THEORY OF HIERARCHICAL COUPLING

A.I. Olemskoi, A.D. Kiselev

Department of Physical Electronics, Sumy State University
2, Rimskii-Korsakov St., 244007 Sumy, UKRAINE
E-mail: Alexander@olem.sumy.ua

* Permanent address: Department of Pure and Applied Mathematics,
Chernigov Technological Institute, 250027 Chernigov, UKRAINE
E-mail: kisel@elit.chernigov.ua

Abstract

Recursion relation between intensity of hierarchical objects at neighbouring levels of a hierarchical tree, strength of coupling between them and level distribution of nodes of the hierarchical tree is proposed. Regular (including Fibonacci), degenerate and irregular trees are considered. It is shown that the strength of hierarchical coupling is an exponentially, logarithmically or power law decreasing function of distance from a common ancestor, respectively.

PACS numbers: 47.53+n, 64.60.Ak, 89.90.+n
Keywords: hierarchical trees, ultrametric space, recursion relation

1 Formulation of the problem

Despite widespread occurrence of hierarchy in social life and recognizing of its importance to other systems [1], the theory of hierarchically subordinated ensembles has been mainly evolved as a necessary part needed to understand dynamics of spin glasses [2], [3]. The key point is that the hierarchically subordinated objects form ultrametric space. Geometrically, the latter can be conceived of as a Cayley tree (see Fig.1). Degree of hierarchical coupling between objects, $w$, corresponding to the nodes of given level depends on the distance between them defined by the number of steps $m$ to a common ancestor, so that the ultrametric space is equipped with metrics, $\zeta \propto m$ ($\zeta$ is the distance). The primary goal of this work is to show how the function $w(\zeta)$ can be derived for different types of the hierarchical trees.

Let $z_k$ be an intensity of a hierarchical object at the level $k$, assuming that the intensity $z_k$ increases by going from the level $k$ to the nearest higher level $k-1$ (it looks like climbing the career ladder). Mathematically, it can be expressed in terms of the simplest recursion relation

$$z_{k-1} = z_k + N_k^{-1} w(z_k),$$

where $N_k$ is the number of nodes at level $k$ and $w(z_k)$ is the required function of hierarchical coupling. In the case of regular tree, shown in Fig.1a, we have the exponentional dependence
of $N_k$ on $k$ which is typical of fractal objects:

$$N_k = j^k,$$

(2)

where $j$ is the branching ratio of tree (the case of $j = 2$ is shown in Fig.1a). Another example is the degenerate tree with the only branching node per level (see Fig.1b), so that

$$N_k = (j - 1)k + 1.$$

(3)

For the Fibonacci tree (Fig.1c) the number of nodes on $k$–th level, $N_k = F(k+2)$, is determined by the Fibonacci numbers $F(k)$, $k = 0, 1,...$, that obey the equation $F(k+2) = F(k+1) + F(k)$ with $F(1) = F(2) = 1$. For $k \gg 1$ the latter implies that $F(k+2) \approx q\tau^k$, where $q \approx 1.17082$, $\tau = (\sqrt{5} + 1)/2 \approx 1.61803$ is so–called golden mean. As a result, the number of nodes on $k$–th level of the Fibonacci tree reads

$$N_k \approx q\tau^k, \quad k \gg 1.$$

(4)

Lastly, for the irregular tree like that depicted in Fig.1d a power law approximation can be suggested:

$$N_k = nk^a, \quad n > 0, \quad a < 1.$$

(5)

Note that Eq.(1) was originally used for description of resonance hierarchy in nonlinear oscillations [4] and enjoys the property of self–similarity that is distinguishable feature of hierarchical systems. In order to clarify the point, suppose that $z_k \propto \xi^k$ with $\xi \leq 1$ being the similarity parameter and $w(z)$ is a homogeneous function, $w(\xi z) = \xi^\alpha w(z)$. Then in the limiting case $k \gg 1$, when $z_{k-1} \sim z_k$, Eqs.(1), (2) give the conventional relation

$$\alpha = 1 - D, \quad D \equiv \ln j / \ln \xi^{-1}$$

(6)

linking exponent $\alpha$ of a physical quantity with fractal dimension $D \leq 1$ of a self-similar object type of rugged coastline [5]. Moreover, since the function $w(z)$ can be assumed to be of the form $w = W z^\alpha$, substituting expression $z_k = x_k \xi^k = x_k j^{-k/D}$ in Eqs.(1), (2) provides the recursion relation for $x_k$ in the following form:

$$x_{k-1} = \phi(x_k), \quad \phi(x) \equiv \xi(x + W x^{1-D}).$$

(7)

The map $\phi(x)$ has two fixed points: the stable one $x_s = 0$ and the critical one

$$x_c \equiv \left( \frac{W}{j^{1/D} - 1} \right)^{1/D}.$$

(8)

As a result, we obtain the following homogeneous functions

$$z_k = x_c j^{-k/D},$$

(9)

$$w_k = W^{1/D} \left( j^{1/D} - 1 \right)^{-\Delta} j^{-\Delta k},$$

(10)

where

$$\Delta = (1 - D)/D$$

(11)

is the decrement that determines the scale of hierarchical coupling in ultrametric space.
2 Recursion relation in the continuum approximation

When \( k \to \infty \) the continuum approximation can be used, so that the finite difference \( z_k - z_{k-1} \) is replaced with the derivative \( \frac{dz}{dk} \). Eq.(1) then can be rewritten in the integral form

\[
\int \frac{dz}{w(z)} = - \int \frac{dk}{N_k} \equiv (\ln j)^{-1} \int \frac{d\zeta}{N(\zeta)},
\]

(12)

where \( \zeta \equiv (k_0 - k) \ln j \), \( k \leq k_0 \).

(13)

According to the above discussion, for small \( z \) the function \( w(z) \) can be taken in the form

\[
w(z) = Wz^{1-D}, \quad z \to 0.
\]

(14)

It is not difficult then to solve Eq.(12) for different trees with the node numbers \( N_k \) defined by Eqs.(2)–(5) and with Eq.(13) taken into consideration.

2.1 Regular and Fibonacci trees

In the case of regular tree the dependencies of level intensity \( z(\zeta) \) and strength of hierarchical coupling \( w(\zeta) \) on the distance \( \zeta \) in ultrametric space read

\[
z = W^{-1/(1-D)} \left[ (1 - u) + ue^{\zeta - \zeta_0} \right]^{1/D}, \quad u \equiv DW^{1/(1-D)}/\ln j, \quad \zeta_0 \equiv k_0 \ln j;
\]

(15)

\[
w = \left[ (1 - u) + ue^{\zeta - \zeta_0} \right]^{\Delta}, \quad \zeta \leq \zeta_0.
\]

(16)

In Eq.(16) and hereafter \( w(\zeta) \) is assumed to meet the condition \( w(\zeta_0) = 1 \). So, if the distance \( \zeta \) to a common ancestor increases the functions \( z(\zeta) \) and \( w(\zeta) \) reveal exponential increase with increments \( D^{-1} \) and \( \Delta \), correspondingly. Increment of hierarchical subordination amplification (11) becomes zero for the system with \( D = 1 \) (from Eq.(14) the latter means ideal hierarchical subordination). Both of the decrements \( D^{-1} \) and \( \Delta \) increase indefinitely as the fractal dimension \( D \) decreases to zero.

Starting from Eq.(7), it is straightforward to analyse solutions of Eq.(1) and to make a comparison between the results obtained in the continuum limit and exact ones. Referring to Fig.2, where the graphs of functions \( \phi(x) \) and \( \phi^{-1}(x) \) are depicted, it is seen that if initial value \( x_0 \) obeys the condition \( x_0 < x_c \) just a few steps needed for \( x \) to approach zero. When \( x_0 = x_c \), the solution is defined by Eq.(8) and in the case of \( x_0 > x_c \) solutions increase indefinitely. It can be shown that in the latter case \( z_k \) exponentially decays to a constant. At this stage, there is a good agreement between the qualitative conclusions of the continuum approximation and exact ones (see Fig.3). It is interesting to note that Eq.(15) can be derived from the exact solution given by Eq.(9) only if \( j^{1/D} - 1 \approx \ln j/D \), that corresponds to the limiting case where \( \ln j \ll D \).

In order to clarify the above points note that within the framework of the continuum approximation the recursion relation (7) takes the form of well–known Landau–Khalatnikov equation

\[
\frac{dx_k}{dk} = - \frac{\partial V}{\partial x_k},
\]

(17)
\[ V = -\frac{x^2}{2} + \int \phi(x) \, dx, \]  
\[ (18) \]

where \( V \) is the effective potential. Inserting the dependence \( \phi(x) \) from Eqs.(7) into Eq.(18) yields the expression for the function \( V(x) \):

\[ V = \frac{Wj^{-1/D}}{2 - D} x^{2-D} - \frac{1 - j^{-1/D}}{2} x^2, \]  
\[ (19) \]

where the second equation (6) is taken into account. According to Fig.4, potential \( V(x) \) increases in region \( x < x_c \), where the value \( x_c \) is given by Eq.(8), and then decreases indefinitely. It is noteworthy to point the analogy with the theory of creation and growth of new phase precipitations, where at precritical size of embryos surface tension results in increase of free energy that then decreases indefinitely due to thermodynamical stimulus of phase transition \[6\].

So, one can speak about instability of hierarchical system with respect to increase in the level number under the initial intensity \( x_0 \) exceeds the critical value \( x_c \). This represents the well–known fact of the bureaucracy self–reproduction in social hierarchy.

Note that the solution of Eq.(17) gives \( z_k \) in the form

\[ z_k = \left( z_0^D - x_c^D (1 - e^{-D(1-\xi)k}) \right)^{1/D} e^{(1-\xi+\ln \xi)k}, \quad \xi \equiv j^{-1/D}, \]  
\[ (20) \]

that is equivalent to Eq.(15) when the above mentioned condition \( \ln j \ll D \) is met, so that \( \ln j \approx j - 1 \), \( \ln \xi \approx \xi - 1 \).

A comparison between Eq.(2) and Eq.(4) shows that for sufficiently large \( k \) the case of Fibonacci tree (Fig.1c) can be reduced to the above–considered regular tree with \( j \) and \( W \) replaced by \( \tau \) and \( W/q \), respectively. So the fractal dimension \( D \) is fixed and equals \( \ln \tau / \ln 2 \approx 0.6942 \) \[7\].

## 2.2 Degenerate tree

In the case of degenerate tree, where \( N_k \) is defined by Eq.(3), Eqs.(12), (14) give

\[ z = W^{-1/(1-D)} \left[ 1 - u \ln \left( 1 + \frac{j - 1}{\ln j} (\zeta_0 - \zeta) \right) \right]^{1/D}, \quad u \equiv \frac{DW^{1/(1-D)}}{j - 1}; \]  
\[ (21) \]

\[ w = \left[ 1 - \ln \left( 1 + \frac{j - 1}{\ln j} (\zeta_0 - \zeta) \right) \right]^\Delta. \]  
\[ (22) \]

When this result is compared with that of Eqs.(15), (16), it is apparent that going from the regular tree to degenerate one results in logarithmic slowing down of the foregoing exponential amplification of hierarchical subordination.

## 2.3 Irregular tree

Let us consider the intermediate case of a tree, characterizing by power law growth in level number (Eq.(5), Fig. 1d). The result can be written as follows:

\[ z = W^{-1/(1-D)} \left[ 1 - u(1 - \zeta/\zeta_0)^{1-a} \right]^{1/D}, \quad u \equiv \frac{DW^{1/(1-D)}k_0^{1-a}}{n(1-a)}, \quad a < 1; \]  
\[ (23) \]
\[ w = \left[ 1 - u(1 - \zeta/\zeta_0)^{1-a} \right]^{\Delta}. \]  

From Eqs.(23), (24) it is clear that the level intensity and the strength of hierarchical subordination show a power law dependence on the distance \( \zeta \).

### 3 Discussion

The above consideration show both the level intensity \( z \) and the strength of hierarchical coupling \( w \) decrease as the level number \( k \) increases and, conversely, they are increasing functions of the distance \( \zeta \) in ultrametric space. In this connection, it should be emphasized that the dependence \( w_k \) characterizes the degree of allience of hierarchical objects at the reference level, whereas \( w(\zeta) \) corresponds to the strength of hierarchical subordination.

A distinguishing feature of the regular tree is the fastest rate of change of \( z(\zeta) \) and \( w(\zeta) \). As it is seen from Eq.(10), the number of hierarchically subordinated level is bounded by the finite value

\[ \kappa = (\Delta \ln j)^{-1} \equiv D[(1 - D) \ln j]^{-1}. \]  

In other words, depth of hierarchical subordination \( \kappa \) is finite for regular tree, and in this sense it implies the weak hierarchical coupling. However, for the special case of totalitarian hierarchy, where the hierarchical coupling is ideal (in Eq.(14) \( D = 1 \)), depth of the subordination is infinitely large (\( \kappa = \infty \)). Nevertheless, according to Eq.(9) intensity of the hierarchical objects still decays exponentially and the totalitarian system, though being ideally subordinated, is doomed to inefficiency. Social experiments that lend support to this conclusion are well known.

With passage to irregular tree, that is supposed to be of widespread occurrence in nature, instead of exponentially fast decay inherent to regular tree the hierarchical coupling exhibits a power law dependence. The slowest, logarithmic law corresponds to degenerate hierarchy with the only branching nodes per level that can be realized as a selection system. For both irregular and degenerate trees depth of the hierarchical subordination is infinite, and the hierarchical coupling of such trees is strong.

It should be emphasized that the above strength of hierarchical coupling is fixed with the condition \( w(\zeta_0) = 1 \) related to the top level. Since real hierarchical system usually is built up from top downwards, such choice of normalization is preferred to the condition \( w(\zeta = 0) = 1 \). It seems, that given the choice the expressions inside the square brackets in Eqs.(15), (16), (23), (24) may take negative values at \( \zeta = 0 \) under the parameter \( u > 1 \). But since we are interested in the supercritical case where \( z_0 > x_c \) and, as is seen from comparison between Eq.(15) and Eq.(21), \( u \equiv (x_c/z_0)^D \), the condition \( u < 1 \) is fulfilled.

As is evident from the foregoing, the fractal dimension \( D \), that governs the force of hierarchical coupling \( w(\zeta) \) at given configuration of a tree, plays an important part in the theory. Throughout this paper, it was adopted that force of hierarchical coupling is characterized by the only single value of \( D \). It is not difficult to extend the consideration to the case of multifractal coupling. To do this one has to introduce additional parameter \( q \in (-\infty, \infty) \), so that the strength \( w_q(\zeta) \) is distributed over \( q \) with density \( \rho(q) \) and the fractal dimension \( D(q) \) is in the range between the maximum dimension \( D_{-\infty} \) and the minimum one \( D_{+\infty} \) [7]. For example, \( \rho \) for ideal hierarchical coupling is given by

\[ \rho(q) = |D'(q_0)|^{-1} \delta(q - q_0), \]  

5
where the prime denotes the derivative with respect to $q$ and $q_0$ is a solution of equation $D(q) = 1$. As a result, the total strength is defined by

$$w(\zeta) = \int_{-\infty}^{\infty} w'_q(\zeta) \rho(q) dq,$$

where Eqs.(16),(22),(24) with $D$ replaced by $D(q)$ can be used as a kernel of Eq.(27). Dependencies $D(q)$ and $\rho(q)$ for a given multifractal can be found after solving the respective problem (see [7]).

4 ACKNOWLEDGMENTS

One of us (A.I.O.) is grateful to the International Science Foundation for financial support under the Grant SPU072044 (ISSEP).
References


FIGURE CAPTIONS

Fig. 1 Basic types of hierarchical trees (the level number is indicated at left, corresponding number of nodes - at right): a) regular tree with \( j = 2 \); b) degenerate tree with \( j = 3 \); c) Fibonacci tree; d) irregular tree for \( n = 1 \) at \( a = 2 \).

Fig. 2 a) The plots of \( \phi(x) \) (dashed line) and \( \phi^{-1}(z) \) (solid line) at \( W = 0.5 \) and \( D = 0.6 \) (behaviour of \( x \) under successive iterations of the map \( \phi^{-1}(x) \) is shown by arrows for \( x_0 < x_c \) and \( x_0 > x_c \)).

Fig. 3 The \( k \)-dependencies of \( \ln z_k \), where the solutions of Eq.(7) are obtained numerically (circle dots) and in the continuum approximation (solid line) at \( W = 1 \), \( j = 2 \) and \( D = \ln j = 0.6931 \).

Fig. 4 The effective potential (19) as a function of \( x \) at \( W = 1 \), \( j = 2 \) and \( D = 0.6 \).
FIGURE 1
FIGURE 3

Computer simulation
Continuum approximation

\[ \ln z_k \]

\[ k \]