2}$$

$$N_\mu = [(\alpha_A + \mu)! (\alpha_A - \mu)!]^{-1/2} \quad (3.2)$$

$$N_\lambda = [(\alpha_B + \lambda)! (\alpha_B - \lambda)!]^{-1/2}$$

$$\lambda = \lambda_a - \lambda_{\bar{a}}, \quad \mu = \lambda_b - \lambda_{\bar{b}}, \quad \lambda' + \lambda_2 = \mu'.$$

Here, α_A and α_B denote the trajectory parameters. The factor $\beta_{\lambda_a \lambda_{\bar{a}}}(\lambda_b \lambda_{\bar{b}})$ represents the coupling of the Regge pole A(B) to the particles $\bar{a}$ and a ($\bar{b}$ and b) ²⁶⁾. $\gamma_{\lambda_2 \mu' \lambda'}$ is an unknown function describing the coupling of particle 2 to the two Regge poles A and B. The angle φ is the angle between the normals to the planes $a1$ and $b3$ in the rest frame of particle 2 ^{8), 3)}. The signature factors and other known factors are included in $\Delta(t_{a1}, t_{b3})$.

In (3.1) we have assumed the factorization of the residue functions.

Let us denote the angles used in the Reggeization procedure by χ_A and χ_B , χ_A being the angle of particle $\bar{1}$ and 2 in the c.m.s. of particles a and $\bar{1}$ and χ_B being the angle between particles $\bar{3}$ and 2 in the c.m.s. of particles b and $\bar{3}$. It can be seen that $\cos \chi_A$ ($\cos \chi_B$) goes to infinity if S_{12} (S_{23}) goes to infinity except for

$$t_{a1}^{\min} \geq t_{a1} \geq t_{a1}^0 \quad (3.3a)$$

$$(t_{b3}^{\min} \geq t_{b3} \geq t_{b3}^0) \quad (3.3b)$$

where $|\cos \chi_A|$ ($|\cos \chi_B|$) is bounded by 1. This region shrinks like $1/S_{12}$ ($1/S_{23}$) for S_{12} (S_{23}) going to infinity. For $m_1^2 > m_a^2$ ($m_3^2 > m_b^2$) this region reduces to $t_{a1} = t_{a1}^{\min}$ ($t_{b3} = t_{b3}^{\min}$) even at finite S_{12} (S_{23}). If t_{a1} and/or t_{b3} are those regions (3.3a), (3.3b) the simple formula (3.1) cannot be used (see Section 3c). In fact, one cannot use (3.1) even for t_{a1} and/or t_{b3} in larger regions, such that $\cos \chi_A < C$ and/or $\cos \chi_B < C$, where C is some constant. How large the constant C has to be, does not seem to be possible to be decided a priori in any convincing way. From now on, we shall call these regions the "forward cones", (see, however, ^{27),28}).

For t_{a1} outside the forward cone one obtains ²⁹⁾ from (2.12) and (3.1)

$$\rho_{nn^+}(t_{a1}) = \frac{\sum_{\lambda_a} \mathcal{N}_{n+\lambda_a} \mathcal{N}_{n^++\lambda_a} \beta_{\lambda_a n}(t_{a1}) \beta_{\lambda_a n^+}(t_{a1})}{\sum_{\lambda_a, n} |\mathcal{N}_{n+\lambda_a} \beta_{\lambda_a n}(t_{a1})|^2} \quad (3.4)$$

Notice that the ρ_{nn^+} in (3.4) no longer depend on S and S_{23} ³⁰⁾.

3. b. COMPARISON WITH TWO-BODY PROCESSES AND DISCUSSION

Let us now consider the two-body process

$$a + b \rightarrow 1 + 2 \quad (3.5)$$

having in common with the three-body reaction (2.1) the vertex $a1A$.

The canonical density matrix elements are given by ²⁰⁾

$$\rho_{nn^+}(s, t) = N \sum_{\substack{\lambda_a \lambda_b \\ \lambda_2}} F_{\lambda_a \lambda_{\bar{1}}=n}^{\lambda_b \lambda_2}(s, t) F_{\lambda_a \lambda_{\bar{1}}=n}^{*\lambda_b \lambda_2}(s, t) \quad (3.6)$$

In the high energy region of the direct channel the two-body helicity amplitudes are given by

$$F_{\lambda_a \lambda_1}^{\lambda_b \lambda_2}(s, t) = \eta(t) \mathcal{N}_\lambda \mathcal{N}_\mu \beta_{\lambda_a \lambda_{\bar{1}}}(t) \beta_{\lambda_b \lambda_2}(t) S^{\alpha_A(t)} \quad (3.7)$$

$\eta(t)$ being a known function of t which contains the signature factor, etc. The factors $\mathcal{N}_\lambda$ and $\mathcal{N}_\mu$ are defined as in (3.2) ^{*)}. For t outside the forward cone ³¹⁾ we obtain from (3.6) and (3.7) for the density matrix elements of particle 1 ³⁰⁾

$$\rho_{nn^+}(t) = \frac{\sum_{\lambda_a} \mathcal{N}_{\lambda_a+n} \beta_{\lambda_a n}(t) \mathcal{N}_{\lambda_a+n^+} \beta_{\lambda_a n^+}(t)}{\sum_{\lambda_a n} |\mathcal{N}_{\lambda_a+n} \beta_{\lambda_a n}(t)|^2} \quad (3.8)$$

^{*)} The products $\beta_{\lambda_a \lambda_{\bar{1}}} \beta_{\lambda_b \lambda_2}$ are the residues of the parity conserving helicity amplitudes at the Regge pole. For this reason we have factored out the $\mathcal{N}_\lambda$'s.

Now, it follows from the (assumed) factorization property of the residue functions that the residue functions in (3.1) and (3.7) are equal. We, therefore, deduce from (3.4) and (3.8) the equality of the density matrix elements in two and three-body processes:

$$\rho_{nn^+}(t) = \rho_{nn^+}(t_{a1} = t) \quad (3.9)$$

Equation (3.9) is only valid for both t and t_{a1} outside their forward cones.

3. c. CONCLUDING REMARKS

1) From conservation of angular momentum follows that the two-body amplitude $F_{\lambda_a \lambda_1}^{\lambda_b \lambda_2}(s, t^{\min}) = 0$ if $\lambda - \mu \neq 0$, where $\lambda = \lambda_a - \lambda_1$, $\mu = \lambda_b - \lambda_2$. This implies that $\rho_{nn^+}(t^{\min}) = 0$ if $n \neq n^+$. Since a similar statement is not true for the three-body amplitude (except for both $t_{a1} = t_{a1}^{\min}$ and $t_{b3} = t_{b3}^{\min}$ and $\lambda_2 = 0$) formula (3.9) is certainly not valid for $t = t^{\min}$ and/or $t_{a1} = t_{a1}^{\min}$. This result also follows from the Regge model since, although the amplitudes have a Regge type behaviour inside the forward and backward cones ³²⁾, the formulae (3.1) and (3.7) are not valid there. This can easily be seen in the two-body case starting from the Reggeized helicity amplitudes free of kinematical singularities ^{33), 12)}. One obtains ³⁴⁾ instead of (3.7) ³⁵⁾

$$F_{\lambda_a \lambda_1}^{\lambda_b \lambda_2}(s, t) \propto s^{\alpha(t) - \max\{|\lambda|, |\mu|\}} \left(\cos \frac{\theta_t}{2}\right)^{|\lambda + \mu|} \left(\sin \frac{\theta_t}{2}\right)^{|\lambda - \mu|} \quad (3.10)$$

For $\cos \theta_t/2$ and $\sin \theta_t/2$ large, the product of the three terms in (3.10) reproduces the behaviour $s^{\alpha(t)}$ in (3.7). However, for t inside the forward cone, (3.10) shows that the helicity amplitudes are no longer the product of two factors, each of them depending only on the helicities of either the lower or the upper vertex. Therefore, formula (3.8) does not follow³⁶⁾. For the three-body amplitudes, although the kinematical singularities are not known, one cannot expect (3.1) to be valid in the forward cones.

2) A crucial assumption in deriving Eqs. (3.4), (3.8) and (3.9) is the factorization property of the residue functions. It is known^{33),37)} that the usual threshold behaviour of the residue functions are, in general, incompatible with factorization if all the possible kinematical singularities of the amplitude are present. The factorization property is, however, generally accepted^{8),28),33),37),38),39)}. It imposes some restrictions on the kinematical singularities of the amplitudes at $t=0$. It has been shown by Leader³⁷⁾ that these restrictions have in some cases (boson exchange) the nice property of allowing to avoid a conspiracy which otherwise would be present as a consequence of the constraint relations.

3) The comparison between the density matrix elements in two-body and in three-body processes [Eq. (3.9)] can only be done for resonances produced in the upper or lower vertex (see the Figure); the vertex responsible for production of the resonance to be analyzed can be determined through the ordering of the longitudinal momenta in the final state as discussed in^{3),4)}. Although the energies available at present are perhaps not high enough to expect the contribution of the leading Regge trajectory to dominate, we think it is worth while to test formula (3.9)⁴⁰⁾. The comparison should, of course, be done in channels for which the density matrix in the two-body reaction as calculated from the leading trajectory alone is somehow in agreement with experiment^{28),41)}. Since the density matrices do not depend on the energy one can enlarge the statistics by joining experiments performed at different energies.

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APPENDIX A

The expressions of the angles ϑ_d , φ_d , ϑ_c and φ_c in terms of the five invariants (2.2) are

$$\cos \vartheta_d = \frac{S_{23}^2 + S_{23}(2S_{12} - m_2^2 - m_3^2 - m_1^2 - s) + (m_2^2 - m_3^2)(m_1^2 - s)}{\sqrt{\lambda(S_{23}, m_3^2, m_2^2)} \sqrt{\lambda(S_{23}, m_1^2, s)}} \quad (\text{A.1})$$

$$(\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz)$$

$$\begin{aligned} \cos \varphi_d &= \frac{[\vec{p}_a \times \vec{p}_b][\vec{p}_3 \times \vec{p}_1]}{||[\vec{p}_a \times \vec{p}_b]|| ||[\vec{p}_3 \times \vec{p}_1]||} \bigg|_{\vec{p}_2 + \vec{p}_3 = 0} \\ &= \frac{\varepsilon_{\mu\nu\sigma\tau} p_a^\mu p_b^\nu (p_2 + p_3)^\sigma \varepsilon_{\alpha\beta\gamma} p_3^\alpha p_1^\beta (p_2 + p_3)^\gamma}{\sqrt{\varepsilon_{\mu\nu\sigma\tau} p_a^\mu p_b^\nu p_1^\sigma \varepsilon_{\alpha\beta\gamma} p_a^\mu p_b^\nu p_1^\sigma} \sqrt{\varepsilon_{\mu\nu\sigma\tau} p_3^\mu p_1^\nu p_2^\sigma \varepsilon_{\alpha\beta\gamma} p_3^\mu p_1^\nu p_2^\sigma}} \quad (\text{A.2}) \end{aligned}$$

$$\cos \vartheta_c = \frac{S_{23}^2 + S_{23}(2t_{b3} - m_2^2 - m_3^2 - t_{a1} - m_b^2) + (m_2^2 - m_3^2)(t_{a1} - m_b^2)}{\sqrt{\lambda(S_{23}, m_3^2, m_2^2)} \sqrt{\lambda(S_{23}, m_b^2, t_{a1})}} \quad (\text{A.3})$$

$$\begin{aligned}
\cos \varphi_c &= \frac{[\vec{p}_a \times \vec{p}_1][\vec{p}_3 \times \vec{p}_5]}{||[\vec{p}_a \times \vec{p}_1]|| ||[\vec{p}_3 \times \vec{p}_5]||} \Big|_{\vec{p}_2 + \vec{p}_3 = 0} \\
&= \frac{\varepsilon_{\mu\nu\varrho\sigma} p_a^\mu p_1^\nu (p_2 + p_3)^\varrho \varepsilon_{\alpha\beta\gamma} p_3^\sigma p_5^\alpha (p_2 + p_3)^\gamma}{\sqrt{\varepsilon_{\mu\nu\varrho\sigma} p_a^\mu p_1^\nu p_5^\varrho \varepsilon_{\alpha\beta\gamma} p_a^\sigma p_1^\beta p_5^\gamma} \sqrt{\varepsilon_{\mu\nu\varrho\sigma} p_3^\mu p_5^\nu p_2^\varrho \varepsilon_{\alpha\beta\gamma} p_3^\sigma p_5^\beta p_2^\gamma}} \\
&= \frac{N}{\sqrt{D_1} \sqrt{D_2}}
\end{aligned} \tag{A.4}$$

where

$$\begin{aligned}
N &= N(s, s_{12}, s_{23}, t_{a1}, t_{b3}) \\
&= \begin{vmatrix} s + t_{b3} - s_{12} - m_b^2 & -s + m_a^2 + m_b^2 & s_{12} + t_{a1} - t_{b3} - m_1^2 \\ -s + s_{12} + s_{23} - m_2^2 & s + t_{a1} - s_{23} - m_a^2 & m_1^2 + m_2^2 - s_{12} \\ -t_{b3} + m_b^2 + m_3^2 & -2m_b^2 & s_{23} + t_{b3} - t_{a1} - m_3^2 \end{vmatrix} \tag{A.5}
\end{aligned}$$

$$\begin{aligned}
D_1 &= D_1(s, s_{23}, t_{a1}) \\
&= \begin{vmatrix} 2m_a^2 & t_{a1} - m_a^2 - m_1^2 & s - m_a^2 - m_b^2 \\ t_{a1} - m_a^2 - m_1^2 & 2m_1^2 & s_{23} + m_a^2 - s - t_{a1} \\ s - m_a^2 - m_b^2 & s_{23} + m_a^2 - s - t_{a1} & 2m_b^2 \end{vmatrix} \tag{A.6}
\end{aligned}$$

and

$$\mathcal{D}_2 = \mathcal{D}_2(s, s_{23}, t_{a1}, t_{b1})$$

$$\begin{vmatrix} 2m_3^2 & t_{b3} - m_b^2 - m_3^2 & s_{23} - m_2^2 - m_3^2 \\ t_{b3} - m_b^2 - m_3^2 & 2m_b^2 & -s_{23} - t_{b3} + t_{a1} + m_3^2 \\ s_{23} - m_2^2 - m_3^2 & -s_{23} - t_{b3} + t_{a1} + m_3^2 & 2m_2^2 \end{vmatrix} \quad (\text{A.7})$$

APPENDIX B

Let us consider the state $|p_1^0 \lambda_1; p_2^0 \lambda_2\rangle$, which represents the direct product of the states $|p_1^0 \lambda_1\rangle$ and $|p_2^0 \lambda_2\rangle$ with helicities λ_1 and λ_2 and masses m_1 and m_2 , respectively, and three-momenta $\vec{p}_1^0, \vec{p}_2^0$ such that $\vec{p}_1^0 = -\vec{p}_2^0$ points in the direction of the positive z axis. We now apply to $|p_1^0 \lambda_1; p_2^0 \lambda_2\rangle$ the Wigner projection operator ⁴²⁾ which selects a state $|Q_{12}^0 j m, \lambda_1 \lambda_2\rangle$ with total angular momentum j , its projection m on the positive z direction, and a momentum $Q_{12}^0 = \{\vec{Q}_{12}^0, w\}$ such that $\vec{Q}_{12}^0 = 0$ and

$$w = \sqrt{m_1^2 + |\vec{p}_1^0|^2} + \sqrt{m_2^2 + |\vec{p}_1^0|^2}$$

$$|Q_{12}^0 j m \lambda_1 \lambda_2\rangle = C N_j \int_{-1}^1 d(\cos \vartheta) \int_0^{2\pi} d\varphi D_{\lambda_m}^{*j}(\varphi, \vartheta, 0) R_{\varphi \vartheta 0} |p_1^0 \lambda_1; p_2^0 \lambda_2\rangle \quad (B.1)$$

$$\left(\lambda = \lambda_1 - \lambda_2, N_j = \sqrt{\frac{2j+1}{4\pi}}, C = \sqrt{\frac{|\vec{p}_1^0|}{4w}} \right)$$

where r is a rotation with Euler angles $\varphi, \vartheta, 0$ and R is the corresponding operator.

From (B.1) we define the following state

$$|Q_{12} j m \lambda_1 \lambda_2; q_3 \lambda_3\rangle = [Z |Q_{12}^0 j m \lambda_1 \lambda_2\rangle] \otimes |q_3 \lambda_3\rangle \quad (B.2)$$

where the three-vector $\vec{q}_3$ is along the negative z direction and Z is a Lorentz transformation in the positive z direction such that $\vec{Q}_{12} = -\vec{q}_3$.

Evaluating $ZR_{\varphi\vartheta_0}|p_1^0\lambda_1;p_2^0\lambda_2\rangle$ we find

$$\begin{aligned} ZR_{\varphi\vartheta_0}|p_1^0\lambda_1;p_2^0\lambda_2\rangle &= e^{-i\varphi(\lambda_1-\lambda_2)} ZR_{\vartheta_0}|p_1^0\lambda_1;p_2^0\lambda_2\rangle \\ &= e^{-i\varphi(\lambda_1-\lambda_2)} e^{-i\pi S_2} \sum_{v_1 v_2} d_{v_1\lambda_1}^{S_1}(\beta_1^d) d_{v_2\lambda_2}^{S_2}(\beta_2^d) |q_1 v_1; q_2 v_2\rangle \end{aligned} \quad (B.3)$$

where S_1 and S_2 are the spins of particles 1 and 2 and the momenta q_1 and q_2 are such that $\vec{q}_1 + \vec{q}_2 = -\vec{q}_3$ [see Wick¹⁰⁾, formula (21) where the definition of the angles β_1^d and β_2^d is given⁴³⁾].

From (B.1), (B.2) and (B.3) one finds

$$\begin{aligned} |Q_{12} j m \lambda_1 \lambda_2; q_3 \lambda_3\rangle &= C e^{-i\pi S_2} N_j \int_{-1}^1 d(\cos\vartheta) \int_0^{2\pi} d\varphi d_{m, \lambda_1 - \lambda_2}^j(\vartheta) \\ &\quad e^{i\varphi[m - (\lambda_1 - \lambda_2)]} \sum_{v_1 v_2} d_{v_1\lambda_1}^{S_1}(\beta_1^d) d_{v_2\lambda_2}^{S_2}(\beta_2^d) |q_1 v_1; q_2 v_2; q_3 v_3\rangle \end{aligned} \quad (B.4)$$

$$(|q_1 v_1; q_2 v_2; q_3 v_3\rangle \equiv |q_1 v_1; q_2 v_2\rangle \otimes |q_3 v_3\rangle)$$

Using the orthonormality properties of the rotation matrices²⁴⁾ one easily verifies that the inverse of formula (B.4) is

$$\begin{aligned} |q_1 \lambda_1; q_2 \lambda_2; q_3 \lambda_3\rangle &= C^{-1} e^{i\pi S_2} \sum_{j m} N_j e^{-i\varphi[m - (v_1 - v_2)]} d_{m, v_1 - v_2}^j(\vartheta) \\ &\quad d_{\lambda_1 v_1}^{S_1}(\beta_1^d) d_{\lambda_2 v_2}^{S_2}(\beta_2^d) |Q_{12} j m v_1 v_2; q_3 \lambda_3\rangle. \end{aligned} \quad (B.5)$$

With the following definitions

$$C_{\lambda_2 \lambda_b}^{\lambda_1 \lambda_2 \lambda_3} = \langle q_1 \lambda_1; q_2 \lambda_2; q_3 \lambda_3 | T | q_a \lambda_a; q_b \lambda_b \rangle \quad (\text{B.6})$$

and

$$Q_{\lambda_a \lambda_b}^{(jm \lambda_1 \lambda_2) \lambda_3} = \langle Q_{12} j m \lambda_1 \lambda_2; q_3 \lambda_3 | T | q_a \lambda_a; q_b \lambda_b \rangle, \quad (\text{B.7})$$

where T is the T matrix operator, we obtain from (B.4) and (B.5) the formulae (2.8) and (2.7) respectively.

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- 13) G. Bros, H. Epstein, V. Glaser, Commun.Math.Phys. 1, 240 (1965).
- 14) A very recent preprint of Bali et al. ⁴⁴⁾ introduces a different set
of variables based on Toller's group theoretical analysis of scat-
tering amplitudes. These variables seem to be very well suited for
the description of multiperipheral models at high energies.
- 15) Unless stated otherwise, when writing g and G we always mean
 $g(S, t_{a1}, S_{23})$ and $G(S, t_{a1}, S_{23}, \cos \vartheta_d, \varphi_d)$.
- 16) Again, f and F mean $f(S, t_{a1}, S_{23})$ and $F(S, t_{a1}, S_{23}, \cos \vartheta_c, \varphi_c)$.
- 17) G. Cohen-Tannoudji et al. ¹²⁾ have recently obtained a crossing
relation which differs from (2.9) by a phase; this, however, has no
effect on our formulae.
- 18) See also Section 2b., where this is shown analytically for the high
energy region of the direct channel.
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 25) Note, that the expression (2.12) contains infinite sums over j and m . The dependence of f on these two indices is essentially given through the rotation matrices $d_{m, \lambda_2 - \lambda_3}^j(\mathcal{Q}_c)$ [see Eq. (2.8)]. Owing to the expressions of the d functions and due to the peaking of the amplitude F , only a few terms in the j, m sums contribute appreciably.
 26) The product $\mathcal{N}_\lambda \beta_{\lambda_a \lambda_{\bar{a}}}$ in (3.1) is denoted $\beta_{\lambda_a \lambda_{\bar{a}}}$ in 8). The reason why we factor out the $\mathcal{N}_\lambda$ will be clear later on.
 27) B. Diu and M. Le Bellac, Orsay preprint, TH. 191 (1967).
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 29) In Eq. (2.19) the integration over F goes from $t_{b3} = t_{b3}^{\min}$ to $t_{b3} = t_{b3}^{\max}$ and near $t_{b3} = t_{b3}^{\min}$ we cannot use Eq. (3.1). However, since t_{a1} is outside its forward cone, one can use near $t_{b3} = t_{b3}^{\min}$ a factorizable expression

$$F_{\lambda_a \lambda_{\bar{a}}}^{\lambda_b \lambda_{\bar{b}}} = \mathcal{N}_\lambda \beta_{\lambda_a \lambda_{\bar{a}}}(t_{a1}) S_{12}^{\alpha_\lambda(t_{a1})} \eta_{\lambda_2 \lambda_3 \lambda_{\bar{b}}}(t_{a1}, t_{b3}, \varphi, S_{23})$$

Therefore, Eq. (3.2) follows.

- 30) The factors $\mathcal{N}_\lambda$ in Eqs. (3.4) and (3.8) contain the trajectory parameters $\alpha(t)$. Also, the residue functions contain such trajectory parameters 7). Together they give rise to dips in some density matrix elements at a value of t such that $\alpha(t) = 0$. Of course, which density matrix elements contain the dip and which do not, depends on the way the α factors are attributed to the sense and nonsense states. Four such ways, compatible with factorization of the residue functions, are discussed in 45).

- 31) Again, this formula is not valid for t in the forward cone (see the discussion of $\cos \chi_A$ and $\cos \chi_B$ on page 10). For the moment, we restrict ourselves to processes in which either $m_a \neq m_1$ or $m_b \neq m_2$ or both [see footnote ³⁶⁾].
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- 35) Formula (3.10) has been used to show ^{27),28)} that in vector meson production if the leading Regge trajectory has natural parity, the process can be dominated, inside the forward cone, by a lower lying trajectory with unnatural parity.
- 36) For $m_a = m_1$ and $m_b = m_2$ there is no forward cone and formula (3.7) is valid everywhere. However, the crossed-channel helicity amplitudes have to satisfy constraint relations at $t = t^{\min} = 0$ ^{12), 39), 46), 47)}. These constraints imply $\phi_{nn^+}(0) = 0$ if $n \neq n^+$.
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- 40) The angular distribution for the decay of a resonance produced, say, in the upper vertex of the Figure, has the same form as in two-body processes ²²⁾ provided the Gottfried-Jackson angle ϑ and the Treiman-Yang angle φ are defined here in the following way:
 ϑ is the angle in the rest system of particle 1 between the direction of motion of the incoming particle a and one of the decay particles, say α .
 φ is the azimuthal angle, again in the rest system of 1, between the plane $a\alpha$ and the plane defined by 2 and $\vec{p}_b - \vec{p}_3$.
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