Nucleon generalized polarizabilities within a relativistic Constituent Quark Model

B. Pasquini\textsuperscript{a} and G. Salmè\textsuperscript{b}
\textsuperscript{a}Dipartimento di Fisica Nucleare e Teorica, Università di Pavia, and
Istituto Nazionale di Fisica Nucleare, Sezione di Pavia, Pavia, Italy
\textsuperscript{b}Istituto Nazionale di Fisica Nucleare, Sezione Sanità,
Viale Regina Elena 299, I-00161 Roma, Italy

Abstract

Nucleon generalized polarizabilities are investigated within a relativistic framework, defining such quantities through a Lorentz covariant multipole expansion of the amplitude for virtual Compton scattering. The key physical ingredients in the calculation of the nucleon polarizabilities are the Lorentz invariant reduced matrix elements of the electromagnetic transition current, which can be evaluated from off-energy-shell helicity amplitudes. The evolution of the proton paramagnetic polarizability, $\beta_{\text{para}}(q)$, as a function of the virtual-photon three-momentum transfer $q$, is explicitly evaluated within a relativistic constituent quark model by adopting transition form factors obtained in the light-front formalism. The discussion is focussed on the role played by the effects due to the relativistic approach and to the transition form factors, derived within different models.

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1 Introduction

One of the major goals of present day researches is to gain an understanding of the compositeness of the nucleon in terms of the fundamental quark-gluon dynamics. An outstanding role in unravelling the nucleon structure is played by the electromagnetic (em) interactions which provide clean information on the internal nucleon degrees of freedom. In particular, a great deal of attention has been recently devoted to virtual Compton scattering (VCS) off the nucleon, which is the natural complement to real Compton scattering and form factor measurements. VCS will be analyzed experimentally by means of the $ep \rightarrow e'p'\gamma$ reaction in the different kinematic domains accessible at Jefferson Lab [1] and at MIT-Bates [2], while initial data below the pion production threshold have been taken at MAMI laboratory [3].

The virtual nature of the initial photon opens the possibility of investigating the scattering amplitude independently as a function of the momentum and energy transfers, allowing to access a much greater variety of observables than in real Compton scattering. Depending on the photon energy, VCS can be employed to disentangle different aspects of the nucleon structure: first, it provides important insights in the perturbative QCD description of the nucleon wave function in the hard-scattering regime [4]. Second, diffractive VCS has recently attracted new attention in deep inelastic kinematics in order to get information about some parton distribution inaccessible in standard inclusive measurements [5, 6]. Other energy domains of particular physical interest are the region of low-lying resonances, where VCS becomes sensitive to the excitations of the nucleon, and the threshold regime, where one can measure new em observables which generalize the usual electric ($\alpha$) and magnetic ($\beta$) polarizabilities. In particular, at low final photon energy and arbitrary momentum transfer, it has been shown [7, 8] that the nucleon structure information of the VCS amplitude can be parametrized in terms of a small set of independent generalized polarizabilities. Many authors have theoretically investigated the generalized polarizabilities within the framework of different approaches, such as the non relativistic constituent quark (CQ) model [7], effective Lagrangian models [9] and field-theoretical models like the linear sigma model [10] and heavy baryon chiral perturbation theory [11].

Our aim is to study the generalized polarizabilities of the nucleon within a relativistic scheme based on a Lorentz covariant multipole expansion of the VCS amplitude, which corresponds to classify the initial and final photon interaction vertices in terms of Lorentz invariant reduced matrix elements of the em transition current. These quantities summarize all the dynamical information relevant to the VCS process and, in our model, are obtained using the em transition form factors calculated by adopting a relativistic CQ model, formulated within the light-front hamiltonian dynamics [12], and an off-energy-shell prescription for the baryon em current. In section 2, we describe the appropriate formalism to define the generalized polarizabilities, while in section 3 the results for the evolution of the proton paramagnetic polarizability, $\beta_{\text{para}}(q)$, as a function of the virtual-photon three-momentum transfer $q$, is presented. The sensitivity to both relativistic effects, introduced by the Lorentz covariance constraint, and transition form factors, obtained in a relativistic as well as non-relativistic framework, is discussed. The conclusions are drawn in the final section.
2 General formalism

The amplitude for the VCS process $\gamma^*N \rightarrow \gamma N$ is given by

$$T = \varepsilon^\mu_\nu(q') H^{\mu\nu} \varepsilon_\nu(q),$$  \hspace{1cm} (1)

where $H^{\mu\nu}$ is the Compton tensor which describes the interaction between a nucleon and an ingoing virtual photon with polarization $\varepsilon^\mu(q)$ and four-momentum $q^\mu \equiv (\omega, \vec{q})$, followed by the emission of a real final photon with polarization $\varepsilon'^\nu(q')$ and four-momentum $q'^\mu \equiv (\omega' = |\vec{q}'|, \vec{q}')$. To leading order in perturbation theory, $H^{\mu\nu}$ reads

$$H^{\mu\nu} = -i \int d^4ze^{iq'^z} \langle \vec{p}_f \sigma' | T(I^\mu(z), I^\nu(0)) | \vec{p}_i \sigma \rangle,$$  \hspace{1cm} (2)

where $I^\mu$ is the hadronic em current and $T(\ , \ )$ denotes the time-ordered product. In Eq. (2), $p^\mu_i(f) \equiv (E_i(f), \vec{p}_i(f))$ and $\sigma$ ($\sigma'$) are the four-momentum and the third component of the spin of the initial (final) nucleon, respectively, while the normalization of the states is $\langle \vec{p} \sigma | \vec{p}' \sigma' \rangle = (2\pi)^3 E/m \delta_{\sigma,\sigma'} \delta(\vec{p} - \vec{p}')$. According to the analysis performed in Ref. [7], the Compton tensor can be split into the sum of the nucleon Born term $H^{\mu\nu}_B$ and a residual contribution $H^{\mu\nu}_R$

$$H^{\mu\nu} = H^{\mu\nu}_B + H^{\mu\nu}_R,$$  \hspace{1cm} (3)

where $H^{\mu\nu}_B$ includes the gauge invariant contribution from the intermediate nucleon propagation and from the nucleon-antinucleon excitation and is entirely determined by the Dirac and Pauli form factors, while the residual part, $H^{\mu\nu}_R$, provides the new information on the nucleon structure. The latter one essentially contains two contributions, the first one from the nucleon excited states, and the second one from the contact term corresponding to the seagull diagram derived from the non-relativistic reduction of the em transition operator. In our model, the seagull term is not present, since we assume a relativistic scheme for describing the em interaction and therefore the residual part reduces to the contribution from the baryon resonant states only. It is customary (cf. [7]) to parametrize $H^{\mu\nu}_R$ in terms of generalized polarizabilities obtained by performing a multipole expansion of the initial and final photon. This procedure corresponds to classify the two-step photon-nucleon interaction in terms of the various angular momentum and parity transfers. In the low energy VCS limit, at finite $q \equiv |\vec{q}|$ and $\omega' \rightarrow 0$, the generalized polarizabilities are defined as

$$P(\rho' L', \rho L)^S(q) = \left[ \frac{1}{\omega' L' \omega L} H^{\mu\nu}_R(\rho' L', \rho L)^S(q, \omega') \right]_{\omega' = 0},$$  \hspace{1cm} (4)

where $\rho(\rho')$ indicates the type of the multipole transition ($\rho = 0$ : Coulomb; $\rho = 1$ : magnetic; $\rho = 2$ : electric; $\rho = 3$ longitudinal) and $L(L')$ is the angular momentum of the ingoing (outgoing) photon; $S$ refers to the spin $(S = 1)$ and non spin-flip $(S = 0)$ character.
of the transition. In Eq. (4), $H^{(\rho' L', \rho L)S}_R$ are the reduced multipoles introduced in Ref. [7] and related to $H^{\mu\nu}_R$ as follows

$$H^{(\rho' L', \rho L)S}_R = \sum_{\sigma, \sigma'} \sum_{M, M'} (-1)^{L + M + 1/2 + \sigma'} (\frac{1}{2} - \sigma') (\frac{1}{2} \sigma | S, s)$$

$$\times \langle L' M', L - M | S, s \rangle \frac{1}{4\pi} \int \hat{q} \, d\hat{q}' V^*_\mu(\rho' L' M', \hat{q}') H^{\mu\nu}_R V_\nu(\rho LM, \hat{q}), \quad (5)$$

where $V^\nu(\rho LM, \hat{q})$ are the four-dimensional basis defined in terms of the spherical harmonics, $Y_{LM}$, and the vector spherical harmonics, $\tilde{Y}_{LM}$, by

$$V^\nu(0LM, \hat{q}) \equiv \langle Y_{LM}(\hat{q}), \tilde{0} \rangle,$$

$$V^\nu(\rho LM, \hat{q}) \equiv \langle 0, \sum_\ell C^\rho_{\ell L} \tilde{Y}_{LM}(\hat{q}) \rangle, \quad (\rho = 1, 2, 3), \quad (6)$$

with the coefficients $C^\rho_{\ell L}$ given by

$$C^1_{\ell L} = \delta_{\ell, L},$$

$$C^2_{\ell L} = \sqrt{\frac{L + 1}{2L + 1}} \delta_{\ell, L - 1} + \sqrt{\frac{L}{2L + 1}} \delta_{\ell, L + 1},$$

$$C^3_{\ell L} = \sqrt{\frac{L}{2L + 1}} \delta_{\ell, L - 1} - \sqrt{\frac{L + 1}{2L + 1}} \delta_{\ell, L + 1}. \quad (7)$$

In Eq. (5), by gauge invariance of the Compton tensor, the longitudinal ($\rho = 3$) multipole transition may be expressed by the Coulomb ($\rho = 0$) one, while in the Siegert limit, the electric ($\rho = 2$) and the Coulomb multipoles are related by current conservation. In particular, for the virtual photon ($\rho$) the difference between the electric and Coulomb contribution is given in terms of mixed polarizabilities, $\hat{P}^{(\rho' L', L)S}$, whereas for the real photon ($\rho'$) this term can be neglected to leading order in $\omega'$ [7]. As a result, by retaining in $H^{(\rho' L', \rho L)S}_R$ only the linear terms in $\omega'$ and imposing the selection rules of the angular momentum and parity conservation, one finds only 10 independent generalized polarizabilities [7], which are further reduced to 6 by charge conjugation symmetry [8].

In order to achieve a fully consistent relativistic calculation of the nucleon excitation contribution to the generalized polarizabilities, it is necessary to develop a proper Lorentz covariant treatment of the multipoles $H^{(\rho' L', \rho L)S}_R$. The sum of the contributions of the nucleon excited states in the direct ($s$) and crossed ($u$) channel leads to the following definition of the tensor $H^{\mu\nu}_R$

$$H^{\mu\nu}_R = \sum_{X \neq N, N} \left( \frac{m_X}{E_X} \langle N_{\frac{1}{2}}^2 \sigma' | \tilde{p}_f | I^\nu(0) | X J_X \mu_X; \tilde{p}_X \rangle \langle X J_X \mu_X; \tilde{p}_X | I^\mu(0) | N_{\frac{1}{2}}^2 \sigma; \tilde{p}_i \rangle \right) \left( \frac{E_f - E_X + \omega'}{E_f - E_X + \omega'} + \frac{m_X}{E_X} \langle N_{\frac{1}{2}}^2 \sigma'; \tilde{p}_f | I^\nu(0) | X J_X \mu_X; \tilde{p}_X' \rangle \langle X J_X \mu_X; \tilde{p}_X' | I^\mu(0) | N_{\frac{1}{2}}^2 \sigma; \tilde{p}_i \rangle \right), \quad (8)$$

4
where the sum covers the whole excitation spectrum of the nucleon, without the nucleon (N) and antinucleon (N) contribution taken into account in $H_B^{\mu \nu}$, and the on-mass-shell four-momenta of the intermediate states in the direct and crossed channel are denoted as $p_X^\rho \equiv (E_X = \sqrt{p_X^2 + m_X^2}, \vec{p}_X = \vec{p}_X^0 + \vec{q}')$ and $p_X^\mu \equiv (E_X = \sqrt{p_X^2 + m_X^2}, \vec{p}_X = \vec{p}_X^i - \vec{q}'$), respectively.

The key physical ingredients in $H_B^{\mu \nu}$ are the matrix elements of the baryonic em current $I^\mu(0)$ between the initial or final nucleon and each intermediate resonance state. The multipole decomposition defined in Eq. (5) allows us, in each matrix element, to separate purely geometrical factors, as described by the various angular momentum and parity transfers, from dynamical aspects concerning the baryon structure. Such a separation can be achieved by developing a formalism consistent with the constraint of relativistic covariance [13]. In order to accomplish this, first of all, we perform the covariant multipole decomposition of the matrix elements describing the nucleon-resonance transition in the direct channel, then we generalize the results to the final interaction vertex and to the crossed term by using the complex conjugation and the crossing symmetry properties, respectively.

The expansion of the $N - X$ transition matrix element onto the multipole basis of Eq. (6) reads

$$V_\mu(\rho LM; \vec{q}) \langle X J_X \mu_X; \vec{p}_X|I^\mu(0)|N \frac{1}{2} \sigma; \vec{p}_i \rangle =$$

$$\sum_{\ell s} \sum_{m m_s} B_{\ell L}^{\rho s} \langle \ell m, s m_s|L M \rangle Y_\ell^s(\vec{q}) \langle X J_X \mu_X; \vec{p}_X|I^s_{m_s}(0)|N \frac{1}{2} \sigma; \vec{p}_i \rangle,$$

where the coefficients $B_{\ell L}^{\rho s}$ are defined by

$$B_{\ell L}^{\rho s} = \delta_{s,0}, \quad B_{\ell L}^{\rho s} = -\delta_{s,1} C_{\ell L}^\rho, \quad (\rho = 1, 2, 3),$$

and $I^s_{m_s}$ are the spherical components of the current, $I^s_{m_s} = (-1)^s I \cdot \xi_{m_s}^s$, with the four-vector basis given by

$$\xi_0^s = (1, 0, 0, 0), \quad \xi_{\pm 1}^1 = \mp \frac{1}{\sqrt{2}}(0, 1, \pm i, 0), \quad \xi_0^1 = (0, 0, 0, 1).$$

By Lorentz covariance of the current operator and transformation properties of the baryon states, the current matrix elements evaluated in a generic frame may be related to the corresponding ones calculated in the rest frame of the excited resonance by the well known formula (see e.g. [14])

$$\langle X J_X \mu_X; \vec{p}_X|I^s_{m_s}(0)|N \frac{1}{2} \sigma; \vec{p}_i \rangle = \sqrt{E_{0i} \over E_i} \sum \Lambda_{m_{m_s}, m_s}^{s \tilde{s}} \left( \vec{p}_X \over m_X \right)$$

$$\times D_{\sigma' \sigma}^{1/2}(R_{\ell L}(\vec{p}_X, \vec{p}_i)) \langle X J_X \mu_X; \vec{p}_X|I^s_{m_s}(0)|N \frac{1}{2} \sigma'; \vec{p}_{0i} \rangle,$$

where, here and in the following, the sum is understood over repeated indices. In Eq. (12), $p_{0i}^\mu = (E_{0i}, \vec{p}_{0i})$ is the nucleon four-momentum in the resonance rest frame. The spherical
components of the rotationless boost operator, \( \Lambda_{m_s \bar{m}_s} (\bar{p}_X/m_X) \), can be explicitly expressed through \( SL(2, \mathbb{C}) \) operators in the form
\[
\Lambda_{m_s \bar{m}_s} (\frac{\bar{p}_X}{m_X}) = (-1)^s (\xi_{m_s})^* \Lambda^\nu_{\mu} (\xi_{\bar{m}_s})^\mu = (-1)^{m_s+\frac{1}{2}} \text{Tr} \left\{ \sigma_{m_s}^s L \left( \frac{\bar{p}_X}{m_X} \right) \sigma_{\bar{m}_s}^s \bar{L} \left( \frac{\bar{p}_X}{m_X} \right) \right\} ,
\]
where \( L(\bar{p}_X/m_X) = (E_X + m_X + \bar{p}_X \cdot \vec{\sigma})/\sqrt{2m_X(E_X + m_X)} \) and \( \sigma_{m_s}^s \) are the spherical components of the four-vector defined in terms of the Pauli matrices as \( \sigma_\mu \equiv (1, \vec{\sigma}) \). The rotationless boost operator maps into the rest frame of the baryon state, viz.
\[
(m_X, \vec{0}) = \hat{\Lambda}^{-1} \left( \frac{\bar{p}_X}{m_X} \right) p_X, \quad p_{0i} = \hat{\Lambda}^{-1} \left( \frac{\bar{p}_X}{m_X} \right) p_i,
\]
while \( R_L \) is the corresponding Wigner rotation given by
\[
R_L (\bar{p}_X, \vec{p}_i) = L^{-1} \left( \frac{\vec{p}_{0i}}{m_N} \right) L^{-1} \left( \frac{\bar{p}_X}{m_X} \right) L \left( \frac{\vec{p}_i}{m_N} \right) = \frac{(E_X + m_X)(E_i + m_N) - \bar{p}_X \cdot \vec{p}_i - i\bar{\sigma} \cdot (\bar{p}_X \times \vec{p}_i)}{\sqrt{2(E_X + m_X)(E_i + m_N)(E_X E_i - \bar{p}_X \cdot \vec{p}_i + m_N m_i)}} .
\]
As shown in Refs. [14, 15], the current matrix elements in the resonance rest frame can be parametrized in terms of Lorentz invariant reduced matrix elements according to
\[
\langle X J_X \mu_X; \vec{0} | I^*_m \rangle (0) | N_{\frac{1}{2}}^1 \sigma; \vec{p}_{0i} \rangle = \frac{(-1)^s}{\sqrt{2}} \sum \langle \ell m s m_\ell | J M \rangle \times \langle \frac{1}{2} \sigma J M | J_X \mu_X \rangle Y^*_{\ell m}(\vec{p}_{0i}) \langle X J_X | I_{\ell s J}(Q^2) | N_{\frac{1}{2}}^1 \rangle ,
\]
where the reduced matrix elements \( \langle X J_X | I_{\ell s J}(Q^2) | N_{\frac{1}{2}}^1 \rangle \) square four-momentum transfer, \( Q^2 = -q \cdot q \). Furthermore they are constrained by the selection rules dictated by parity and angular momentum conservation, viz.
\[
(-1)^{\ell + s} = \Pi \Pi_X, \quad |J_X - \frac{1}{2}| \leq J \leq J_X + \frac{1}{2},
\]
with \( \Pi \) and \( \Pi_X \) denoting the intrinsic parity of the nucleon and baryon resonance, respectively. Finally, by using Eqs. (12) and (16) we obtain the Lorentz covariant multipole decomposition for the matrix element in a generic frame, at the same \( Q^2 \).

The corresponding results for the matrix elements describing the \( X - N \) transition can be easily derived by means of the relation
\[
\langle N_{\frac{1}{2}}^1 \sigma; \vec{p}_i | I^*_m \rangle X J_X \mu_X; \bar{p}_X \rangle = (-1)^{m_\ell} \langle X J_X \mu_X; \bar{p}_X | I^*_m | N_{\frac{1}{2}}^1 \sigma; \vec{p}_i \rangle^*,
\]
which, in terms of the reduced matrix elements, reads
\[
\langle N_{\frac{1}{2}}^1 | I_{\ell s J}(Q^2) | X J_X \rangle = (-1)^{J_X - \frac{1}{2}} \frac{J_X}{\sqrt{2}} \langle X J_X | I_{\ell s J}(Q^2) | N_{\frac{1}{2}}^1 \rangle ,
\]

\[
6
\]
where $\hat{J}_X = \sqrt{2J_X + 1}$. Then, by collecting the results of Eqs. (5), (8), (9), (12) and (16), and after a straightforward angular momentum algebra, the contribution of the direct term (s) to the reduced multipoles takes the form

$$
H_{s}^{(\rho^s L^s, \rho L^s)} = \frac{1}{32\pi \sqrt{2S}} \sum_X \sum (-1)^{\tilde{s} + s' + 1 + L + \ell + S + c + K + J'} b \tilde{b} \dd' \dd \tilde{\dd'} a \tilde{a} a' \bar{L} \bar{L}' \bar{J} \bar{J}' \hat{c} \hat{J}_X
$$

$$
\times B_{L_L}^{\rho s} B_{L'_{\ell'}}^{\rho' s'} \left\{ \begin{array}{ccc}
\tilde{s} & s & b \\
L & \ell & d \\
J & J' & J_X
\end{array} \right\} \left\{ \begin{array}{ccc}
\tilde{s}' & s' & \tilde{b}' \\
L' & \ell' & d' \\
J' & L' & a'
\end{array} \right\} \left\{ \begin{array}{ccc}
J & L & a \\
J' & L' & a' \\
c & S & K
\end{array} \right\}
$$

$$
\times \int d\dd \int d\dd' \frac{m_X}{E_X} \sqrt{\frac{E_{\dd'}}{E_{\dd}}} \left( -1 \right)^k O^K_k (S, c; \bar{p}_X, \bar{p}_L, \bar{p}_f) T^K_k (\nu, \nu'; \tilde{q}, \tilde{p}_o; \tilde{q}', \tilde{p}_o; \tilde{p}_X) \frac{N \frac{1}{2}}{E_f - E_X + \omega'}
$$

$$
\times \langle N \frac{1}{2} \mid I_{[L, s, J]}(Q^2) \mid X J_X \rangle \langle X J_X \mid I_{[L', s', J']}(0) \mid N \frac{1}{2} \rangle
$$

(20)

where the tensor $T^K_k$ is obtained by coupling the spherical representation of the Lorentz boost and the spherical harmonics derived by the multipole expansion of the em field at the first and second interaction vertex and depends on the indices $\nu \equiv (a, b, s, \bar{s}, d, L, \ell)$ and $\nu' \equiv (a', b', s', \bar{s}', d', L', \ell')$. The tensor $O^K_k$ is related to the Wigner rotations $R_L(\bar{p}_X, \bar{p}_L)$ and $R_L^T(\bar{p}_L, \bar{p}_f)$ acting on the initial and final nucleon state, respectively. The explicit expressions for the tensors $T^K_k$ and $O^K_k$ can be found in Appendix. The analogous expression for the contribution of the $u$ channel follows from Eq. (20) by using the crossing symmetry property, corresponding to the $q^\mu \leftrightarrow -q'^\mu$, $V_\nu(\rho LM, \dd) \leftrightarrow V_\nu^*(\rho' L'M', \dd')$, and reads explicitly

$$
H_{u}^{(\rho^u L^u, \rho L^u)} = \frac{1}{32\pi \sqrt{2S}} \sum_X \sum (-1)^{\tilde{s} + s' + 1 + L + \ell + S + c + K + J'} b \tilde{b} \dd' \dd \tilde{\dd'} a \tilde{a} a' \bar{L} \bar{L}' \bar{J} \bar{J}' \hat{c} \hat{J}_X
$$

$$
\times B_{L_L}^{\rho s} B_{L'_{\ell'}}^{\rho' s'} \left\{ \begin{array}{ccc}
\tilde{s} & s & b \\
L & \ell & d \\
J & J' & J_X
\end{array} \right\} \left\{ \begin{array}{ccc}
\tilde{s}' & s' & \tilde{b}' \\
L' & \ell' & d' \\
J' & L' & a'
\end{array} \right\} \left\{ \begin{array}{ccc}
J & L & a \\
J' & L' & a' \\
c & S & K
\end{array} \right\}
$$

$$
\times \int d\dd \int d\dd' \frac{m_X}{E_X} \sqrt{\frac{E_{\dd'}}{E_{\dd}}} \left( -1 \right)^k O^K_k (S, c; \bar{p}_X, \bar{p}_L, \bar{p}_f) T^K_k (\nu, \nu'; \tilde{q}, \tilde{p}_o; \tilde{q}', \tilde{p}_o; \tilde{p}_X) \frac{N \frac{1}{2}}{E_i - E_X - \omega'}
$$

$$
\times \langle N \frac{1}{2} \mid I_{[L, s, J]}(Q^2) \mid X J_X \rangle \langle X J_X \mid I_{[L', s', J']}(0) \mid N \frac{1}{2} \rangle
$$

(21)

Eqs. (20) and (21) exhibit the required Lorentz covariance, obtained by the decomposition of the em current operator into spherical tensors which transform irreducibly under the Lorentz group. The main result is that the geometrical and kinematical aspects can be completely factored into the 6-j and 9-j symbols and the Lorentz transformation coefficients given by the tensors $T^K_k$ and $O^K_k$, while the dynamical content is summarized into the
Lorentz invariant reduced matrix elements of the em current. The latter ones contain all the relevant information on the baryon structure and represent the quantities to be calculated by models. First of all, it is essential to note that in the VCS process, below the pion threshold, only off-mass-shell intermediate baryons are involved; while in the actual calculation an off-energy-shell prescription for the baryon current has been introduced. To this end, we followed the same spirit of the approach proposed by De Forest [16] for the electron-nucleus scattering, namely the resonance four-momentum is on its mass-shell and only the three-momentum is conserved, with \( \vec{p}_X = \vec{p}_f + \vec{q}' \) in the direct channel and \( \vec{p}'_X = \vec{p}_i - \vec{q}' \) in the crossed one. Within such an approach, we have evaluated off-energy-shell helicity amplitudes defined by

\[
S_{1/2}^X = \langle X J_{1/2}; 0| I_0^0| N 1/2; -|p_{01}| \hat{z} \rangle,
\]

\[
L_{1/2}^X = \langle X J_{1/2}; 0| I_0^0| N 1/2; -|p_{01}| \hat{z} \rangle,
\]

\[
A_{1/2}^X = -\langle X J_{1/2}; 0| I_1^1| N 1/2 - 1/2; -|p_{01}| \hat{z} \rangle,
\]

\[
A_{3/2}^X = -\langle X J_{3/2}; 0| I_1^1| N 1/2; -|p_{01}| \hat{z} \rangle.
\]

(22)

By using Eqs. (16) and (22), the reduced matrix elements have been expressed in terms of the off-energy-shell helicity amplitudes as follows

\[
\langle X J_X \parallel I_{(s,t)}(Q^2)\parallel N_{1/2} \rangle = \Pi_X 2^\frac{8 \pi (2\ell + 1)}{(2J + 1)}
\times \left[ \langle \ell 0 s 0| J 0 \rangle \langle \frac{1}{2} \frac{1}{2} J 0| J_X \frac{1}{2} \rangle \left( \delta_{s,0} S_{1/2}^X + \delta_{s,1} L_{1/2}^X \right) \right.

\]

\[
\left. - \langle \ell 0 s 1| J 1 \rangle \langle \frac{1}{2} \frac{1}{2} J 1| J_X \frac{3}{2} \rangle A_{3/2}^X + \langle \frac{1}{2} - \frac{1}{2} J 1| J_X \frac{1}{2} \rangle A_{1/2}^X \right],
\]

(23)

where \( L_{1/2}^X \) can be eliminated in favour of \( S_{1/2}^X \) by current conservation.

### 3 The paramagnetic polarizability

As a first application of the formalism developed in the previous section, we analyze the role of both relativistic and transition form factor effects in the calculation of the dipole magnetic polarizability of the nucleon. In the center of mass frame of the final nucleon-photon system and in the limit of \( \omega' \rightarrow 0 \), the polarizabilities depend only on the three-momentum of the virtual photon. The \( q \) evolution of diagonal magnetic contribution represents a direct generalization of the \( \beta \) polarizability for the virtual photon case, viz.

\[
\beta(q) = -\alpha_{QED} \sqrt{\frac{3}{8}} P^{(11,11)0}(q),
\]

(24)
where $\alpha_{\text{QED}} \simeq 1/137$ and the real Compton scattering limit is recovered for $q \to 0$.

As is well known, the magnetic polarizability consists of a positive paramagnetic contribution which is largely cancelled by a negative diamagnetic term due to the polarization of the pion cloud surrounding the nucleon. Since we are interested in evaluating only the nucleon resonance excitation contribution, we limit ourselves to discuss the paramagnetic term $\beta_{\text{para}}(q)$. Moreover, in our exploratory analysis we consider only the overwhelming contribution from the $\Delta(1232)$ to the dipole magnetic excitation of the nucleon \(^1\).

The reduced matrix elements of the $N - \Delta(1232)$ transition current have been explicitly calculated in the low energy VCS limit by using the off-energy-shell prescription described in Sect. 2 and the following relativistic current operator

$$I_{\Delta}^{\mu\nu} = \sqrt{\frac{2}{3}} \left[ G_1^\Delta(Q^2) K_{1}^{\mu\nu} + G_2^\Delta(Q^2) K_{2}^{\mu\nu} + G_3^\Delta(Q^2) K_{3}^{\mu\nu} \right],$$  \hspace{1cm} (25)

where $G_i^\Delta(Q^2)$ and $K_{\mu\nu}^{\mu\nu}$ are the kinematic-singularity free form factors and the tensors as defined in Ref. [17], respectively. In particular, for the $\Delta(1232)$ excitation the helicity amplitudes are

$$A_{1/2}^\Delta = \frac{1}{\mathcal{E}} \sqrt{\frac{2}{3}} q \left[ G_1^\Delta(Q^2)(\omega - E_N - m_N) + G_2^\Delta(Q^2) \omega m_\Delta - G_3^\Delta(Q^2) Q^2 \right],$$

$$A_{3/2}^\Delta = \frac{1}{\mathcal{E}} \sqrt{\frac{2}{3}} q \left[ -G_1^\Delta(Q^2)(\omega + E_N + m_N) - G_2^\Delta(Q^2) \omega m_\Delta + G_3^\Delta(Q^2) Q^2 \right],$$

$$S_{1/2}^\Delta = \frac{1}{\mathcal{E}} \sqrt{\frac{2}{3}} q^2 \left[ G_1^\Delta(Q^2) + G_2^\Delta(Q^2) m_\Delta + G_3^\Delta(Q^2) \omega \right],$$  \hspace{1cm} (26)

where $\mathcal{E} = \sqrt{2m_N(E_N + m_N)}$ with $E_N = \sqrt{q^2 + m_N^2}$. Note that within this off-shell prescription the constraint imposed by current conservation in the Siegert limit of the helicity amplitude is satisfied, viz.

$$\lim_{q \to 0} \left( \sqrt{3} A_{1/2}^\Delta - A_{3/2}^\Delta \right) = \lim_{q \to 0} \left( \frac{\omega}{q} \sqrt{6} S_{1/2}^\Delta \right),$$  \hspace{1cm} (27)

and the corresponding relation for the reduced matrix elements (see Eq. (23)) in the limit of $q \to 0$ reads

$$\lim_{q \to 0} \left[ \sqrt{\frac{3}{5}} \langle \Delta^{3/2}_{1/2} \parallel I_{[1,1,2]}(0) \parallel N^{1/2} \rangle + \sqrt{\frac{2}{5}} \langle \Delta^{3/2}_{3/2} \parallel I_{[3,1,2]}(0) \parallel N^{1/2} \rangle \right] =$$

$$\lim_{q \to 0} \left[ \frac{\omega}{q} \sqrt{\frac{3}{2}} \langle \Delta^{3/2}_{1/2} \parallel I_{[2,0,2]}(0) \parallel N^{1/2} \rangle \right].$$  \hspace{1cm} (28)

\(^1\)Within such an approximation the proton and neutron magnetic polarizabilities are equal since the transition form factors involved in the calculations are the same.
In order to perform a consistent relativistic calculation, within a Lorentz covariant formalism as well as a relativistic approach for describing the internal nucleon dynamics involved in the scattering process, we have evaluated the magnetic polarizability using the $G_i^\Delta$ form factors obtained in Ref. [12] within the light-front hamiltonian dynamics. In particular such form factors have been calculated i) by adopting baryon eigenstates of the relativized mass operator proposed in Ref. [18], which quite well reproduces the baryon mass spectroscopy, and ii) by using a relativistic CQ one-body em current which contains Dirac and Pauli CQ form factors as well.

The reduced multipoles have been calculated in the center of mass frame of the final photon-nucleon system and as a consequence the boost transformations in Eqs. (20) and (21) only affect the crossed channel term, where the $\Delta(1232)$ is excited with momentum $\vec{p}_\Delta = -\vec{q} - \vec{q}'$, while they reduce to the identity in the direct term, where the $\Delta(1232)$ is in its rest frame. The explicit expression of the $s$ channel term in the resonance rest frame becomes

$$H_s^{(\ell' L', \rho L) S} = \frac{1}{4 \pi \sqrt{2}} \sum_{X} \sum (-1)^{s + s' + 1 + L' + S} \hat{J}_{X} B_{\ell L}^{\rho S} B_{\ell' L'}^{s' s'} \left\{ \begin{array}{ccc} \frac{1}{2} & \frac{1}{2} & S \\ L & L' & J_{X} \end{array} \right\}$$

$$\times \frac{\langle N \frac{1}{2} || I_{[1,1,l]}(0)|| \sqrt{\omega'^2 + m_N^2 - m_X + \omega'} \rangle}{\langle \Delta \frac{1}{2} || I_{[1,1,1]}(Q^2)|| N \frac{1}{2} \rangle}$$

while the corresponding contribution to the dipole magnetic polarizability reads

$$P_s^{(11,11)0} = \frac{-1}{4 \pi \sqrt{3}} \left( \frac{1}{\omega'^q} \left\{ \frac{1}{2} \langle \Delta \frac{3}{2} || I_{[1,1,1]}(Q^2)|| N \frac{1}{2} \rangle \right\}_{\omega'=0} \right)$$

Eqs. (29) and (30) recover the results of Ref. [7] obtained in a non-relativistic framework, while in the $u$-channel contribution the covariant formalism introduces a mixing between electric, Coulomb and magnetic multipole transitions, as a typical relativistic effect due to the boost transformations from a frame to another one.

In Fig. 1, the curves corresponding to i) our calculation, obtained by using the relativistic formalism presented in Sect. 2 and the $N - \Delta(1232)$ form factors of Ref. [12]; ii) the non-relativistic CQ model of Ref. [7]; and iii) the effective Lagrangian model of Ref. [9] are compared. The data point at the real-photon limit has been obtained by taking the transition form factors extracted from the experimental helicity amplitudes of PDG [19] as input in our formalism. On the one hand, it should be pointed out that such a data point, equal to $(13.1 \pm 1.6) \times 10^{-4}$ fm$^3$, is in fair agreement with the value obtained in Ref. [20] by using the experimental data on pion photoproduction. On the other hand, following the analysis of Refs. [21, 22], the disagreement with the theoretical calculations could be ascribed to the lacking of a fully consistent treatment of the final state interaction of the $N\pi$ system in the extraction of the experimental helicity amplitudes. The main difference between the model calculations of $\beta_{\text{para}}(q)$ stems from the different $q$ dependence of the transition form
factors combined with the relativistic effects due to the Lorentz transformations. The role of these different ingredients is disentangled in Figs. 2 and 3, where the results of the non-relativistic CQ model of Ref. [7] have been considered as a reference, since such a model is more close to our approach in terms of CQ. In Fig. 2, the comparison with our results, obtained as in Fig. 1, for the separate contributions from the direct and crossed channels is shown. In the direct term, since the Lorentz transformations are equal to the identity, the different behaviour as a function of $q$ is mainly given by the $q$ dependence of the transition form factors. In particular, the effects due to the relativistic transition form factors are more relevant at the real-photon point and for $q > 0.9$ GeV/c, where it is found a slower fall-off than the gaussian one predicted by the non-relativistic CQ model. The relativistic effects are more significant in the contribution of the crossed term, where the action of the Lorentz transformations affects the results mainly at increasing values of the momentum transfer.

A more detailed study of the effects of the Lorentz transformations can be done with the aid of the results shown in Fig. 3, where the predictions of the fully non-relativistic CQ model of Ref. [7] are compared with the calculations obtained with our relativistic formalism, Eqs. (20) and (21), but the same non-relativistic form factors as in the model of Ref. [7]. As expected, the two calculations of the direct term contribution overlap, whereas the discrepancy in the crossed channel contribution gives a direct evidence of the importance of the Lorentz transformations, whose effects grow for increasing values of $q$.

4 Summary

We have investigated the nucleon generalized polarizabilities within a relativistic framework. In particular, we have formally derived Lorentz covariant expressions for the reduced multipoles of the residual part of the Compton tensor (see Eqs. (20) and (21)). These multipoles determine the generalized polarizabilities in the limit of vanishing real-photon energy. As a first application, we have calculated the $\Delta(1232)$ contribution to the nucleon paramagnetic polarizability, analyzing in details the role of the relativistic formalism and the em transition form factors, obtained within different models. On the one hand, by working in the center of mass of the final photon-nucleon system, the Lorentz covariance produces effects only in the $u$ channel contribution, by introducing model-independent kinematical corrections which become important at values of the three-momentum transfer $q \geq 0.5$ GeV/c. On the other hand, the $q$ evolution of the nucleon paramagnetic polarizability is sizably different if one adopts the transition form factors obtained within a relativistic (light-front) CQ model or within a non-relativistic CQ model, even at low values of the momentum transfer.

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6 Appendix

In this Appendix the explicit form of the tensors $O^K_k$ and $T^K_k$ (see Eqs. (20) and (21)) is given.

In particular, the tensor $O^K_k(S,c;\vec{p}_X,\vec{p}_i,\vec{p}_f)$ is defined in terms of the Wigner rotations $R = R_L(\vec{p}_X,\vec{p}_i)$ and $R^\dagger = R^\dagger_L(\vec{p}_X,\vec{p}_f)$ (cf. Eq. (15)) according to

$$O^K_k(S,c;\vec{p}_X,\vec{p}_i,\vec{p}_f) = \sum_{\zeta,\gamma} \langle S\zeta,c\gamma|Kk \rangle \text{Tr} \left\{ R^{\sigma}_S R^{\dagger}_L \right\},$$

(31)

where the indices $S$ and $c$ can assume only two values, $(0,1)$, and $\sigma_S^{\zeta(\gamma)}$ are the spherical components of the operator $\sigma_\mu = (I,\vec{\sigma})$.

The tensor $T^K_k$ is defined as

$$T^K_k(\nu,\nu';\hat{q},\hat{p}_o;\vec{p}_X,\vec{p}_i) = \sum_{a,a'} \langle a\alpha,a'\alpha'|Kk \rangle \Phi^a_{\alpha}(\vec{p}_X/m_X) \Phi^{a'}_{\alpha'}(\vec{p}_X/m_X),$$

(32)

where $\nu = (a,\varepsilon)$ and $\nu' = (a',\varepsilon')$, while $\varepsilon$ and $\varepsilon'$ stand for the set of indices $(b,s,\bar{s};d,\ell,L)$ and $(b',s',\bar{s}';d',\ell',L')$, respectively. In Eq. (32), the tensor $\Phi^a_{\alpha}$ is explicitly given by

$$\Phi^a_{\alpha}(\vec{p}_X/m_X) = \sum_{\delta,\beta} \langle d\delta,b\beta|a\alpha \rangle \{ Y_\ell(\hat{q}) \otimes Y_L(\hat{p}_o) \}_{d,\delta} \Lambda^b_{\alpha}(s,\bar{s};\vec{p}_X/m_X),$$

(33)

where the bipolar spherical harmonics are defined by

$$\{ Y_\ell(\hat{q}) \otimes Y_L(\hat{p}) \}_{d,\delta} = \sum_{m,\lambda} \langle \ell\delta,m\lambda|d\delta \rangle Y^\ell_m(\hat{q}) Y^L_m(\hat{p}),$$

(34)

and $\Lambda^b_{\alpha}(s,\bar{s};\vec{p}_X/m_X)$ is given in terms of the spherical components of the Lorentz boost (see Eq. (13))

$$\Lambda^b_{\alpha}(s,\bar{s};\vec{p}_X/m_X) = \sum_{m,s} (-1)^\bar{m}_s \langle s m_s,\bar{s} - \bar{m}_s|b\beta \rangle \Lambda^{s\bar{s}}_{m_s\bar{m}_s}(\vec{p}_X/m_X).$$

(35)

References


Figure 1: The ∆(1232) contribution to the nucleon paramagnetic polarizability vs the three-momentum transfer \( q \). Solid line: results obtained by using our relativistic formalism (cf. Eqs. (20) and (21)) and the transition form factors evaluated within a relativistic CQ model [12]; dotted line: the non-relativistic CQ model of Ref. [7]; dot-dashed line: the effective Lagrangian model of Ref. [9]. The data point at the real-photon point (slightly displaced in order to show the error bar) is the result obtained from the experimental helicity amplitudes of PDG [19] (see text).

Figure 2: The direct and crossed term of the ∆(1232) contribution to the nucleon paramagnetic polarizability vs the three-momentum transfer \( q \). The solid and dashed lines are the direct and crossed term, respectively. Thick lines correspond to the predictions obtained by using our relativistic formalism (cf. Eqs. (20) and (21)) and the transition form factors evaluated within a relativistic CQ model [12], while thin lines are the results obtained within the non-relativistic CQ model of Ref. [7].

Figure 3: The same as Fig. 2, but using in the covariant formalism the transition form factors of the non-relativistic CQ model of Ref. [7]. The results corresponding to the direct channel contribution calculated by our relativistic formalism and by the fully non-relativistic approach of Ref. [7] overlap.