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KINEMATICAL SINGULARITIES IN CROSS-SECTIONS
AND DENSITY MATRICES

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A B S T R A C T

An analytic continuation procedure is proposed for cross-sections and density matrix elements to discuss their possible kinematical singularities. As expected, the s channel cross-section even when expressed in terms of the t channel amplitudes is free from kinematical singularities in the t variable whereas the density matrix elements in the Jackson frame can be singular at the t channel thresholds and pseudothresholds.

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In recent applications of high-energy models (especially the Regge pole model) an important question consists in taking properly into account the kinematical singularities of the scattering amplitudes ^{1),2),3)}. It was noticed, however, by Lin ⁴⁾ that the singularities at the t channel thresholds and pseudothresholds do not appear explicitly in the s channel cross-section. Jackson and Hite ⁵⁾ then explained that these singularities were cancelled by the fulfilment of kinematical constraints at the t channel thresholds and pseudothresholds.

The purpose of this note is an attempt to clarify this argument. In order to study the properties of the cross-section at the t channel thresholds one needs to know how to continue the cross-section outside the physical region. In Ref. ⁴⁾ and Ref. ⁵⁾ the analytic properties of the cross-section were not clearly stated and therefore in our opinion the continuation procedure was rather ambiguous.

1. - We first show that it is possible to define the s channel cross-section to be an analytic function in variable t for any s fixed in the physical region of the s channel. In this region the cross-section is usually defined as ^{*)}

$$\frac{d\sigma}{dt} = \sum_{m_i} M_{m_i}^{(s)}(s_0 + i\varepsilon, t) M_{m_i}^{(s)*}(s_0 + i\varepsilon, t) \quad (1)$$

where $M_{m_i}^{(s)}$ are the s channel scattering amplitudes, m_i denotes the spin projections (e.g., helicities or transversities), s_0 is above the physical threshold in the s channel, ε is positive and t is inside the s channel physical region. Expression (1) is not analytic in any of the complex variables s or t , but

^{*)} In fact, Eq. (1) represents rather a transition probability than a cross-section as we have suppressed for simplicity the flux factor $(64 \pi s p_{in}^2)^{-1}$.

$$F(s, t) = \sum_{m_i} M_{m_i}^{(s)}(s_0 + i\varepsilon, t) M_{m_i}^{(s)*}(s_0 + i\varepsilon, t^*) \quad (2)$$

is an analytic function of t for s_0 fixed in the physical region and for real t it coincides with the cross-section (1) ^{*)}. It is now meaningful to speak about the kinematical singularities of the function $F(s, t)$ which one can uniquely continue and which we shall call the analytic cross-section. It is easy to prove that this analytic cross-section has no kinematical singularities. It has been proved ³⁾ that the s channel helicity amplitudes have no kinematical singularities in variable t , apart from the singularities on the boundary of the physical region which follow from the angular momentum conservation. Since these singularities are exactly the same for $M^{(s)}(s, t)$ and $M^{(s)*}(s, t^*)$, no branch points or poles appear in $F(s, t)$, and since the sum runs over all the amplitudes no kinematical zero appears either.

2. - It is well known that due to the orthogonality of the crossing matrix it is also possible to express the s channel cross-section by the t channel amplitudes ⁶⁾. We have

$$\frac{d\sigma}{dt} = \sum_{m_i} |M_{m_i}^{(s)}(s, t)|^2 = \sum_{m'_i} |M_{m'_i}^{(t)}(s, t)|^2 \quad (3)$$

However, the right-hand side equality was supposed to be valid only inside the s channel physical region as the crossing angles become complex outside this region. It is remarkable that Eq. (3) can be analytically continued for all values of t at a fixed $s = s_0 + i\varepsilon$ and is equal to the analytic cross-section.

*) We owe this remark to Professor M. Froissart. We did not try to define a cross-section which would be analytic also in s ; this is possible but irrelevant for our purposes.

$$\begin{aligned}
F(s,t) &= \sum_{m_i} M_{m_i}^{(s)}(s_0+i\varepsilon, t) M_{m_i}^{(s)*}(s_0+i\varepsilon, t^*) = \\
&= \sum_{m_i} M_{m_i}^{(t)}(s_0+i\varepsilon, t) M_{m_i}^{(t)*}(s_0+i\varepsilon, t^*)
\end{aligned}
\tag{4}$$

To verify that Eq. (4) holds for all values of t , it is sufficient to check that the crossing angles $\chi_i(s,t)$ are real analytic functions, i.e., that

$$\chi_i^*(s, t^*) = \chi_i(s, t) \tag{5}$$

This can be seen from the formulae for $\cos \chi_i$ or $\sin \chi_i$ ^{3), 7)}.

One can now ask the question why $F(s,t)$ has no kinematical singularities while $M^{(t)}(s,t)$ and $M^{(t)}(s,t^*)$ do. To see how the singularities disappear, we shall use now the transversity amplitudes ^{8), 9)}. Their threshold singularities are factorizable and they have no threshold constraints ³⁾. As their singularities are of the square root type, some of them can vanish at the thresholds and this generates threshold relations for helicity amplitudes. The crossing matrix for transversity is diagonal.

$$M_{\tau_3 \tau_1, \tau_4 \tau_2}^{(t)} = \sigma e^{-i(\tau_1 \chi_1 - \tau_2 \chi_2 + \tau_3 \chi_3 - \tau_4 \chi_4)} M_{\tau_3 \tau_1, \tau_4 \tau_2}^{(s)} \tag{6}$$

where σ is a numerical phase factor. Since $M^{(s)}$ is kinematically regular in t , the threshold kinematical singularities of $M^{(t)}$ are given by the phase factor containing the crossing angles. We write

$$M_{\tau_3 \tau_1, \tau_4 \tau_2}^{(t)} = e^{i\chi} R(s, t) \tag{7}$$

with

$$\chi = -\tau_1 \chi_1 + \tau_2 \chi_2 - \tau_3 \chi_3 + \tau_4 \chi_4 \tag{8}$$

4.

and where $R(s, t)$ is a function kinematically regular in t . Then, because of Eq. (5), it becomes obvious that expression

$$M_{\tau_L}^{(t)}(s_0 + i\epsilon, t) M_{\tau_L}^{(t)*}(s_0 + i\epsilon, t^*) \quad (9)$$

is kinematically regular.

3. - We discuss now the density matrix elements. By using $M^*(s, t^*)$ instead of $M^*(s, t)$ we can define the analytic continuation of density matrix elements. These elements when given in the helicity (or transversity) frame are expressed directly by the s channel amplitudes and therefore have no singularities at the t channel thresholds. This cannot be said about the density matrix elements in the Gottfried-Jackson frame ⁶⁾ which are either expressed by the analytically continued t channel amplitudes or, alternatively, obtained from the helicity density matrices by rotating them through an angle about the normal to the production plane. This angle (often called the Jackson angle) is simply related to a crossing angle and therefore singular at the t channel thresholds. To discuss the singularities of the density matrix elements we use again the t channel transversity amplitudes which give us the density matrix elements in a frame analogous to the Jackson frame and which could be called the Jackson-transversity frame ⁹⁾

$$\rho_{\tau_3 \tau'_3}^{(Jt_2)} = \frac{1}{N} \sum_{\tau_1 \tau_2 \tau_4} M_{\tau_3 \tau_1 \tau_4 \tau_2}^{(t)} M_{\tau'_3 \tau_1 \tau_4 \tau_2}^{(t)*}(s, t^*) \quad (10)$$

This frame can be obtained from the ordinary transversity frame by performing the Jackson rotation. The density matrix elements in the conventional Jackson frame are again linear combinations of $\rho_{\tau_3 \tau'_3}^{(Jtr)}$ with numerical coefficients. In the same way, as in Section 2, the singular part of $\rho_{\tau_3 \tau'_3}^{(Jtr)}$ can be factorized according to

$$\rho_{\tau_3 \tau'_3}^{(Jt_2)} = e^{-i\chi_3(\tau_3 - \tau'_3)} G(s, t) \quad (11)$$

where $G(s, t)$ has no kinematical singularities at the t channel thresholds. We assume now that the resonance has been produced in a parity conserving reaction. Then $\tau_3 - \tau'_3$ is an even integer (cf. e.g. Ref. 9)). As $e^{i\chi_3}$ can have at most square root branching points, it follows that, in expression (11) there are no branch points but at most zeros or poles. They can appear only in the off-diagonal elements. In contrast, in the ordinary Jackson frame all the density matrix elements can have such kinematical singularities, even the diagonal ones.

The nature of these singularities is peculiar. The crossing angles χ_i have a branch point at the boundary of the physical region (i.e., in the forward or backward directions). The singular points lie off the boundary (at least in the different mass case) and they appear always in pairs, each of them on a different Riemann sheet but in the same place on the Mandelstam plane. For instance, if one of them is a second-degree kinematical zero, the other is always a second-degree pole.

This discussion can easily be translated into properties of statistical tensors (for a recent discussion of their properties, see Ref. 10)). They have no kinematical singularities or zeros when expressed directly through the s channel amplitudes (helicity, transversity frames). The transversity statistical tensors in the Jackson frame contain the singular factor $e^{iM\chi}$

$$T_M^J = e^{iM\chi} H(s, t) \quad (12)$$

and, as M is an even integer, the above discussion can be repeated here. Only T_0^J is free from threshold singularities (in the Jackson transversity frame).

4. - To conclude, we would like to comment on the methods used to take into account kinematical singularities.

i. - Using t channel helicity amplitudes (cf. Ref. 5))

One writes down the threshold and pseudothreshold singularities and constraints between the amplitudes on the thresholds and pseudothresholds. The final check consists in verifying that the singularities are really cancelled in the cross-section. This seems to be rather tedious as the constraint equations are not straightforward to derive.

ii. - Using t channel transversity amplitudes

This solves the problem of the t threshold singularities which are factorizable and well known ^{3),11)}. However, the transversity amplitudes behave in a complicated way at the boundary of the physical region and this is apparently even more important. Besides having singularities they fulfil some constraints at the boundary of the physical region.

iii. - Using s channel helicity amplitudes (cf. Ref. 12))

This is probably the easiest method as these amplitudes have only the s channel threshold singularities which are far away from the high-energy region and therefore may be neglected in phenomenological considerations.

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R E F E R E N C E S

- 1 - Y. Hara - Phys.Rev. 136, B507 (1964).
- 2 - L.L.Ch. Wang - Phys.Rev. 142, 1187 (1966).
- 3 - G. Cohen-Tannoudji, A. Morel and H. Navelet - Ann.Phys. (to be published).
- 4 - K.Y. Lin - Phys.Rev. 155, 1515 (1967).
- 5 - J.D. Jackson and G.E. Hite - Report UCRL-17959 (1967).
- 6 - K. Gottfried and J.D. Jackson - Nuovo Cimento 33, 309 (1964).
- 7 - T.L. Trueman and G.C. Wick - Ann.Phys. 26, 322 (1964).
- 8 - A. Kotański - Acta Phys.Polon. 29, 699 (1966).
- 9 - A. Kotański - Acta Phys.Polon. 30, 629 (1966).
- 10 - A. Kotański and K. Zalewski - Nucl.Phys. B4, 559 (1968).
- 11 - A. Kotański - CERN Preprint TH. 876 (1968).
- 12 - G. Cohen-Tannoudji, A. Morel and Ph. Salin - CERN Preprint TH. 860 (1968).