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CURRENT ALGEBRA AND $\pi - \pi$ SCATTERING LENGTHS.A DISPERSION RELATION APPROACH.

A. Donnachie

CERN - Geneva

A B S T R A C T

The usual simple linear expansion of the scattering amplitude in s , t and u is used to continue on to the mass shell but not to physical values of s . The continuation to physical s values is made using a linear combination of the partial wave dispersion relations on the first and second Riemann sheets, into which any desired effects of the unitarity cut may easily be inserted. A complicated spectrum of $T=0$ and $T=2$ scattering lengths is obtained. Many of the qualitative features of the solution are in agreement with other calculations in which some account has been taken of the unitarity cut.

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1. - INTRODUCTION

In Weinberg's ¹⁾ initial determination of the $\pi - \pi$ s wave scattering lengths from current algebra and P.D.D.A.C., the basic assumption was made that the unitarity cuts are weak and consequently a simple power series expansion in the Mandelstam variables s , t and u is sufficient to continue the scattering amplitude on to the physical threshold, despite the fact that strictly speaking the convergence of the expansion fails there due to the unitarity cut.

Although the solution obtained is consistent with this assumption in that the scattering lengths are found to be small, and although this approach has been checked up to second order terms in the expansion ²⁾, it is interesting to ask whether this result remains when the effect of the unitarity cut is taken into account explicitly. Several different approaches to this question have been given previously ^{3), 4), 5)}. In these approaches unitarity is included in some approximate way, while exact crossing symmetry is maintained. Our present purpose is to investigate the consequences of applying unitarity exactly and crossing symmetry only approximately.

To apply unitarity exactly it is necessary to make use of partial wave amplitudes rather than total amplitudes. By making use of a linear combination of partial wave dispersion relations on the first and second Riemann sheets it is possible to obtain a simple parametrization of the amplitudes into which different types of final state interaction can easily be inserted. This is developed in detail in Section 2. The current algebra conditions and their consequences for the partial wave amplitudes are discussed in Section 3. The results for the $T=2$ s wave and $T=0$ s wave scattering lengths are given in Sections 4 and 5 respectively. Final conclusions are given in Section 6.

2.

2. - FIRST AND SECOND SHEET DISPERSION RELATIONS

Let the initial pions, isospin indices a and b , have four-momenta k and p respectively and let the final pions, isospin indices c and d , have four-momenta l and q respectively. The matrix element may be written as

$$\langle l d, q b | M | p c, k a \rangle = A \delta_{ab} \delta_{cd} + B \delta_{ad} \delta_{cb} + C \delta_{ac} \delta_{bd} \quad (2.1)$$

the transition amplitudes of definite isospin being given by

$$\begin{aligned} M^{(0)} &= A + B + 3C \\ M^{(1)} &= A - B \\ M^{(2)} &= A + B \end{aligned} \quad (2.2)$$

with the converse

$$\begin{aligned} A &= \frac{1}{2} (M^{(1)} + M^{(2)}) \\ B &= \frac{1}{2} (M^{(2)} - M^{(1)}) \\ C &= \frac{1}{3} (M^{(0)} - M^{(2)}) \end{aligned} \quad (2.3)$$

The amplitudes $M^{(T)}$ ($T=0,1,2$) may be expanded in partial waves $f_l^{(T)}$ by

$$M^{(T)} = \sum_l (2l+1) f_l^{(T)}(s) P_l(x) \quad \begin{cases} l \text{ odd, } T=1 \\ l \text{ even, } T=0,2 \end{cases} \quad (2.4)$$

where

$$f_l^{(T)}(s) = \sqrt{\frac{s}{s-4}} e^{i\delta_l^{(T)}(s)} \sin \delta_l^{(T)}(s) \quad (2.5)$$

It should be noted that the matrix element defined here is normalized differently from that of Weinberg, the relation between them being

$$M = -\frac{1}{32\pi} M \text{ (Weinberg)} \quad (2.6)$$

The continuation of the first sheet function $f_{\ell}^I(s)$ ($=f_{\ell}(s)$) on to the second Riemann sheet is given by

$$f_{\ell}^{II} = \frac{f_{\ell}^I(s)}{1 + 2i \sqrt{\frac{s-4}{s}} f_{\ell}^I(s)} \quad \begin{cases} -\infty \leq s \leq 0 \\ 4 \leq s \leq \infty \end{cases} \quad (2.7a)$$

$$\frac{f_{\ell}^I(s)}{1 - 2i \sqrt{\frac{4-s}{s}} f_{\ell}^I(s)} \quad 0 < s < 4 \quad (2.7b)$$

The singularities of $f_{\ell}^{II}(s)$ are the same as those of $f_{\ell}^I(s)$ (namely cuts from $-\infty$ to 0 and 4 to $+\infty$) with the possible addition of poles arising from zeroes of the denominator of Eq. (2.7). There are two particular cases in which we shall be interested.

1) - A real pole between $s=0$ and $s=4$

This corresponds to the existence of a virtual bound state in the partial wave ℓ and occurs at the point $s=s_p$ if

$$f_{\ell}(s_p) = \frac{1}{2} \sqrt{\frac{s_p}{4-s_p}} \quad (2.8)$$

the pole having residue T_p , given by

$$\frac{1}{T_p} = -\frac{1}{f_{\ell}^2(s_p)} \frac{\partial f_{\ell}^I(s_p)}{\partial s} + \frac{4}{s_p^2} \sqrt{\frac{s_p}{4-s_p}} \quad (2.9)$$

2) - A pair of complex conjugate poles, with $\text{Re } s > 0$

This corresponds to the existence of a resonance in the partial wave l . If for simplicity we represent this by a Breit-Wigner resonance for the first sheet function

$$\frac{e^{i\delta_l} \sin \delta_l}{q} = \frac{\gamma_l}{(s_R - s) - i\gamma_l q} \quad (2.10)$$

then the poles occur at

$$s = \alpha_l \pm i\beta_l \quad (2.11)$$

where

$$\alpha_l = s_R - \frac{1}{2} \gamma'^2, \quad \gamma' = \frac{1}{2} \gamma \quad (2.12)$$

$$\beta_l = \gamma' \sqrt{s_R - 4 - \gamma'^2} \quad (2.13)$$

and have residue

$$R = a_l \pm i b_l \quad (2.14)$$

where

$$a_l = \gamma' \sqrt{s_R - 4 - \gamma'^2} - \frac{\gamma'^3}{4 \sqrt{s_R - 4 - \gamma'^2}} \quad (2.15)$$

$$b_l = \gamma'^2 \quad (2.16)$$

Denoting the inelastic threshold by s_I , then for $4 \leq s < s_I$

$$\operatorname{Re} f_e^{\text{II}}(s) = \operatorname{Re} f_e^{\text{I}}(s) \quad (2.17)$$

and

$$\operatorname{Im} f_e^{\text{II}}(s) = - \operatorname{Im} f_e^{\text{I}}(s) \quad (2.18)$$

so that the linear combination

$$F_e(s) = f_e^{\text{I}}(s) + f_e^{\text{II}}(s) \quad (2.19)$$

does not have the elastic unitarity cut and consequently will satisfy a dispersion relation of the form

$$\begin{aligned} \operatorname{Re} F_e(s) = \text{Pole Terms} &+ \frac{P}{\pi} \int_{s_I}^{\infty} ds' \frac{\operatorname{Im} F_e(s')}{(s'-s)} \\ &+ \frac{1}{\pi} \int_{-\infty}^0 ds' \frac{\rho_e(s')}{(s'-s)} \end{aligned} \quad (2.20)$$

where $2i\rho_e(s)$ is the discontinuity of $F_e(s)$ across the left-hand cut and the pole terms, if they exist, will either have the form

$$\frac{T_P}{(s-s_P)} \quad (2.21)$$

if they arise from a virtual bound state, or

$$\frac{2[a_e(\alpha_e - s) + b_e\beta_e]}{[(\alpha_e - s)^2 + \beta_e^2]} \quad (2.22)$$

if they arise from a resonance.

In principle, the inelastic threshold occurs at $s=16$, but in practice the effective inelastic threshold is very much higher, as can be seen from the comparison of 2π and 4π mass spectra in various experiments, and is of the order of $s=40$. If we set $s_I \simeq 40$, then since we are interested only in evaluating Eq. (2.20) close to threshold (for $1 \leq s \leq 4$), the principal value integral in Eq. (2.20) may be replaced by a constant. Future algebraic manipulation is simplified by subtracting at $s=1$, thus replacing the "high-energy" constant by a "low-energy" subtraction constant. Finally it is necessary to parametrize the function $P_\ell(s)$ occurring in Eq. (2.20). The simplest choice is that of a δ function

$$P_\ell(s) = -\pi T_\ell \delta(s-s_\ell) \quad (2.23)$$

Carrying out these procedures, we obtain the following expression for the s wave amplitudes

$$\text{Re } F_0(s) = A_0 + (s-1) \left\{ P_0(s_P, T_P, s) + R_0(s_R, \gamma_R, s) - \frac{T_0}{(s-s_0)(1-s_0)} \right\} \quad (2.24)$$

where

$$P_0(s_P, T_P, s) = - \frac{T_P}{(s-s_P)(1-s_P)} \quad (2.25)$$

and

$$R_0(s_R, \gamma_R, s) = \frac{2[a_0\{(\alpha_0-s)(\alpha_0-1)-\beta_0^2\} + b_0\beta_0\{2\alpha_0-s-1\}]}{[(\alpha_0-s)^2 + \beta_0^2][(\alpha_0-1)^2 + \beta_0^2]} \quad (2.26)$$

with α_0 , β_0 , a_0 and b_0 being given in terms of s_R , γ_R by Eqs. (2.12), (2.13), (2.15) and (2.16). Either s_p and T_p , or s_R and γ_R are to be considered as free parameters to be injected into the system to study the effects of the unitarity cut, while the parameters a_0 , T_0 and s_0 are to be determined from the requirements of current algebra and P.D.D.A.C.

Finally, it should be noted that the s wave scattering lengths are given by

$$a_0 = f_0(4) = \frac{1}{2} F_0(4) \quad (2.27)$$

In the partial-wave expansion Eq. (2.4) we shall restrict ourselves to s and p waves only, since d waves (and higher) do not appear to become significant until above 1 GeV. With this restriction,

$$\begin{aligned} M^{(0)} &= f_0^{(0)}(s) \\ M^{(1)} &= 3x f_1^{(1)}(s) = \frac{3(s+2t-4)}{(s-4)} f_1^{(1)}(s) \\ M^{(2)} &= f_0^{(2)}(s) \end{aligned} \quad (2.28)$$

3. - CURRENT ALGEBRA AND P.D.D.A.C. REQUIREMENTS

The three conditions given by Weinberg are :

1. when $p^\mu = \ell^\mu$ is on the mass shell and $q^\mu = k^\mu \rightarrow 0$, then

$$\langle \ell d, q b | M | p c, k a \rangle \rightarrow \tilde{M}_{db,ca} + \frac{1}{8\pi} (g_V/F_\pi)^2 2 p \cdot q (\delta_{da} \delta_{bc} - \delta_{bd} \delta_{ac}) \quad (3.1)$$

where $\tilde{M}_{db,ca}$ is an unknown constant ; in this limit

$$\begin{aligned} s &\rightarrow 1 - 2 p \cdot q \\ t &\rightarrow 0 \\ u &\rightarrow 1 + 2 p \cdot q \end{aligned} \quad (3.2)$$

2. the Adler self-consistency argument that M vanishes when any one of the four pion momenta vanishes, and the other three are on the mass shell, i.e.,

$$M = 0 \quad (3.3)$$

when

$$s = t = u = 1 \quad (3.4)$$

3. the unknown constant $\tilde{M}_{db,ca}$ in Eq. (3.1) is proportional to δ_{ba} .

We shall take the usual simple linear expansion of the amplitude in s , t and u which is crossing symmetric, namely

$$\begin{aligned} M = & \delta_{ab} \delta_{cd} [a + B(s+u) + b t + \dots] \\ & + \delta_{ad} \delta_{cb} [a + B(s+t) + b u + \dots] \\ & + \delta_{ac} \delta_{bd} [a + B(u+t) + b s + \dots] \end{aligned} \quad (3.5)$$

but we shall require this to be valid only in the immediate neighbourhood of $s=1$, using it to continue from the off-mass-shell points (where current algebra is applied) on to the nearest on-mass-shell point where we can apply the formalism of Section 2. On the mass shell, Eq. (3.5) becomes

$$\begin{aligned}
 M = & \delta_{ab} \delta_{cd} [a + B(4-t) + bt + \dots] \\
 & + \delta_{ad} \delta_{cb} [a + B(s+t) + b(4-s-t) + \dots] \\
 & + \delta_{ac} \delta_{bd} [a + B(4-s) + bs + \dots]
 \end{aligned}
 \tag{3.6}$$

Substituting Eq. (2.3) in Eq. (2.1), recalling Eq. (2.28) and comparing with Eq. (3.6) yields immediately

$$f_0^{(0)}(s) = 5a + 2B(8-s) + 2b(2+s) \tag{3.7a}$$

$$\frac{3f_1^{(1)}(s)}{(s-4)} = b - B \tag{3.7b}$$

$$f_0^{(2)}(s) = 2a + B(4+s) + b(4-s) \tag{3.7c}$$

and we require these results to be valid only in the neighbourhood of $s=1$.

Following Weinberg, the constants a , B and b can be evaluated from conditions 1., 2. and 3. above, yielding

$$a = -\delta, \quad B = 0, \quad b = +\delta \tag{3.8}$$

where

$$\delta = \frac{1}{8\pi} \left(g_V / F_\pi \right)^2 \tag{3.9}$$

Hence

$$f_0^{(0)} = -\delta(1-2s) \quad (3.10)$$

and

$$f_0^{(2)} = \delta(2-s) \quad (3.11)$$

[Note that if we take Eqs. (3.10) and (3.11) to be the continuation of $f_0^{(0)}$ and $f_0^{(2)}$ on to the unitarity cut, then naturally we reproduce the Weinberg results, namely $f_0^{(0)}(4) = 7\delta \simeq 0.20$ and $f_0^{(2)}(4) = -2\delta \simeq -0.06$.]

Our procedure is now as follows. Knowing the value of the first sheet function f_0^I in the neighbourhood of $s=1$, we can find firstly the second sheet function f_0^{II} in this neighbourhood and then the combined function F_0 , of Eq. (2.19) and its first derivative at the point $s=1$. (It is meaningless to consider higher derivatives of the function F_0 since they will depend, in part, on the higher order terms in the expansion Eq. (3.5), which have been neglected.) These are then compared with the corresponding values of F and its first derivative obtained from the parametrization Eq. (2.24) which allows us to determine the subtraction constant A_0 and the residue of the left-hand cut pole Γ_0 , the latter as a function of the pole position s_0 , which is arbitrary within the limits $-\infty < s_0 < 0$. This means that a range of solutions for the scattering lengths is obtained for any given value of the virtual bound state parameters s_P , Γ_P , or the resonance parameters s_R , γ_R .

4. - T = 2 S WAVE SCATTERING

Since there is no evidence for the existence of a $T = 2$ s wave resonance, we shall not consider this possibility and restrict our discussion to the other two cases, namely when there are no second sheet poles at all and when there is one second sheet pole corresponding to the existence of a virtual bound state.

1) - No second sheet poles

From Eq. (2.24) we have the parametrization

$$F_o^{(2)}(s) = A_o^{(2)} - \frac{(s-1) T_o^{(2)}}{(s-s_o^{(2)})(1-s_o^{(2)})} \quad (4.1)$$

so that

$$F_o^{(2)}(1) = A_o^{(2)} \quad (4.2a)$$

and

$$\frac{\partial F_o^{(2)}(1)}{\partial s} = - \frac{T_o^{(2)}}{(1-s_o^{(2)})^2} \quad (4.2b)$$

From current algebra requirements,

$$F_o^{(2)}(1) = \delta \left\{ 1 + \frac{1}{1-2\sqrt{3}\delta} \right\} \simeq 0.058 \quad (4.3a)$$

and

$$\frac{\partial F_o^{(2)}(1)}{\partial s} = -\delta \left\{ 1 + \frac{1}{1-2\sqrt{3}\delta} \right\} - \frac{10\delta^2}{\sqrt{3}(1-2\sqrt{3}\delta)^2} \simeq -0.063 \quad (4.3b)$$

so that

$$F_o^{(2)}(s) = \frac{0.058 - 0.063(s-1)(1-s_o^{(2)})}{(s-s_o^{(2)})} \quad (4.4)$$

Recalling that $-\infty < s_0^{(2)} < 0$, Eq. (4.4) yields

$$-0.07 < a_0^{(2)} < 0.006 \quad (4.5)$$

2) - A real second sheet pole

From Eq. (2.24) we have the parametrization

$$F_0^{(2)}(s) = A_0^{(2)} + (s-1) \left\{ -\frac{T_P^{(2)}}{(s-s_P^{(2)})(1-s_P^{(2)})} - \frac{T_0^{(2)}}{(s-s_0^{(2)})(1-s_0^{(2)})} \right\} \quad (4.6)$$

with T_P given by Eq. (2.8). One problem here is in deciding what is a reasonable order of magnitude for T_P . Generally speaking, the first sheet function will vary slowly in the region $0 < s < 4$ (except possibly close to the branch points $s=0$ and $s=4$) and so the residue of the second sheet pole will be given approximately by

$$T_P \simeq \frac{s_P^2}{4} \sqrt{\frac{4-s_P}{s_P}} \quad (4.7)$$

We have evaluated the scattering length using the value of the residue of the second sheet pole given by Eq. (4.7). The current algebra requirements are again given by Eq. (4.3). The results are best presented graphically, and this is done in Fig. 1 which shows the upper and lower limits of the $T=2$ s wave scattering length as a function of the position of the virtual bound state, the variation at a given pole position being due to the variation of $s_0^{(2)}$, namely $-\infty < s_0^{(2)} < 0$.

As the pole position approaches the physical threshold, the scattering length naturally becomes infinite. There is also a divergence as the pole position approaches $s=1$, which is symptomatic of the conflict between the current algebra requirements and the pole requirements at $s=1$, where they are incompatible. As the pole approaches $s=0$,

the upper and lower limits approach the values of Eq. (4.5) and attain them at $s=0$. From Fig. 1 it follows that, if a virtual bound state exists, then the $T=2$ scattering length can have any value

$$-\infty < a_0^{(2)} < \infty \quad (4.8)$$

In the absence of a virtual bound state, the inclusion of unitarity with current algebra, in this model, restricts the scattering length to the range given in Eq. (4.5).

5. - T=0 S WAVE SCATTERING

Here we shall consider all three cases, namely no second sheet poles, a virtual bound state pole and resonance poles.

1) - No second sheet poles

The procedure follows exactly that for $T=2$ s-wave scattering. In this case we obtain from the current algebra requirements

$$F_0^{(0)}(1) \simeq 0.058 \quad (5.1a)$$

and

$$\frac{\partial F_0^{(0)}(1)}{\partial s} \simeq 0.120 \quad (5.1b)$$

so that

$$F_0^{(0)}(s) = 0.058 + 0.120 \frac{(s-1)(1-S_0^{(0)})}{(s-S_0^{(0)})} \quad (5.2)$$

yielding

$$0.07 < a_0^{(0)} < 0.21 \quad (5.3)$$

2) - A real second sheet pole

Again the procedure follows exactly that for $T=2$ s-wave scattering, and the results are presented in Fig. 2, where the $T=0$ scattering length is plotted against the position of the virtual bound state. Again there are divergences at $s=1$ and $s=4$ and again the limiting values of Eq. (5.3) are attained at $s=0$. We can again conclude that, if a virtual bound state exists, then the $T=0$ scattering length can have any value

$$-\infty < a_0^{(0)} < \infty \quad (5.4)$$

3) - A pair of complex conjugate poles

From Eqs. (2.24) and (2.26) we have the parametrization

$$F_0^{(0)}(s) = H_0^{(0)} + (s-1) \left\{ \frac{2[a_0^{(0)}(\alpha_0^{(0)} - s)(\alpha_0^{(0)} - 1) - \beta_0^{(0)2}] + b_0^{(0)}\beta_0^{(0)}\{2\alpha_0^{(0)} - s - 1\}}{[(\alpha_0^{(0)} - s)^2 + \beta_0^{(0)2}][(\alpha_0^{(0)} - 1)^2 + \beta_0^{(0)2}]} - \frac{T_0^{(0)}}{(s - s_0^{(0)})(1 - s_0^{(0)})} \right\} \quad (5.5)$$

with $a_0^{(0)}$, $b_0^{(0)}$, $\alpha_0^{(0)}$ and $\beta_0^{(0)}$ given in terms of the resonance position and width by Eqs. (2.12), (2.13), (2.15) and (2.16). The results of the evaluation are shown in Fig. 3 and Fig. 4, where the upper and lower limits on the scattering length are given as functions of the resonance position for different resonance widths. For reasonable values of the resonance position and width, the results do not depart appreciably from the asymptotic values of Eq. (5.3). Of course, if the resonance is taken sufficiently low and/or sufficiently broad, the scattering length can be made as large as one pleases, and if a resonance exists then it is possible to have all scattering lengths such that

$$0.07 < a_0^{(0)} < \infty \quad (5.6)$$

although for reasonable restrictions on the mass and width of the resonance

$$0.07 < a_0^{(0)} \lesssim 0.3 \quad (5.7)$$

6. - CONCLUSIONS

In common with previous calculations ^{3),4),5)}, we find that including the unitarity cuts explicitly can have profound effects on the $\pi - \pi$ scattering lengths obtained from current algebra. In the present model, however, these effects can be large only if a virtual bound state exists. A resonance, of reasonable position and width, does not alter the Weinberg results appreciably, and if no structure at all is specified then the unitarity requirements restrict the scattering lengths to lie in a narrow band about the Weinberg results (narrow on an absolute, but not a relative, scale).

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FIGURE CAPTIONS

- Figure 1 The $a_0^{(2)}$ scattering length as a function of the position of a virtual bound state.
- Figure 2 The $a_0^{(0)}$ scattering length as a function of the position of a virtual bound state.
- Figure 3 The upper and lower limits of the $a_0^{(0)}$ scattering length as a function of resonance position for different resonant widths.

FIG.1 POSITION OF VIRTUAL BOUND STATE

FIG.2 POSITION OF VIRTUAL BOUND STATE

FIG 3