Spherical Self-Similar Solutions in Einstein-Multi-Scalar Gravity

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Abstract

We consider a general non-linear sigma model coupled to Einstein gravity and show that in spherical symmetry and for a simple realization of self-similarity, the spacetime can be completely determined. We also examine some more specific matter models and discuss their relation to critical collapse.

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INTRODUCTION

The problem of gravitational collapse has received a great deal of attention in the past several years. This is due in large part to Choptuik’s results on the collapse of a real scalar field coupled to gravity [1]. Choptuik showed that the strong field regime exhibited rather striking and unexpected behavior. This included the surprising result that black holes can form with arbitrarily small mass and that a scaling relation exists for the black hole mass. In addition, the “critical” solution exactly at the threshold of black hole formation possesses a discrete self-similarity and serves as an intermediate attractor for near-critical solutions in the space of initial data.

Considerable effort has been made to understand these results from both computational and analytic perspectives. The general behavior is now rather well understood and a very nice review article summarizes the relevant literature and current state of the art [2]. However, as pointed out there, progress in understanding critical phenomena from a purely analytic perspective has been slower in coming. This is largely due to the well known difficulties associated with finding closed form solutions to Einstein’s equations in dynamical situations. One of the few closed form solutions to the Einstein equations which appears somewhat related to critical phenomena is the so-called Roberts solution. It had been written down several times in the literature [3] [4], but was rediscovered in an attempt to get an analytic grasp of the features seen in critical collapse [5]. The solution is spherically symmetric and continuously self-similar. It also contains timelike singularities. When these singularities are cut out and replaced with Minkowski space, the spacetime describes a collapsing ball of scalar field. In addition, as a parameter is varied, the solution describes dispersal of the field to infinity or the formation of an apparent horizon.

Our attempt here is to extend this solution to a wider class of matter models. Numerical work has shown that critical behavior arises in a variety of matter models coupled to gravity. Thus one might expect the Roberts solution to be a member of a larger class of these “toy models.”

Indeed, we find that the basic spacetime for scalar field collapse as described by the Roberts solution turns out to extend virtually unchanged to a large class of massless interacting scalar fields. It is this result that we present in the next section and follow with some specific examples.

THE MODEL AND THE SPACETIME SOLUTION

We are interested in considering a generalization of the Einstein-scalar field problem. So we turn our attention to Einstein gravity coupled to multiple interacting scalar fields. The particular form of the theory we will consider is given by the action

\[ S = \int d^4x \sqrt{-g} \{R + 2L_m\}, \tag{1} \]

where the Lagrangian density for the matter fields is given by

\[ L_m = -\frac{\lambda}{2} G_{AB}(\phi^C)\phi_A^{\alpha\alpha}\phi_B^{\alpha\alpha} \tag{2} \]
where we have $N$ scalar fields $\phi^A$ ($A = 1 \ldots N$) and $G_{AB}$ is a set of functions of the scalar fields which must be specified to fix the model. In general, the functions making up $G_{AB}$ take the form of a Riemannian metric on the $N$-dimensional internal space of the scalar fields (the target space). Thus the $N$ real scalar fields, $\phi^A$, are coupled to Einstein gravity with $\lambda$ a dimensionless coupling constant. In fact, any parameters entering the target space metric $G_{AB}$ will also be dimensionless and this in turn will allow solutions to be self-similar. The theory defined by Eq.(2) is thus a particular case of more general Einstein-multi-scalar theories of gravity such as those discussed in [6].

Varying the action with respect to the metric and the matter fields yields the Einstein field equations and the equations of motion for the matter:

\[ R_{\mu\nu} = \lambda G_{AB}(\phi^C) \phi^A_{,\mu} \phi^B_{,\nu}, \]  
\[ \phi^A_{,\mu} = -\phi^B_{,\mu} \phi^C_{,\mu} \Gamma^A_{BC}, \]

where, as usual, a comma and a semicolon denote partial and covariant differentiation, respectively. We also define the Christoffel symbols on the target space

\[ \Gamma^A_{BC} = \frac{1}{2} G^{AD} \left[ \frac{\partial G_{BD}}{\partial \phi^C} + \frac{\partial G_{CD}}{\partial \phi^B} - \frac{\partial G_{BC}}{\partial \phi^D} \right]. \]

Because of the contracted Bianchi identities $T_{\mu\nu;\lambda} g^{\nu\lambda} = 0$, Eqs.(3) - (4) are not independent. In the following, we shall use this fact to solve these equations by properly choosing the independent ones.

Working in spherical symmetry, we can write the metric in double null coordinates

\[ ds^2 = -2 e^{2\sigma} du dv + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right), \]

where $\sigma$ and $r$ are functions of the two null coordinates $u$ and $v$ only. The metric is invariant under the coordinate transformations $\bar{u} = \bar{u}(u), \; \bar{v} = \bar{v}(v)$. In addition to assuming spherical symmetry, we are interested in spacetimes that are continuously self-similar (CSS). Stated invariantly, we are assuming the existence of a homothetic Killing vector field $\xi$ satisfying [7]

\[ \mathcal{L}_\xi g_{\mu\nu} = \xi_{\mu\nu} + \xi_{\nu\mu} = 2 g_{\mu\nu}, \]

where $\mathcal{L}$ denotes the Lie derivative. It is then straightforward to show that the homothetic vector field is

\[ \xi = u \frac{\partial}{\partial u} + v \frac{\partial}{\partial v}. \]

In addition, it follows that the the metric coefficients take the form

\[ \sigma = \sigma(z), \quad r = us(z), \]

where $z \equiv v/u$. We also need to know the form of the scalar fields in the presence of the continuous self-similarity. The simplest ansatz is to assume

\[ \mathcal{L}_\xi \phi^A = 0 \]
such that the scalar fields will depend only on the coordinate $z = v/u$ as with the metric coefficients. This is, in fact, the ansatz we will make here, but it is not necessarily the most general. Depending on the symmetries of the $N$-dimensional target space defined by the scalar fields, one possible generalization is to allow the homothetic motion to be accompanied by a change in the scalar fields which respects the target space symmetries. For example, in the case of a complex scalar field ($N = 2$ and $G_{AB} = \eta_{AB}$, a flat target space), there is a $U(1)$ symmetry which could be incorporated into the self-similarity ansatz for the fields [8]. Another possible generalization is to make a kind of hedgehog ansatz for the scalar fields before imposing self-similarity. In this case, the $\phi^A$ fields will have some nontrivial angular dependance. An example is the self-similar solution of a spherically symmetric $O(4)$ non-linear sigma model which models the relativistic collapse of a global texture [9].

For the ansatz given by Eq.(10), the Einstein field equations (3) take the form

\begin{align}
2z\sigma'(s - zs') + z^2 s'' &= -\frac{\lambda z^2 s}{2} G_{AB} \phi^A \phi^B, \\
2\sigma' s' - s'' &= \frac{\lambda s}{2} G_{AB} \phi^A \phi^B, \\
2(\sigma' + z\sigma'') + 2z s'' &= -\lambda z G_{AB} \phi^A \phi^B, \\
\left[\frac{(s^2)'}{z}\right]' &= \frac{e^{2\sigma}}{z^2}.
\end{align}

where a prime indicates differentiation with respect to the similarity variable $z$. From Eqs.(11) and (12) we find that $\sigma = \text{const}$. By properly rescaling the null coordinates $u$ and $v$, we can set $\sigma = 0$. Thus, without loss of generality, in the following we shall consider only the case where $\sigma = 0$. Substituting into Eq.(14) we find that

\begin{equation}
s^2(z) = \alpha z^2 - z + \beta,
\end{equation}

where $\alpha$ and $\beta$ are two integration constants. Meanwhile, Eqs.(11) - (13) reduce to the single equation

\begin{equation}
G_{AB} \phi^A \phi^B = \frac{\gamma}{s^4},
\end{equation}

where $\gamma \equiv (1 - 4\alpha\beta)/(2\lambda)$. It is interesting to note that without having specified the target space metric $G_{AB}$, we have been able to completely determine the metric coefficients. Thus, spherical symmetry and our simple ansatz for self-similarity completely fix the spacetime geometry to be the same for all of these non-linear sigma models. In particular, the spacetime and its global structure is the same as that of the Roberts solution. \footnote{Strictly speaking, this will be true provided $1 - 4\alpha\beta > 0$ as in the Roberts solutions. However, in the following, we will encounter a situation where this is not necessarily the case.}
SOLUTIONS OF THE MATTER EQUATIONS

To complete the solution, we need to specify the non-linear sigma model by choosing a form for $G_{AB}$. In the following we will restrict ourselves to a 2-dimensional target space of the form

$$G_{AB} = \begin{pmatrix} 1 & 0 \\ 0 & f(\phi) \end{pmatrix} \quad (17)$$

where, for clarity, we define $\phi^1 = \phi$ and $\phi^2 = \psi$.

The matter field equations Eq.(4) can now be written as

$$\left[ s^2 \phi' \right]' = s^2 f(\phi) \frac{\phi'}{s^2} \quad (18)$$
$$\left[ s^2 f(\phi) \psi' \right]' = 0. \quad (19)$$

Clearly, the second equation has the first integral,

$$\frac{d\psi}{dz} = \frac{c_0}{f(\phi)s^2}. \quad (20)$$

where $c_0$ is an integration constant. Inserting the above equation into Eq.(16), we obtain

$$\frac{d\phi}{dz} = \pm \left[ \gamma - \frac{c_0^2}{f(\phi)} \right]^{1/2} \frac{1}{s^2}. \quad (21)$$

To solve Eqs.(20) and (21), we need to specify the function $f(\phi)$. In the following, we shall consider some particular cases.

I \quad $f(\phi) = e^{-\phi}$

This case corresponds to the non-linear $\sigma$-model considered in [8] for the case $\kappa > 0$. In fact, if we set

$$F = \frac{1}{\sqrt{\kappa}} \frac{1 + i\tau}{1 - i\tau}, \quad \tau = \frac{1}{4} \left( \psi + 2ie^{\phi/2} \right), \quad (22)$$

then the Lagrangian density of the complex matter field $F$ takes the form

$$L_m = -\frac{\left| \nabla F \right|^2}{(1 - \kappa |F|^2)^2} = -\frac{1}{16\kappa} \left( \phi^{\alpha} \phi_{,\alpha} + e^{-\phi} \psi_{,\alpha} \psi_{,\alpha} \right), \quad (23)$$

which corresponds to the one given by Eqs.(2) and (17) with

$$\lambda = \frac{1}{8\kappa}, \quad f(\phi) = e^{-\phi}. \quad (24)$$

As noticed in [8], this also corresponds to the Brans-Dicke (BD) theory of gravity considered in [10] with

$$\lambda = \omega + \frac{3}{2}, \quad (25)$$

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where $\omega$ is the BD coupling constant. Note the difference between $\lambda$ used here and the one used in [10].

Substituting Eq.(24) into Eqs.(20) and (21), and then integrating them, we obtain

$$
\phi = 2 \ln \frac{2g}{1 + g^2} + \ln \frac{\gamma}{c_0^2},
$$

$$
\psi = \pm \sqrt{\gamma} \frac{4}{c_0} \frac{1 + g^2}{1 + g^2} + \psi_0,
$$

where

$$
g = c_1 \left( \frac{z - z_+}{z - z_-} \right)^{\pm \sqrt{\kappa}},
$$

$$
z_\pm = \frac{1 \pm \sqrt{1 - 4\alpha\beta}}{2\alpha},
$$

and $c_1$, $\phi_0$ and $\psi_0$ are integration constants. Note that in this case the parameters $\alpha$ and $\beta$ have to be chosen such that $\alpha\beta < 1/4$. Otherwise, no solutions exist.

II $f(\phi) = \sin^2 2\phi$

For this case we have the non-linear $\sigma$-model considered in [8] for $\kappa < 0$. Actually, if we set

$$
F = \tan \frac{\phi}{\sqrt{|\kappa|}} (\sin 2\psi + i \cos 2\psi),
$$

we find that

$$
L_m = - \frac{|\nabla F|^2}{(1 - \kappa |F|^2)^2} = - \frac{1}{|\kappa|} \left( \phi^{\alpha} \phi_{,\alpha} + \sin^2 2\phi \psi^{\alpha} \psi_{,\alpha} \right),
$$

which corresponds to Eqs.(2) and (17) with

$$
\lambda = \frac{2}{|\kappa|}, \quad f(\phi) = \sin^2 2\phi.
$$

Inserting the above expressions into Eqs.(20) and (21), we find the following solution

$$
\cos 2\phi = \left[ 1 - \frac{2\lambda c_0^2}{1 - 4\alpha\beta} \right]^{1/2} \cos g,
$$

$$
\psi = \pm \frac{1}{2} \tan^{-1} \left[ \sqrt{\gamma} \frac{\tan g}{c_0} \right] + \psi_0,
$$

but now

$$
g \equiv \pm \sqrt{|\kappa|} \ln \left( \frac{z - z_+}{z - z_-} \right) + c_1,
$$

where $z_\pm$ are given by Eq.(27), and $c_1$ and $\psi_0$ are other integration constants. Similar to the last case, now the condition $\alpha\beta < 1/4$ has to hold.
Note that when $\kappa = 0$, the self-similar solution of the non-linear $\sigma$-model of Eq.(23) or Eq.(29) with the ansatz (10) is the same as that of Roberts [4]. As a matter of fact, in the latter case we have $f(\phi) = 1$. Then, from Eqs. (20) and (21) we find that $\psi = A\phi$, where $A$ is a constant.

**III** \[ f(\phi) = \frac{1}{A} e^{-\phi}, \lambda = \frac{1}{2} (3 + 2\omega) \]

Finally, we consider the case corresponding to the BD theory considered in [10]. As noted previously, when $\omega > -3/2$, it is equivalent to the non-linear $\sigma$-models with $\kappa > 0$. However, when $\omega < -3/2$, such a correspondence no longer exists. Therefore, in the following we shall consider only the case where $\omega < -3/2$. Before proceeding and by way of justification, we note that the case where $\omega < 0$ is also worthy of study for more than its intrinsic theoretical interest. In particular, for sufficiently negative $\omega$, the theory passes all the experimental and observational constraints as does the case of positive $\omega$ [11]. However, when $\omega < -3/2$, the static spherical spacetimes are quite different from the ones with $\omega > -3/2$. In particular, in the former case “cold” spherical black holes exist [12], while in the latter case, the only static spherical black hole is the Schwarzschild black hole [13].

Inserting the above expressions for $\lambda$ and $f(\phi)$ into Eqs.(20) and (21), we find that solutions exist for both $\alpha \beta < 1/4$ and $\alpha \beta > 1/4$.

**IIIa** \[ 1 - 4\alpha \beta > 0 \]

In this case, we have

$$\phi = \ln \left[ \cos^2 g \right] + \phi_0, \quad \psi = c_0 |3 + 2\omega| \tan g + \psi_0,$$

(33)

where

$$g \equiv \frac{\epsilon}{(8\lambda^2 c_0^2)^{1/2}} \ln \left( \frac{z - z_+}{z - z_-} \right) + c_1,$$

(34)

and $z_\pm$ are given by Eq.(27), and $\epsilon = \text{sign}(\alpha)$.

**IIIb** \[ 1 - 4\alpha \beta < 0 \]

Now we have

$$\phi = -\ln \left[ \sinh^2 \left( -\frac{g}{2} \right) \right] + \phi_0,$$

$$\psi = \pm \frac{2(4\alpha \beta - 1)}{c_0 |2\omega + 3|} \coth \left( -\frac{g}{2} \right) + \psi_0,$$

(35)

where

$$g \equiv \pm \left[ \frac{2c_0^2(2\omega + 3)}{4\alpha \beta - 1} \right]^{1/2} \tan^{-1} \left[ \frac{2\alpha z - 1}{(4\alpha \beta - 1)^{1/2}} \right] + c_1.$$

(36)

For $r$ to be real, we need to impose the condition $\alpha > 0$ and $\beta > 0$ in this case.

Clearly, by giving different $f(\phi)$, we can obtain various solutions for the matter fields $\phi$ and $\psi$. However, these solutions are not all independent. For some particular choices, they may correspond to the same matter fields. This follows, of course, from the diffeomorphism invariance of the target space manifold. For example, the choice $f(\phi) = \sinh^2 2\phi$ gives the
same solution as that given in Case I. This is closely related to the fact that the matter fields described by the complex function \( F \) has a \( SL(2, \mathbb{R}) \) symmetry for the case \( \kappa > 0 \) [8]. But for \( \kappa < 0 \), no such symmetry exists. We would thus expect that the solution given in Case II is unique.

Since the spacetime geometry is independent of the particular matter model, the global structure of the spacetimes should be the same as that of the Roberts solution [5], provided that \( 1 - 4\alpha\beta > 0 \). Of course, the Ricci scalar and the mass will also be unchanged

\[
R = \frac{uv}{r^4} (1 - 4\alpha\beta) \quad M = -\frac{uv}{4r} (1 - 4\alpha\beta).
\] (37)

Again, one can remove the timelike singularities which are present and replace them with Minkowski space, matching at the advanced time \( v = 0 \). In this way, we imagine the scalar fields to be turned on at \( v = 0 \). Thus, the solution with \( \beta = 1 \) resemble critical collapse in the sense that for \( \alpha < 0 \) black holes are formed (supercritical regime), while for the case \( 0 < \alpha < 1/4 \), no black holes are formed and the fields disperse to infinity (subcritical regime) [5]. Note, however, that because of the matching, the solutions are \( C^0 \) in contrast to the smooth, attracting critical solutions which have been found in numerical calculations.

When \( 1 - 4\alpha\beta < 0 \), which is the case IIIb, there is no correspondence to the Roberts solution. From Eq.(37) we can see that to have the mass be non-negative, the solution has to be restricted to the regions \( u > 0, v > 0 \) and \( u < 0, v < 0 \), which will be referred, respectively, as regions I and II. In region II the solution can be considered as representing the gravitational collapse of two scalar fields and a point-like singularity is finally formed at \( u = 0 = v \). This singularity is direction-dependent. When the singularity is approached along different directions, the singular behavior will be different. In particular, if one approaches the singularity along the null hypersurface \( u = 0 \) or the one \( v = 0 \), everything seems regular. On the hypersurfaces \( u = 0, v < 0 \) and \( v = 0, u < 0 \), the two scalar fields carry null data. Using the arguments for the case \( 1 - 4\alpha\beta > 0 \) [5], we can replace the solution in the regions \( u > 0, v < 0 \) and \( u < 0, v > 0 \) by two flat regions. Once this is done, the spacetime needs to be further extended beyond the hypersurfaces \( v = 0, u > 0 \) and \( u = 0, v > 0 \), which are Cauchy horizons. The extension beyond them are not unique. For example, one can match the two flat regions to the solution given by Eq.(35) for \( u > 0, v > 0 \), or make an analytic extension and replace region I by another flat region. However, in either case the singularity at \( u = 0 = v \) is naked. Figure 1 gives a conformal diagram of the spacetime summarizing the above results.

Finally, we would like to note that the conclusion that the spacetime geometry is independent of the choice of the particular scalar field model is based on the fact that the spacetime realizes a fairly simple assumption of self-similarity. Otherwise, it is easy to show that this conclusion does not hold. Therefore, for the general perturbations (which do not have such a symmetry), it would be expected that they will depend on the choice of non-linear sigma model. In particular, there are no \textit{a priori} reasons to expect that the spectrum of the unstable modes of the perturbations of the solution \( \alpha = 0 \) and \( \beta = 1 \) for a general choice of \( G_{AB} \) is the same as that for the particular Roberts solution as was considered in [14]. This point deserves further investigation and will be considered in future work.
ACKNOWLEDGMENT

We would like to thank M.W. Choptuik and S.L. Liebling for valuable discussions. Part of this work was done while one of the authors (AW) was a visitor at the Center for Relativity, the University of Texas at Austin. He would like to express his gratitude to the Department for hospitality. The financial assistance from CNPq, FAPERJ, and UERJ (AW) is gratefully acknowledged. EWH was supported in part by the National Science Foundation under Grant Nos. PHY93-10083 and PHY93-18152.
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FIGURES

FIG. 1. The conformal diagram for the spacetime described by case IIIb. The singularity is at \((u, v) = (0, 0)\) and the Cauchy horizons are at \(u = 0, v > 0\) and \(v = 0, u > 0\). Region I can be replaced either (analytically) by flat space or (continuously) by the solution given by Eq.(35). For more details, see the text.