POLYMERS ENTANGLEMENT AND CHERN-SIMONS FIELD THEORY

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Abstract

In recent times some interesting field theoretical descriptions of the statistical mechanics of entangling polymers have been proposed by various authors. In these approaches, the dynamics of a single test polymer fluctuating in a background of static polymers is considered using mean-field type techniques. The extension to the case in which two or more polymers become dynamical is not straightforward unless severe constraints are imposed on the spatial configurations of the polymers. In this paper we present another approach, based on Chern–Simons field theory, which is able to describe the entanglement of two dynamical polymers in terms of gauge fields and second quantized replica fields.

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1 Introduction

In recent times, some interesting field theoretical descriptions of the statistical mechanics of closed polymers subjected to entanglement have been proposed by various authors [1]–[6]. In all these approaches, which are based on the pioneering works [7],[8], the inclusion of higher order link invariants [9],[10], [11], is yet an unsolved problem. On the other side, these invariants are necessary in order to specify in an unique way the distinct topological states of the polymer chain. The main difficulty in the utilization of higher order link invariants is that they cannot easily be expressed in terms of the variables which characterize the polymer, i.e. its trajectory in the three dimensional ($3-D$) space and its contour length. For this reason, most of the nonnumerical works use the simplest topological invariant, namely the Gauss linking number [2]. Even in this case, a rigorous treatment of the $3-D$ entanglement problem is mathematically difficult and basically only the dynamics of a single polymer fluctuating in a background of static polymers has been discussed until now. The configurations of the background polymers may eventually be “averaged” exploiting mean-field type arguments (see e.g. [1]).

As it has been recently suggested [2],[6], a possible way out from the above difficulties is the introduction of Chern-Simons (C–S) field theory [12] in the treatment of polymer entanglement. In doing this, however, one is faced with the problem of computing expectation values of holonomies around the closed trajectories formed by the polymers in space. As it is is well known, these expectation values are affected in C–S field theories by the presence of unregulated contributions coming from the self-linking of the loops. If the loops have no dynamics, a good regularization is provided by the so-called framing procedure of the curves [10]. We notice however that the framing depends on the form of the loops, which in our case is a dynamical variable, since one has to consider all the possible trajectories of the polymers in $3-D$ space. Therefore, the addition of a framing would complicate the form of the polymer action and eventually spoil the possibility of exploiting the techniques of second quantization. The latter are necessary to arrive at a field theoretical treatment of entangling polymers.

To avoid this problem, we propose in this paper a model with two abelian C–S fields, in which the coupling constants are suitably chosen in order to cancel all the undesired self-linking terms without the need of a framing.
The advantage of introducing auxiliary C–S fields is that they “mediate” the interactions between the polymers, so that the random walk of two dynamical closed polymers subjected to entanglement reduces to the random walk of two single polymers fluctuating in an electromagnetic field. As a consequence, each polymer can be separately treated with the powerful methods developed in refs. [3],[4]. The final result is a gauge field theory describing the dynamics of two polymers chains subjected to entanglement. With respect to refs. [3],[4], the only complication in our approach is that the external magnetic field arising due to the presence of the background polymers is replaced by quantum C–S fields. For instance, we can compute the analogous of all polymer configurational probabilities derived in ref. [4]. This will be done in Section 3.

The material presented in this paper is divided as follows. In Section 2 we briefly review the field theoretical approach developed in refs. [3],[4] to describe the entanglement of one test polymer fluctuating in the background of a static polymer. In Section 3 this approach will be extended to the situation in which both polymers are dynamical. Finally, in the Conclusions some possible generalizations and applications of our treatment will be discussed.

2 Statistical–Mechanical Theory of Polymer Entanglement

Let $P$ be a polymer (see refs. [6] for a general introduction on polymers) represented as a long chain of $N + 1$ segments $\vec{r}_{i+1} - \vec{r}_i$ for $i = 0, \ldots, N$. Each segment has fundamental step length $a$, which is supposed to be very small with respect to the total length of the polymer. Moreover, the junction between adjacent segments is such that they can freely rotate in all directions. In the limit of large values of $N$, the ensemble of $M$ polymers $P_1, \ldots, P_M$ of this kind can be regarded as the ensemble of $M$ particles subjected to a self-avoiding random walk. The whole configuration of a polymer $P$ is thus entirely specified by the trajectory $\mathbf{r}(s)$ of a particle in three dimensional $(3 - D)$ space, with $0 \leq s \leq L$. $L$ is the contour length of the polymer and plays the role of time. We assume that the molecules of the polymer repel each other with a self-avoiding potential $v(\mathbf{r} - \mathbf{r}')$. The potential $v$ must be strong enough to avoid unwanted intersections of the trajectory $\mathbf{r}(s)$ with
itself. In the following, the case of two closed or infinitely long polymers \( P_1 \) and \( P_2 \) of contour length \( L_1 \) and \( L_2 \) respectively will be investigated. We suppose that they describe closed loops or infinitely long Wilson lines \( C_1 \) and \( C_2 \) in \( 3 - D \) space.

In order to take into account the entanglement of \( P_1 \) around \( P_2 \), we introduce the Gauss invariant \( \chi(C_1, C_2) \):

\[
\chi(C_1, C_2) \equiv \frac{1}{4\pi} \oint_{C_1} \oint_{C_2} d\mathbf{r}_1 \times d\mathbf{r}_2 \cdot \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3}
\] (1)

Physically, the Gauss invariant can be interpreted as the potential energy of a particle \( \mathbf{r}_1 \) moving inside a magnetic field \( \mathbf{B} \) generated by the particle \( \mathbf{r}_2 \):

\[
\chi(C_1, C_2) = \oint_{C_1} d\mathbf{r}_1(s_1) \cdot \mathbf{B}(\mathbf{r}_1(s_1))
\] (2)

where

\[
\mathbf{B}(\mathbf{r}_1) = \frac{1}{4\pi} \oint_{C_2} d\mathbf{r}_2(s_2) \times \frac{\mathbf{r}_1 - \mathbf{r}_2(s_2)}{|\mathbf{r}_1 - \mathbf{r}_2(s_2)|^3}
\] (3)

For the moment, we confine ourselves to the situation in which one single test polymer, for instance \( P_1 \), is entangling with a static polymer \( P_2 \). Thus \( P_2 \) has no dynamics and its configuration \( \mathbf{r}_2(s_2) \) is fixed. The configurational probability \( G_m(\mathbf{r}_1, L_1; \mathbf{r}_{0,1}, 0) \) to find \( P_1 \) with one end at a point \( \mathbf{r}_1 \) starting the other end at \( \mathbf{r}_{0,1} \) after intersecting a number \( m = 0, 1, 2, \ldots \) of times with \( P_2 \) can be expressed in terms of path integrals as the Green function of a particle subjected to a self-avoiding random walk:

\[
G_m(\mathbf{r}_1, L_1; \mathbf{r}_{0,1}, 0) = \mathcal{D}\mathbf{r}_1(s_1) \delta(\chi(C_1, C_2) - m) \times \\
\exp \left\{ - \int_0^{L_1} ds_1 \mathcal{L}_0 - \frac{1}{2a^2} \int_0^{L_1} ds_1 \int_0^{L_1} ds_1' v(\mathbf{r}(s_1) - \mathbf{r}(s_1')) \right\}
\] (4)

where

\[
\mathcal{L}_0 = \frac{3}{2a} r_1^2
\] (5)

is the Lagrangian of the free random walk. The case of closed loops or infinitely long polymers is recovered by taking suitable boundary conditions in the path integral (4). For instance, we require that \( \mathbf{r}_1 = \mathbf{r}_{0,1} \) for a closed
loop. To simplify the computations, it is convenient to work with the chemical potential $\lambda$ conjugated to the topological charge $m$. Thus we take the Fourier transform of $G_m(r_1, L_1; r_{0,1}, 0)$ with respect to $m$:

$$G_m(r_1, L_1; r_{0,1}, 0) = \int \frac{d\lambda}{2\pi} e^{-i\lambda m} G_\lambda(r_1, L_1; r_{0,1}, 0)$$

Comparing with eq. (4), the Green function $G_\lambda(r_1, L_1; r_{0,1}, 0)$ is given by:

$$G_\lambda(r_1, L_1; r_{0,1}, 0) = \int_{r_{0,1}}^{r_1} D r_1(s_1) \exp \left\{ - \int_0^{L_1} ds_1 L_0 \right\} \times \exp \left\{ - \int_0^{L_1} ds_1 L_0 - \frac{1}{2a^2} \int_0^{L_1} ds_1 \int_0^{L_1} ds_1' v(r(s_1) - r(s_1')) + i\lambda \chi(C_1, C_2) \right\}$$

Let us notice that the above path integral does not describe a Markovian random walk due to the presence of the non-local self-avoiding term. To reduce it to a Markovian random walk, we introduce gaussian scalar fields $\phi(r)$ with propagator:

$$\langle \phi(r)\phi(r') \rangle = \frac{1}{a^2} v(r - r')$$

Thus we have from eq. (7)

$$G_\lambda(r_1, L_1; r_{0,1}, 0) = \langle G_\lambda(r_1, L_1; r_{0,1}, 0|\phi, B) \rangle_\phi$$

In (9) the symbol $\langle \rangle_\phi$ denotes average over the auxiliary fields $\phi$ and

$$G_\lambda(r_1, L_1; r_{0,1}, 0|\phi, B) = \int_{r_{0,1}}^{r_1} D r_1(s_1) \exp \left\{ - \int_0^{L_1} ds_1 (L_\phi + i\phi_1(s_1) \cdot \lambda B(r_1(s_1))) \right\}$$

where we have put

$$L_\phi = L_0 + i\phi(r_1(s_1))$$

We remark that the Green function $G_\lambda(r_1, L_1; r_{0,1}, 0|\phi, B)$ is formally that of a particle diffusing under the magnetic field $B$ defined in eq. (3) and an imaginary electric potential $i\phi$. Thus it can be shown to satisfy the Schrödinger-like equation:

$$\left\{ \frac{\partial}{\partial L_1} - \frac{a}{6} (\nabla + i\lambda B)^2 + i\phi \right\} G_\lambda(r_1, L_1; r_{0,1}, 0|\phi, B) = \delta(L_1) \delta(r_1 - r_{0,1})$$
The Laplace transformed of the above equation with respect to the contour length $L_1$ is

$$\left\{ z_1 - \frac{a}{6}(\nabla + i\lambda B)^2 + i\phi \right\} G_\lambda(r_1, r_{0,1}; z_1|\phi, B) = \delta(r_1 - r_{0,1})$$  \hspace{1cm} (13)$$

where

$$G_\lambda(r_1, r_{0,1}; z_1|\phi, B) = \int_0^\infty dL_1 e^{-z_1 L_1} G_\lambda(r_1, L_1; r_{0,1}, 0|\phi, B)$$  \hspace{1cm} (14)$$

The variable $z_1$ plays the role of the chemical potential conjugate to the contour length $L_1$. From now on, we set $D = \nabla + i\lambda B$. Starting from eq. (13) and integrating over the auxiliary fields $\phi$ by means of the replica trick, one can express the Laplace transformed $G_\lambda(r_1, r_{0,1}; z_1)$ of the Green function (7) in term of second quantized fields. Skipping the details of the derivation, that can be found in ref. [4], we just state the result that one obtains in the case of a self-avoiding potential of the kind $v(r) = a^2 v_0 \delta(r)$:

$$G_\lambda(r_1, r_{0,1}; z_1) = \lim_{n \to 0} \int \prod_{\omega=1}^n D\psi^{*\omega} D\psi^{\omega} \psi^{*\varpi}(r_1) \psi^{\varpi}(r_{0,1}) e^{-F[\Psi]}$$  \hspace{1cm} (15)$$

In the above equation the fields $\psi^{*\omega}, \psi^{\omega}, \omega = 1, \ldots, n$, are complex replica fields and $\Psi = (\psi^1, \ldots, \psi^n)$. Moreover, $\varpi$ is an arbitrarily chosen integer in the range $1, \ldots, n$ and the polymer free energy $F[\Psi]$ is given by:

$$F[\Psi] = \int d^3r \left\{ \frac{a}{6} ||D\Psi||^2 + z_1 ||\Psi||^2 + v_0 ||^2 + z_1 ||\Psi||^4 \right\}$$  \hspace{1cm} (16)$$

where $||D\Psi||^2 = \sum_\omega (D\psi^{\omega})^2$ and $||\Psi||^2 = \sum_\omega |\psi^{\omega}|^2$. The generalization of the above formulas to an arbitrary number $M$ of static polymers $P_2, \ldots, P_M$ is straightforward, but not the inclusion of their dynamics. Even in the case of two polymers, the analogous of the Schrödinger equation (13) becomes complicated due to the presence of the nontrivial interactions between the polymers introduced by the Gauss invariant (1). This makes the derivation of the Green function $G_\lambda(r_1, L_1; r_{0,1}, 0|\phi, B)$ in terms of second quantized fields extremely difficult, apart from a few cases in which the trajectories of the polymers are strongly constrained. On the other side, without including the dynamics of all the polymers, there is always the difficulty of defining those configurations of the static polymers which are physically relevant. Indeed, we see from eqs. (15–16) that the polymer free energy $F[\Psi]$
depends on the magnetic potential $B$ through the covariant derivative $D$. As we will see in the next section, the introduction of auxiliary Chern–Simons fields will remove these problems.

### 3 Polymers Entanglement via Chern–Simons Fields

We study in this Section the dynamics of two polymers $P_1$ and $P_2$ subjected to entanglement. In analogy with the previous Section, we consider the configurational probability $G_m(\{r\}, \{L\}; \{r_0\}, 0)$ of finding the polymer $P_1$ with ends in $r_1$ and $r_{0,1}$ and the polymer $P_2$ with ends in $r_2$ and $r_{0,2}$. Moreover, we require that $P_1$ intersects $P_2$ $m$ times. Here we have put $\{r\} = r_1, r_2$, $\{L\} = L_1, L_2$ etc. From now on, the indices $\tau, \tau', ..., = 1, 2$ will be used to distinguish the two different polymers. The self-avoiding potential of the previous Section must be extended in the present case in order to take into account the reciprocal repulsions among the molecules of the two different polymers. Thus we choose a self-avoiding potential of the kind:

$$v_{\tau\tau'}(r_\tau(s_\tau) - r_{\tau'}(s_{\tau'})) = v^0_{\tau\tau'} v(r_\tau(s_\tau) - r_{\tau'}(s_{\tau'}))$$  \hspace{1cm} (17)

where $v^0_{\tau\tau'}$ is a symmetric $2 \times 2$ matrix and $v(r)$ is a strongly repulsive potential. As seen in the previous Section, the presence of self-avoiding potentials of this kind leads to random walks which are not Markovian. To solve this problem, we introduce here auxiliary scalar fields with gaussian action and propagator:

$$\langle \phi_\tau(x) \phi_{\tau'}(y) \rangle = \frac{1}{a^2} v^0_{\tau\tau'} v(x - y)$$  \hspace{1cm} (18)

In future we will make use of the following formula:

$$\int \prod_{\tau=1}^{2} D\phi_\tau \exp\left\{-\frac{a^2}{2} \int d^3 x d^3 y \phi_\tau(x) M^{\tau\tau'}(x, y) \phi_{\tau'}(y) - i \sum_{\tau=1}^{2} \int d^3 x J_\tau(x) \phi_\tau(x) \right\} =$$

$$\exp\left\{-\frac{1}{2a^2} \int_{0}^{L_\tau} ds_\tau \int_{0}^{L_{\tau'}} ds_{\tau'} v_{\tau\tau'}(r_\tau(s_\tau) - r_{\tau'}(s_{\tau'})) \right\}$$  \hspace{1cm} (19)

where $M^{\tau\tau'}(x, y)$ is the inverse of the matrix $v_{\tau\tau'}(x - y)$ and

$$J_\tau(x) = \int_{0}^{L_\tau} ds_\tau \delta^{(3)}(x - r_\tau(s_\tau))$$  \hspace{1cm} (20)
At this point, we introduce suitably chosen abelian C–S fields in order to write the contribution in the path–integral (7) given by the Gauss invariant in a more suitable way that will allow the decoupling of the two polymers. With the introduction of the C–S fields, our treatment of the polymer entanglement problem departs from that of Section 2.

We will consider for our purposes abelian C-S field theories with action:

$$A_{CS}(A, \kappa) = \frac{\kappa}{8\pi} \int d^3x \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho$$

with \(\mu, \nu, \rho = 1, 2, 3\). \(\kappa\) is a real coupling constant and \(\epsilon^{\mu\nu\rho}\) is the completely antisymmetric tensor in \(3 - D\). The above action can be also written in another useful form:

$$A_{CS}(A, \kappa) = \frac{\kappa}{8\pi} \int d^3r A \cdot (\nabla \times A)$$

where \(r = (x^1, x^2, x^3)\). The C–S propagator in the Feynman gauge is:

$$G_{\mu\nu}(x, y) = i \frac{\kappa \epsilon_{\mu\nu\rho}}{|x - y|^3}$$

The observables of the theory are gauge invariant operators built out of the basic fields \(A_\mu\). A complete set is given by the holonomies around closed curves

$$W(C, \gamma) \equiv \exp \left\{ -i \gamma \oint_C A_\mu dx^\mu \right\}$$

The vacuum expectation value of two of these observables \(W(C_1, \gamma_1)\) and \(W(C_2, \gamma_2)\) is:

$$\langle W(C_1, \gamma_1) W(C_2, \gamma_2) \rangle_A = \exp \left\{ -i \left( \frac{2\pi}{\kappa} \right) \left[ \gamma_1^2 \lambda(C_1, C_1) + \gamma_2^2 \lambda(C_2, C_2) + 2\gamma_1 \gamma_2 \lambda(C_1, C_2) \right] \right\}$$

where \(\lambda(C_\tau, C_\tau), \tau = 1, 2\) is the so-called self-linking number of the loop \(C_\tau\). To reproduce the term of eq. (7) containing the Gauss invariant \(\chi\), we need two Chern–Simons fields \(a_\mu\) and \(b_\mu\) with actions \(A_{CS}(a, \kappa)\) and \(A_{CS}(b, -\kappa)\) respectively. Using eq. (25) and setting for instance

$$\gamma_1 = \frac{\kappa}{4\pi} \quad \gamma_2 = \frac{\lambda}{4}$$

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it is in fact possible to see that
\[
\langle \mathcal{W}(C_1, \gamma_1)\mathcal{W}(C_2, \gamma_2) \rangle_a \langle \mathcal{W}(C_1, -\gamma_1)\mathcal{W}(C_2, \gamma_2) \rangle_b = \exp \left\{-i\lambda\chi(C_1, C_2) \right\}
\]
(27)
The right hand side of the above equation is exactly the contribution due to the polymer entanglement appearing in eq. (7). We are now ready to write the expression of the Green function \(G_\lambda(\{r\}, \{L\}; \{r_0\}, 0)\) for two entangling polymers. First of all, let us put:
\[
G_\lambda(\{r\}, \{L\}; \{r_0\}, 0) = \langle G_\lambda(\{r\}, \{L\}; \{r_0\}, 0|\phi, a, b) \rangle_{\phi, ab}
\]
(28)
where \(\langle \rangle_{\phi, ab}\) denotes the average with respect to the fields \(\phi, a, b\) and
\[
G_\lambda(\{r\}, \{L\}; \{r_0\}, 0) = \prod_{\tau=1}^2 \int_{r_{0,\tau}}^{r_\tau} \mathcal{D}r_\tau(s_\tau) \exp \left\{-\int_0^{L_\tau} \mathcal{L}_\phi \right\} \times
\]
\[
\exp \left\{-i\gamma_\tau \int_0^{L_\tau} A^\tau(r_\tau(s_\tau)) \cdot dr_\tau(s_\tau) \right\}
\]
(29)
The parameters \(\gamma_\tau\) appearing in the above equation are defined as in eq. (26) and the fields \(A^\tau\) are given by the relation:
\[
A^\tau = a + (-1)\tau b \quad \tau = 1, 2
\]
(30)
Exploiting formulas (19) and (25) in order to perform the two independent integrations over the fields \(\phi_\tau, a, b\) in eq. (28), one finds that:
\[
G_\lambda(\{r\}, \{L\}; \{r_0\}, 0) =
\]
\[
\int_{r_{0,1}}^{r_1} \mathcal{D}r_1(s_1) \int_{r_{0,2}}^{r_2} \mathcal{D}r_2(s_2) \exp \left\{-\int_0^{L_1} ds_1 \mathcal{L}_0(r_1(s_1)) - \int_0^{L_2} ds_2 \mathcal{L}_0(r_2(s_2)) \right\} \times
\]
\[
\exp \left\{-\frac{1}{2\tilde{a}^2} \sum_{\tau=1}^2 \int_0^{L_\tau} ds_\tau \int_0^{L_\tau} ds_\tau' v_{\tau\tau'}(r_\tau(s_\tau) - r_\tau(s_\tau')) - i\lambda\chi(C_1, C_2) \right\}
\]
(31)
By inverse Fourier transformation in \(\lambda\) as in eq. (6), we obtain from eq. (31):
\[
G_m(\{r\}, \{L\}; \{r_0\}, 0) =
\]

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G_{\lambda}({\{r}\}, {\{L\}}, {\{r_0\}}, 0|\phi_\tau, A^\tau) = G_\lambda(r_1, L_1; r_{0,1}, 0|\phi_1, A^1)G_\lambda(r_2, L_2; r_{0,2}, 0|\phi_2, A^2)

(33)

where, for \(\tau = 1, 2\):

\[
G_\lambda(r_{\tau}, L_{\tau}; r_{0,\tau}, 0|\phi_\tau, A^\tau) = \int_{r_{0,\tau}}^{r_{\tau}} Dr_{\tau}(s_\tau) \exp \left\{ - \int_0^{L_{\tau}} [L_{\phi_\tau} - i\gamma_\tau A^\tau \cdot dr_{\tau}(s_\tau)] \right\}
\]

(34)

Each of the Green functions \(G_\lambda(r_{\tau}, L_{\tau}; r_{0,\tau}, 0|\phi_\tau, A^\tau), \tau = 1, 2\), is the Green function of a particle diffusing under the vector potential \(A^\tau\) and the imaginary electromagnetic field \(\phi_\tau\). As a consequence, it can be written as the solution of a Schrödinger-like equation like (12). In analogy with the previous Section, it is convenient to introduce the chemical potentials conjugate to \(L_{\tau}\) and to consider the Laplace transformed Green function:

\[
G_\lambda({\{r}\}, {\{r_0\}}; {\{z\}}|\phi_\tau, A^\tau) = \int_0^\infty \int_0^\infty dz_1 dz_2 e^{-(z_1 L_1 + z_2 L_2)} G_\lambda({\{r}\}, {\{L\}}; {\{r_0\}}, 0|\phi_\tau, A^\tau)
\]

(35)

From eqs. (33–34) we have:

\[
G_\lambda({\{r}\}, {\{r_0\}}; {\{z\}}|\phi_\tau, A^\tau) = G_\lambda(r_{1,0}; z_1|\phi_1, A^1)G_\lambda(r_{2,0}; z_2|\phi_2, A^2)
\]

(36)
where the functions $G\lambda(r_\tau, r_{0,\tau}; z_\tau | \phi_\tau, A^\tau)$ have already been defined in eq. (14). For each value of $\tau = 1, 2$, they explicitly satisfy eq. (13), which we report here for convenience:

$$\left\{ z_\tau - \frac{a}{6} D_\tau^2 + i \phi_\tau \right\} G\lambda(r_\tau, r_{0,\tau}; z_\tau | \phi_\tau, A^\tau) = \delta(r_\tau - r_{0,\tau})$$  

(37)

The covariant derivatives $D_\tau$ are defined as follows: $D_\tau = \nabla + i\gamma_\tau A^\tau$. The solution of eq. (37) in terms of complex scalar fields $\psi^*_\tau, \psi_\tau$ is:

$$G\lambda(r_\tau, r_{0,\tau}; z_\tau | \phi_\tau, A^\tau) = \frac{1}{Z_\tau} \int D\psi^*_\tau D\psi_\tau \psi^*_\tau(r_\tau) \psi_\tau(r_{0,\tau}) e^{-F[\psi_\tau]}$$  

(38)

where, setting $|D_\tau \psi_\tau|^2 = (D_\tau \psi_\tau)^\dagger \cdot D_\tau \psi_\tau$, the free energy $F[\psi_\tau]$ is given by:

$$F[\psi_\tau] = \int d^3 r \left[ \frac{a}{6} |D_\tau \psi_\tau|^2 + (z_\tau + i \phi_\tau) |\psi_\tau|^2 \right]$$  

(39)

Finally, the partition function $Z_\tau$ is of the kind:

$$Z_\tau = \int D\psi^*_\tau D\psi_\tau e^{-F[\psi_\tau]}$$  

(40)

As we see from eq. (39), $F[\psi]$ is nothing but the Ginzburg-Landau free energy of a superconductor in a fluctuating magnetic field. We are now ready to perform the average over the auxiliary fields $\phi_\tau$ in the Green function (36). This integration is however highly non trivial. As a matter of fact, using eq. (38) to express the original Green function (36) in terms of second quantized fields, one immediately realizes that the integrand is not gaussian due to the presence of the factors $Z_\tau^{-1}$. To solve this problem, we exploit the replica trick. Accordingly, let us introduce $2n$ replica fields $\psi^*_\tau^\omega, \psi^\omega_\tau$, with $\tau = 1, 2$ and $\omega = 1, \ldots, n$. In terms of these fields, the Green function (36) can be written as follows:

$$G\lambda(\{r\}, \{r_0\}, \{z\} | \phi_\tau, A^\tau) = \lim_{n \to 0} \int \prod_{\omega=1}^n \prod_{\tau=1}^2 D\psi^*_\tau^\omega D\psi^\omega_\tau \psi^*_\tau^\omega(r_\tau^\omega) \psi_\tau^\omega(r_{0,\tau}^\omega) e^{-F[\psi^\omega_\tau]}$$  

(41)

where $\omega$ is an arbitrary integer chosen in the range $1 \leq \omega \leq n$. According to the replica trick, we will also assume that the limit for $n$ going to zero commutes with the integrations in the fields $A^\tau$ and $\phi_\tau$. In this way, the integral
over the auxiliary fields \( \phi_r \) becomes gaussian and can be easily performed. Supposing in analogy with the previous Section that the self-avoiding potential is of the kind:

\[
v_{ij}(r) = v^0_{ij}\delta(r)
\]

we have after some calculations that:

\[
\langle G_\lambda(\{r\}, \{r_0\}, \{z\} | \phi_r, A^\tau) \rangle_{\phi_r, a, b} = \\
\lim_{n \to 0} \int \prod_{\tau=1}^2 D\psi^\tau D\psi_0^\tau DaDb\psi_0^\tau(r_\tau)\psi_\tau^\tau(r_{0,\tau})\exp \{-\mathcal{A}(A^\tau, \psi^\tau)\}
\]

where, using the same notation of eqs. (15–16), the action \( \mathcal{A}(A^\tau, \psi^\tau) \) is:

\[
\mathcal{A}(A^\tau, \psi^\tau) = i\mathcal{A}_{CS}(a, \kappa) + \mathcal{A}_{CS}(b, -\kappa) + \\
\sum_{\tau=1}^2 \left[ \frac{a}{6} \||\nabla_\tau \psi_\tau||^2 + z_\tau \||\psi_\tau||^2 \right] + \sum_{\tau, \tau'=1}^2 \||\psi_\tau||^2 \nu^0_{\tau\tau'} \||\psi_{\tau'}||^2
\]

Eq. (43) is the generalization of eq. (15), expressing in terms of fields the configurational probability for two entangling polymers \( P_1 \) and \( P_2 \) to have their ends in \( r_\tau \) and \( r_{0,\tau} \) for \( \tau = 1, 2 \) respectively. This probability is given in the space of the chemical potentials \( \lambda, z_\tau \) conjugated to the topological number \( m \) and the contour length \( L_\tau \). The expression of the above configurational probability in the \( m \) space is obtained by taking the inverse Fourier transformation of the Green function (43):

\[
G_\lambda(\{r\}, \{r_0\}, \{z\} | \phi_r, A^\tau) \rangle_{\phi_r, a, b} = \\
\int \frac{d\lambda}{2\pi} e^{-i\lambda m} \langle G_\lambda(\{r\}, \{r_0\}, \{z\} | \phi_r, A^\tau) \rangle_{\phi_r, a, b}
\]

To this purpose, we split the action (44) into three parts:

\[
\mathcal{A}(A^\tau, \psi^\tau) = \mathcal{A}_0(A^\tau, \psi^\tau) + \lambda \sum_{\tau=1}^2 \int d^3r_\tau (r_\tau \cdot A^\tau (r_\tau) + \frac{a}{6} \sum_{\tau=1}^2 \int d^3r A_\tau \cdot A_\tau \||\psi_\tau||^2
\]

where

\[
\mathcal{A}_0(A^\tau, \psi^\tau) = i\mathcal{A}_{CS}(a, \kappa) + \mathcal{A}_{CS}(b, -\kappa) + \\
\sum_{\tau=1}^2 \left[ \frac{a}{6} \||\nabla_\tau \psi_\tau||^2 + z_\tau \||\psi_\tau||^2 \right] + \sum_{\tau, \tau'=1}^2 \||\psi_\tau||^2 \nu^0_{\tau\tau'} \||\psi_{\tau'}||^2
\]
\[ i(r) = \frac{a}{3} \frac{1}{2i} \sum_{\tau=1}^{2} (\Psi_{\tau}^{*} \nabla \Psi_{\tau} - \Psi_{\tau} \nabla \Psi_{\tau}^{*}) = \frac{a}{3} \frac{1}{2i} \sum_{\tau=1}^{2} \sum_{\omega=1}^{n} (\psi_{\tau}^{\omega*} \nabla \psi_{\tau}^{\omega} - \psi_{\tau}^{\omega} \nabla \psi_{\tau}^{\omega*}) \]  

(48)

Performing the Gauss integral in (45) and neglecting irrelevant constant factors, we have:

\[
G_{m}(\{r\}, \{r_{0}\}, \{z\}) = \lim_{n \to 0} \int D\psi_{\tau}^{\omega*} D\psi_{\tau}^{\omega} D\phi \sum_{\tau=1}^{2} \int d^{3}r_{\tau} \exp \left\{ -\frac{1}{4} K^{-1} \left( m - i \sum_{\tau=1}^{2} \int d^{3}r iE(r) \cdot A_{\tau}^{\tau}(r) \right) \right\} K^{-\frac{1}{2}} 
\]

(49)

where

\[
K = \frac{a}{6} \sum_{\tau=1}^{2} \int d^{2}r A_{\tau} \cdot A_{\tau} ||\Psi_{\tau}||^{2} \]

(50)

4 Conclusions

In this paper a C–S based model for polymer entanglement has been proposed and applied to the description of two entangling closed polymers \( P_{1} \) and \( P_{2} \). As we have seen in Section 3, the two polymers, whose action in eq. (32) was complicated by the reciprocal interactions introduced by the Gauss invariant, become completely decoupled after introducing two C–S field theories (see eq. (46)). Each polymer \( P_{\tau}, \tau = 1, 2, \) interacts only with the auxiliary fields \( A_{\tau}, \phi_{\tau} \) and its action is formally that of a polymer moving in the background of the “electromagnetic” field \( (A_{\tau}, i\phi_{\tau}) \). In this way, the application of the methods explained in Section 2 can also be applied to the case in which both polymers have a dynamics. One advantage of having included the dynamics of the second polymer is that the fixed magnetic field \( B \) of eq. (15) has been replaced by the quantum C–S fields of eqs. (43) and (49). Thus, there is no need to give a physical distribution for the configurations of the static polymers or to average over them using mean-field type techniques. On the other side, the complications introduced in our approach by the fact that both polymers are non-static, are minimal. The expression of the Green function in eq. (43) differs from that of eq. (32) only by the presence of two sets of replica fields instead of one.
The inclusion of the C–S fields in the treatment of the polymer entanglement problem opens the possibility of taking into account the higher topological invariants of knot theory [10], [11]. A simple possibility to achieve that is to replace the abelian fields $a, b$ with their nonabelian counterparts. The higher order invariants would appear in the higher order corrections to eq. (27), which can be expressed in terms of knot invariants as shown for instance in refs. [13]. Let us notice however that the nonabelian generalization of the above results is not straightforward, since one must verify that the unwanted contributions coming from the self-linking of the polymers disappear. Indeed, the elimination of the undesired Gauss self-linking terms occurring in eq. (27) is valid only at the first order approximation with respect to the C–S coupling constant $\kappa$ and it still has to be carefully checked at higher orders. A solution to the problem would be the introduction of a suitable framing such that

$$\chi_{\text{framed}}(C, C) = 0$$ (51)

On the other side, as already explained above, to get rid of the self-linking contributions with the help of a framing is not so easy as in pure C–S field theory. In fact, a framing like that in eq. (51) is necessarily depending on the form of the loop $C$, which in turn is a dynamical variable in our case. Thus, a framing of this kind would terribly complicate the form of the Schrödinger equation (37), preventing its solution in terms of second quantized fields.

Concluding, we have shown here that how it is possible to couple abelian C–S fields to polymers avoiding all self-linking contributions. As an example, all the polymer configurational probabilities derived in [4] for one single test polymer chain have been generalized to the case of two dynamical polymers. In the future we hope that it will be possible to extend the present approach also to the case of nonabelian C–S fields and to the situations in which there is an arbitrary number of polymers. Finally, we remark that our results could be also applied to other physical systems such as vortex rings and dislocation lines embedded in a solid, in which topological constraints among entangled one-dimensional excitations in a continuum play essential roles.

References


