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JOINT DECAY CORRELATIONS IN DOUBLE RESONANCE PRODUCTION

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A B S T R A C T

The problem of correlations between decay angular distributions in double resonance production is treated, in particular for the reactions $\pi p \rightarrow \rho N^*$ and $\pi p \rightarrow f^0 N^*$. It is found that in general the combined correlation effects are a priori independent of the individual decay distributions. By considering production in the forward direction, we show that certain combined correlation effects follow from the individual decay distributions, provided that an Adair analysis argument is valid. Such an argument is expected to hold for highly peripheral reactions in which the density matrix elements are slowly varying with production angle. Comparison with experiment is made for the ρN^* correlation, and excellent agreement is found. Suggestions for the analysis of the $f^0 N^*$ reaction are proposed, based on a sample calculation of the observed effect. The success of the absorption model in predicting the correlation is discussed, and more stringent experimental tests are suggested.

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I. - INTRODUCTION

In this paper we consider the problem of correlations between the decay angular distributions of the final state resonances in the reactions

$$0^- + 1/2^+ \rightarrow 1^- + 3/2^+ \\ \hookrightarrow (0^- 0^-) \hookrightarrow (1/2^+ 0^-) \quad (A)$$

$$0^- + 1/2^+ \rightarrow 2^+ + 3/2^+ \\ \hookrightarrow (0^- 0^-) \hookrightarrow (1/2^+ 0^-) \quad (B)$$

Examples of reaction A are $\pi p \rightarrow \rho N^* \rightarrow (\pi^+ \pi^- \pi^+ p)$ and $Kp \rightarrow K^* N^* \rightarrow (K \pi \pi p)$. An example of reaction B is $\pi p \rightarrow f^0 N^* \rightarrow (\pi^+ \pi^- \pi^+ p)$. The experimental data for these reactions show a correlation between the decay angles for the two resonances ^{1),2),3),4)}. The general problem of joint decay distributions has been studied by Pilkuhn and Svensson ⁵⁾, and attempts have been made to explain the observed effects in reaction A using the absorptive peripheral model ^{6),7)}. The predictions of the model, assuming pion exchange for the ρN^* and $K^* N^*$ reactions, were found to be in good agreement with the experimental decay distributions. Not only were the predicted individual decay distributions correct, but also the observed combined correlation was in quantitative agreement with the absorption model prediction. However the absorption model also predicts that some other combined correlation effects should be present, and the preliminary experimental results on ρN^* do not agree with these predictions ⁴⁾.

In an attempt to understand the success of the absorption model, we have asked the following question. Given a pair of individual decay distributions (or equivalently, the spin-density matrices), is there any general argument which shows that these imply the existence of combined correlations? The answer, as will be shown, is no; the

combined correlation effects are a priori independent of the individual decay distributions. For any pair of individual density matrices, there is no purely kinematic argument which either requires or prohibits the presence of joint decay correlations. However, in the special case of production in the forward direction, it is possible to see how the correlations arise, by using an Adair type argument. The result is that while the individual decay distributions do not constrain combined correlations by kinematics alone, a simple dynamical argument can be used to show that some combined correlations follow plausibly from the individual decay distributions. This latter conclusion is restricted to the $\cos \theta_V - \cos \theta_{N^*}$ correlation, the only one observed experimentally. Hence we find that the success of the absorption model with the $\cos \theta_V - \cos \theta_{N^*}$ correlations can be attributed to its success with the individual decay distribution, and to the fact that it respects the Adair argument. The problem of other possible correlations is treated, and it is found that a stringent test of the absorption model is provided by the existence of such effects. Again a forward production angle analysis is used to give an intuitive picture of the combined correlation effects.

The plan of the paper is as follows. In Section II a brief description of the general formulation of joint decay distributions is presented, including the conventions used. The joint decay distribution for reaction A is given explicitly in Section III, and the forward angle analysis and comparison with experiment are in Section IV. The $\cos \theta_c - \cos \theta_d$ correlation for reaction B is analysed in Section V. Section VI contains a brief discussion of absorption model predictions and suggestions for experimental analysis.

We wish to acknowledge that many of the results presented here are essentially not original, being reworkings of earlier treatments of the problem. We feel, however, that the simple forms obtained, and the consequent understanding of the correlation effects are of interest. The general results presented do not depend on any absorption model assumptions, although reference is made to absorption model predictions for the joint decay distribution.

II. - GENERAL FORMULATION OF JOINT DECAY DISTRIBUTIONS

The decay angular distribution of a resonance is determined by the spin density matrix, as shown, for example, by Gottfried and Jackson⁸⁾. For the problem of two resonances, a joint spin density matrix can be defined which determines the combined decay angular distribution, the decay angles being measured in the rest frames of the two resonances^{5),6)}. Details of the derivation are given in Ref. 5), as are references to other formulations of the problem. We follow the conventions of Ref. 6), and consider the reaction

$$O^- + 1/2^+ \rightarrow M + B$$

with M and B meson and baryon resonances. The particles are denoted by a, b, c, d respectively. For simplicity we consider two-body decays of particles c and d . The spherical polar angles of the line of flight of the decay products in the rest frame of particle c are (θ_c, φ_c) . The z axis is taken in the direction of particle a , with the y axis being the normal to the production plane. This is shown in Fig. 1; the normal is defined by $\vec{a} \times \vec{c}$ in the c.m. system. In the particle d rest frame (θ_d, φ_d) are similarly defined, with the b direction as z axis.

Let the individual spin density matrices for particles c and d , $\rho_{mm'}^c$ and $\rho_{nn'}^d$, describe the spin populations of the resonances, each in its rest frame. The joint spin density matrix $R_{mm';nn'}$ ⁹⁾ describes the combined spin population, and includes any correlations between the spins of c and d . In Refs. 5),6) and 7), it is shown how the $R_{mm';nn'}$ are obtained from the c.m. helicity amplitudes. The individual density matrices are given by

$$\rho_{mm'}^c = \sum_n R_{mm';nn} \quad (1a)$$

$$\rho_{nn'}^d = \sum_m R_{mm;n'n'} \quad (1b)$$

4.

which require the trace condition

$$\sum_{m,n} R_{mm;nn} = 1 \quad (2)$$

The combined decay angular distribution is then

$$W(\theta_c \varphi_c; \theta_d \varphi_d) = (N_c N_d)^{-1} \sum_{\substack{m m' \\ n n'}} R_{mm';nn'} X_{mm'}^{S_c}(\theta_c \varphi_c) X_{nn'}^{S_d}(\theta_d \varphi_d) \quad (3)$$

where N_c, N_d are normalizing factors, and the angular functions $X_{mm'}^{S_c}(\theta, \varphi)$ are as defined in Ref. 8). Assuming parity conservation in the production and decay processes, Pilkuhn and Svensson 5) have used the symmetries of the $R_{mm';nn'}$ and $X_{mm'}^S$ to simplify Eq. (3). Their result is

$$W(\theta_c \varphi_c; \theta_d \varphi_d) = (N_c N_d)^{-1} \left\{ 2 \sum_{m,n} R_{mm,nn} X_{mm}^{S_c}(\theta_c \varphi_c) X_{nn}^{S_d}(\theta_d \varphi_d) + 4 \sum_{\substack{m m' \\ n n'}} \text{Re}(R_{mm';nn'}) \text{Re}(X_{mm'}^{S_c}(\theta_c \varphi_c) X_{nn'}^{S_d}(\theta_d \varphi_d)) \right\} \quad (4)$$

Here the first sum takes only one of the two combinations (m,n) and $(-m,-n)$. The second sum excludes $m=m', n=n'$, and takes only one of the four combinations

$$\begin{aligned} & (m, m'; n, n') \\ & (m', m; n', n) \\ & (-m, -m'; -n, -n') \\ & (-m', -m; -n', -n) \end{aligned}$$

Only the real part of the joint density matrix appears in the decay distribution.

The individual decay distribution for one resonance is obtained by integrating the joint distribution over the decay angles of the other resonance. Using

$$\int_{-1}^1 d(\cos \theta) \int_0^{2\pi} d\phi X_{mm'}^S(\theta, \phi) = N \delta_{mm'} \quad (5)$$

we find

$$W_c(\theta_c, \phi_c) = N_c^{-1} \sum_{mm'} \rho_{mm'}^c X_{mm'}^{S_c}(\theta_c, \phi_c) \quad (6)$$

with $W_d(\theta_d, \phi_d)$ defined similarly. The joint distribution may then be written as

$$W(\theta_c, \phi_c; \theta_d, \phi_d) = W_c(\theta_c, \phi_c) W_d(\theta_d, \phi_d) + I(\theta_c, \phi_c; \theta_d, \phi_d) \quad (7)$$

where

$$I(\theta_c, \phi_c; \theta_d, \phi_d) = (N_c N_d)^{-1} \times \sum_{mm', nn'} (R_{mm', nn'} - \rho_{mm'}^c \rho_{nn'}^d) X_{mm'}^{S_c}(\theta_c, \phi_c) X_{nn'}^{S_d}(\theta_d, \phi_d) \quad (8)$$

By using the symmetries as before, we can write Eq. (8) as

$$I(\theta_c, \phi_c; \theta_d, \phi_d) = (N_c N_d)^{-1} \left\{ \begin{aligned} & 2 \sum_{mn} (R_{mm, nn} - \rho_{mm}^c \rho_{nn}^d) X_{mm}^{S_c}(\theta_c, \phi_c) X_{nn}^{S_d}(\theta_d, \phi_d) \\ & + 4 \sum_{\substack{mm' \\ nn'}} \left[(R_{mm', nn'} - \rho_{mm'}^c \rho_{nn'}^d) \right. \\ & \quad \left. \times \operatorname{Re} \left(X_{mm'}^{S_c}(\theta_c, \phi_c) X_{nn'}^{S_d}(\theta_d, \phi_d) \right) \right] \end{aligned} \right\} \quad (9)$$

where the sums are as in Eq. (4). The condition that no combined correlations occur is that $I(\theta_c, \varphi_c; \theta_d, \varphi_d) = 0$. If a linearly independent set is chosen from the $\text{Re}(x_{mm}^{sc}(\theta_c, \varphi_c) x_{nn}^{sd}(\theta_d, \varphi_d))$, the condition for the absence of correlations is that the coefficients of these terms must vanish.

III. - JOINT DECAY DISTRIBUTION FOR THE REACTION

The individual decay distributions for reaction A are given in Ref. ⁸⁾, and the explicit form of the joint decay distribution appears in Refs. ^{5), 10)}. In the latter the angular distribution is given in terms of orthogonal functions, so the conditions for the vanishing of interference effects are easily obtained. We write the expansion using the Legendre polynomial $P_2(\theta) = \frac{1}{2}(3\cos^2\theta - 1)$. The individual angular distributions are :

$$W_c(\theta_c, \varphi_c) = (1/4\pi) \left(1 + (3\rho_{00}^c - 1) P_2(\theta_c) - 3(\rho_{1-1}^c \sin^2\theta_c \cos 2\varphi_c + \sqrt{2} \operatorname{Re} \rho_{10}^c \sin 2\theta_c \cos \varphi_c) \right) \quad (10a)$$

$$W_d(\theta_d, \varphi_d) = (1/4\pi) \left(1 + (1 - 4\rho_{33}^d) P_2(\theta_d) - 2\sqrt{3}(\operatorname{Re} \rho_{3-1}^d \sin^2\theta_d \cos 2\varphi_d + \operatorname{Re} \rho_{31}^d \sin 2\theta_d \cos \varphi_d) \right) \quad (10b)$$

The interference term is decomposed as in Ref. ⁵⁾,

$$I(\theta_c, \varphi_c; \theta_d, \varphi_d) = (1/16\pi^2) \left\{ I_1(\theta_c, \theta_d) + I_2(\theta_c, \theta_d, \varphi_d) + I_3(\theta_c, \varphi_c; \theta_d) + I_4(\theta_c, \varphi_c; \theta_d, \varphi_d) \right\} \quad (11)$$

In writing the expansions of these interference terms in orthogonal functions, we use Eqs. (1) and (2) to simplify the expressions. For example,

$$I_1(\theta_c, \theta_d) = 2 P_2(\theta_c) P_2(\theta_d) \left\{ (R_{11,33} + R_{11,-3-3}) - (R_{11,11} + R_{11,-1-1}) + 2 R_{00,11} - 2 R_{00,33} - \frac{1}{2} (3\rho_{00}^c - 1)(1 - 4\rho_{33}^d) \right\}$$

(12)

From the four quantities $(R_{11;33} + R_{11;-3-3})$, $(R_{11;11} + R_{11;-1-1})$, $R_{00;11}$ and $R_{00;33}$ we can form ρ_{00}^c and ρ_{33}^d , as well as satisfy the trace condition, Eq. (2). Hence only one of these four quantities need appear explicitly, and we find

$$I_1(\theta_c; \theta_d) = 12(\rho_{00}^c \rho_{33}^d - R_{00;33}) P_2(\theta_c) P_2(\theta_d) \quad (13a)$$

Using similar methods we obtain

$$I_2(\theta_c; \theta_d \varphi) = 6\sqrt{3} P_2(\theta_c) \{ (\rho_{00}^c \operatorname{Re} \rho_{3-1}^d - \operatorname{Re} R_{00;3-1}) \sin^2 \theta_d \cos 2\varphi_d + (\rho_{00}^c \operatorname{Re} \rho_{31}^d - \operatorname{Re} R_{00;31}) \sin 2\theta_d \cos \varphi_d \} \quad (13b)$$

$$I_3(\theta_c \varphi_c; \theta_d) = 12 P_2(\theta_d) \{ (\rho_{1-1}^c \rho_{11}^d - R_{1-1;11}) \sin^2 \theta_c \cos 2\varphi_c + (1/\sqrt{2}) (2 \operatorname{Re} \rho_{10}^c \rho_{11}^d - \operatorname{Re} (R_{10;11} + R_{10;-1-1})) \sin 2\theta_c \cos \varphi_c \} \quad (13c)$$

$$I_4(\theta_c \varphi_c; \theta_d \varphi_d) = 3\sqrt{3} \{ \sin^2 \theta_c \sin^2 \theta_d [(\operatorname{Re} R_{1-1;3-1} - \rho_{1-1}^c \operatorname{Re} \rho_{3-1}^d) \cos 2(\varphi_c + \varphi_d) + (\operatorname{Re} R_{1-1;-13} - \rho_{1-1}^c \operatorname{Re} \rho_{31}^d) \cos 2(\varphi_c - \varphi_d)] + \sin^2 \theta_c \sin 2\theta_d [(\operatorname{Re} R_{1-1;31} - \rho_{1-1}^c \operatorname{Re} \rho_{31}^d) \cos(2\varphi_c + \varphi_d) + (\operatorname{Re} R_{1-1;-13} - \rho_{1-1}^c \operatorname{Re} \rho_{31}^d) \cos(2\varphi_c - \varphi_d)] + (1/\sqrt{2}) \sin 2\theta_c \sin^2 \theta_d [(\operatorname{Re} (R_{10;3-1} + R_{10;-1-3}) - 2 \operatorname{Re} \rho_{10}^c \operatorname{Re} \rho_{3-1}^d) \cos(\varphi_c + 2\varphi_d) + (\operatorname{Re} (R_{10;-13} + R_{10;-31}) - 2 \operatorname{Re} \rho_{10}^c \operatorname{Re} \rho_{31}^d) \cos(\varphi_c - 2\varphi_d)] + (1/\sqrt{2}) \sin 2\theta_c \sin 2\theta_d [(\operatorname{Re} (R_{10;31} - R_{10;-1-3}) - 2 \operatorname{Re} \rho_{10}^c \operatorname{Re} \rho_{31}^d) \cos(\varphi_c + \varphi_d) + (\operatorname{Re} (R_{10;13} - R_{10;-3-1}) - 2 \operatorname{Re} \rho_{10}^c \operatorname{Re} \rho_{31}^d) \cos(\varphi_c - \varphi_d)] \} \quad (13d)$$

In order that the individual decays be uncorrelated all of the coefficients appearing in $I_{1,2,3,4}$ must be zero. Each of these is some combination of the forms $(\text{Re}(R_{mm';nn'}) - \text{Re}\rho_{mm'}^c, \text{Re}\rho_{nn'}^d)$ and while the $\text{Re}\rho_{mm'}^c$ and $\text{Re}\rho_{nn'}^d$ are determined from the individual decay distributions, $\text{Re} R_{mm';nn'}$ is not. Furthermore $\text{Re} R_{mm';nn'}$ is not constrained by Eq. (1) so as to make any of the terms non-vanishing. Hence the combined correlations may or may not occur, independent of the individual decay distributions.

The only correlation for which definite limits can be obtained from the individual decays is $I_1(\theta_c, \theta_d)$. The coefficient appearing is $(\rho_{00}^c \rho_{33}^d - R_{00;33})$. The three quantities ρ_{00}^c , ρ_{33}^d and $R_{00;33}$ are non-negative, being diagonal density matrix elements. For $R_{00;33}$ the inequality holds :

$$0 \leq R_{00;33} \leq \min\left\{\frac{1}{2}\rho_{00}^c, \rho_{33}^d\right\} \quad (14a)$$

from which we obtain the inequality

$$\begin{aligned} -\min\{2\rho_{00}^c \rho_{33}^d, \rho_{00}^c \rho_{33}^d\} &\leq (\rho_{00}^c \rho_{33}^d - R_{00;33}) \\ &\leq \rho_{00}^c \rho_{33}^d \end{aligned} \quad (14b)$$

It is clear that while the strength of the $\cos\theta_c - \cos\theta_d$ correlation is limited by the individual decays, the correlation may be of either sign or zero, for any individual density matrices. The $\cos\theta_c - \cos\theta_d$ correlation is determined by the quantity $(\rho_{00}^c \rho_{33}^d - R_{00;33})$, and the angular function $P_2(\theta_c) P_2(\theta_d)$. This latter quantity assumes its maximum value at the points $\cos\theta_{c,d} = \pm 1$, so that the correlation effect appears most strongly at the corners of the $\cos\theta_c - \cos\theta_d$ plot, as shown for example in Ref. ²⁾. If $\rho_{00}^c \rho_{33}^d - R_{00;33}$ is positive, the corners are over-populated, compared to the product distribution $w_c(\theta_c) w_d(\theta_d)$. We refer to this situation as positive interference, and use this term henceforth.

The joint density matrix element $R_{00;33}$ represents the probability for finding particle c with spin projection 0 along the direction of the incident meson in the c rest frame and particle d with spin projection $\frac{3}{2}$ along the direction of the target nucleon in the d rest frame. Let us now consider the distribution in $\cos \theta_d$ for $|\cos \theta_c| = 1$. This is, surprisingly, independent of ρ_{33}^d and is $\propto (1 + (1 - 4R_{00;33}/\rho_{00}^c)P_2(\theta_d))$. If $R_{00;33} = 0$ this gives $1 + P_2(\theta_d)$, or a $(1 + 3\cos^2 \theta_d)$ decay distribution. It is well known that this corresponds to a spin $\frac{3}{2}$ particle with only $s_z = \pm \frac{1}{2}$ substates populated. To see how such a distribution occurs, we observe that $|\cos \theta_c| = 1$ decays can occur only from particle c states with $s_z = 0$. Thus the $\cos \theta_d$ distribution for $|\cos \theta_c| = 1$ depends only on $R_{00;11}$ and $R_{00;33}$. If $R_{00;33} = 0$, the spin $\frac{3}{2}$ particle is then restricted to have spin projections $\pm \frac{1}{2}$, and the $(1 + 3\cos^2 \theta_d)$ distribution follows for $|\cos \theta_c| = 1$. This illustration is relevant, for $R_{00;33}$ must vanish in the forward direction, as we show in the next Section.

IV. - JOINT DECAY DISTRIBUTIONS AT THE FORWARD DIRECTIONS, AND
COMPARISON WITH EXPERIMENT

The special case of production in the forward direction, i.e., $\theta_{cm} = 0^\circ$, is of interest, since the joint decay distribution is then less complicated. The spin and helicity density matrices coincide at $\theta_{cm} = 0^\circ$, and the $R_{mm';nn'}$ can be formed easily from the c.m. helicity amplitudes. For reaction A there are only three non-vanishing independent helicity amplitudes at $\theta_{cm} = 0^\circ$, which we take to be $f_{1\frac{3}{2};\frac{1}{2}}$, $f_{0\frac{1}{2};\frac{1}{2}}$ and $f_{-1-\frac{1}{2};\frac{1}{2}}$ ¹²⁾. Constructing the individual spin density matrices we find that at $\theta_{cm} = 0^\circ$ the off-diagonal elements vanish and that

$$\rho_{00}^c = (2/N) \left| f_{0\frac{1}{2};\frac{1}{2}} \right|^2 \quad (15a)$$

$$\rho_{33}^d = (1/N) \left| f_{1\frac{3}{2};\frac{1}{2}} \right|^2 \quad (15b)$$

$$N = 2 \left(\left| f_{0\frac{1}{2};\frac{1}{2}} \right|^2 + \left| f_{1\frac{3}{2};\frac{1}{2}} \right|^2 + \left| f_{-1-\frac{1}{2};\frac{1}{2}} \right|^2 \right) \quad (15c)$$

For the joint density matrix we find $R_{00;33} = 0$, and of the elements appearing in Eqs. (13b), (13c), (13d) the only non-vanishing ones are

$$R_{10;31} = f_{1\frac{3}{2};\frac{1}{2}} f_{0\frac{1}{2};\frac{1}{2}}^* / N \quad (16a)$$

$$R_{1-1;3-1} = f_{1\frac{3}{2};\frac{1}{2}} f_{-1-\frac{1}{2};\frac{1}{2}}^* / N \quad (16b)$$

The interference terms I_j have the following forms for $\theta_{cm} = 0^\circ$:

$$I_1(\theta_c; \theta_d) = 12 \rho_{00}^c \rho_{33}^d P_2(\theta_c) P_2(\theta_d) \quad (17a)$$

$$I_2(\theta_c; \theta_d \phi_d) = 0 \quad (17b)$$

$$I_3(\theta_c, \varphi_c; \theta_d) = 0 \quad (17c)$$

$$I_4(\theta_c, \varphi_c; \theta_d, \varphi_d) = 3\sqrt{3} \left(\text{Re } R_{1-1;3-1} \sin^2 \theta_c \sin^2 \theta_d \cos 2(\varphi_c + \varphi_d) \right. \\ \left. + (1/\sqrt{2}) \text{Re } R_{10;31} \sin 2\theta_c \sin 2\theta_d \cos(\varphi_c + \varphi_d) \right) \quad (17d)$$

From Eqs. (15) and (16) we can obtain the inequalities :

$$|\text{Re } R_{10;31}| \leq \left[\frac{1}{2} \rho_{00}^c \rho_{33}^d \right]^{1/2} \quad (18a)$$

$$|\text{Re } R_{1-1;3-1}| \leq \left[\frac{1}{2} (1 - \rho_{00}^c - 2\rho_{33}^d) \rho_{33}^d \right]^{1/2} \quad (18b)$$

It is apparent that for the forward production direction, only the correlations of the form $\theta_c = \theta_d$ and those including $(\varphi_c + \varphi_d)$ are not required to vanish from kinematic arguments.

The fact that correlations involving the angle $(\varphi_c + \varphi_d)$ need not vanish at $\theta_{cm} = 0^\circ$ can be understood by considering the definitions of φ_c and φ_d . These angles are defined in the particle c and particle d rest frames respectively as the azimuthal angles of the decays, with the production plane chosen as the x-z plane. At $\theta_{cm} = 0^\circ$, the production plane degenerates into a line, and the angles φ_c and φ_d are no longer defined. In order that the joint decay distribution be well defined at all production angles, the various coefficients of φ_c and φ_d vanish at $\theta_{cm} = 0^\circ$. However, the angle $(\varphi_c + \varphi_d)$ can be defined in the exact forward direction without reference to the production plane, since it is the dihedral angle between the decay planes of c and d, as seen, for example, in the c.m. frame.

A sketch of the decay configuration for the reaction $\pi^+p \rightarrow \rho^0 N^{*++} \rightarrow (\pi^+\pi^-\pi^+\pi^+)$ at $\theta_{cm} = 0^\circ$ is shown in Fig. 2. If the decay angles (θ_c, φ_c) are taken as those of the π^+ in the ρ^0 frame, with (θ_d, φ_d) those of the proton in the N^* frame, then at $\theta_{cm} = 0^\circ$, $(\varphi_c + \varphi_d)$ is the planar angle between the half-plane containing the π^+ and the half-plane containing the proton. Since $(\varphi_c + \varphi_d)$ is well defined at all production angles the corresponding coefficients in the joint decay distribution are not constrained to vanish at $\theta_{cm} = 0^\circ$. This is just the result found from the analysis of the helicity amplitudes at $\theta_{cm} = 0^\circ$ (13).

By considering the special case of $\theta_{cm} = 0^\circ$ we have obtained some restrictions on the joint decay distribution. The $\cos \theta_c - \cos \theta_d$ correlation is determined uniquely from the individual decay distributions, and is necessarily of the positive type which favours the values $|\cos \theta_c| = |\cos \theta_d| = 1$. It is possible for the quantity $\rho_{00}^c \rho_{33}^d$ to vanish in the forward direction, but it is not necessary. The other restriction is that only the terms involving $(\varphi_c + \varphi_d)$ can occur at $\theta_{cm} = 0^\circ$, and all others must vanish. From the inequalities (18), the strength of these $(\varphi_c + \varphi_d)$ terms is limited by the individual density matrix elements. Furthermore if one assumes that the helicity amplitudes have a common phase, the inequalities become equalities, and the strength of the $\cos(\varphi_c + \varphi_d)$ and $\cos 2(\varphi_c + \varphi_d)$ terms in Eq. (13d) is determined from the individual decay distributions. The assumption of common phase is by no means necessary, but it is made, for example, in single particle exchange models, even with absorption.

These forward angle results are purely kinematical ; to use them we must make the additional assumption of peripheralism. This means that the reaction amplitudes are appreciable only at low momentum transfer, and that the density matrix elements are slowly varying with production angle in the diffraction peak. By such an assumption the exact kinematic results at $\theta_{cm} = 0^\circ$ are taken to be approximately valid at small production angles. This assumption, that reaction amplitudes which vanish kinematically in the forward direction do not become large at

small angles, is dynamical, and may be wrong. But a check upon such an assumption is provided by the behaviour of the differential cross-section and density matrix elements at small production angle. If these show no sharp variation with angle, aside from the diffraction peak of the cross-section, the assumption may be correct. A reaction which satisfies these conditions is $\pi^+ p \rightarrow \rho^0 N^{*++}$, and we attempt to explain the experimental results in terms of forward angle predictions and peripheralism.

The experimental data on the reaction $\pi^+ p \rightarrow \rho^0 N^*(\pi^+ \pi^- \pi^+ p)$ at 8 GeV/c have been reported by the Aachen-Berlin-CERN collaboration ^{2), 14)}. Following the method of Goldhaber et al. ¹⁾ the combined correlation effect was observed by classifying events as polar or equatorial in $\cos \theta_c$ if $|\cos \theta_c| \geq$ or ≤ 0.4 . The $\cos \theta_d$ distributions for these two classes are compared; it is found that the $\cos \theta_d$ distribution for equatorial c's is nearly flat, while that for polar c's is enhanced near $|\cos \theta_d| = 1$. Similar results are observed when the $\cos \theta_c$ distributions for polar and equatorial $\cos \theta_d$ are compared. In order to see how these effects reflect the interference term $I_1(\theta_c; \theta_d)$ of Eq. (13a) we reduce Eq. (7) to the θ_c, θ_d distributions by integrating over φ_c and φ_d . This yields

$$W(\theta_c; \theta_d) = W_c(\theta_c) W_d(\theta_d) + 1/4 I_1(\theta_c; \theta_d) \quad (19a)$$

where

$$W_c(\theta_c) = \frac{1}{2} (1 + (3\rho_{00}^c - 1) P_2(\theta_c)) \quad (19b)$$

$$W_d(\theta_d) = \frac{1}{2} (1 + (1 - 4\rho_{33}^d) P_2(\theta_d)) \quad (19c)$$

Defining $\delta = 12(\rho_{00}^c \rho_{33}^d - R_{00;33})$ we can integrate Eq. (19a) over polar and equatorial choices in θ_c and θ_d separately, to obtain the following distributions.

Polar d

$$\text{Equatorial } d \quad w_{c,p}(\theta_c) = Z_d W_c(\theta_c) + 0.084 \delta P_2(\theta_c) \quad (20a)$$

$$w_{c,e}(\theta_c) = (1-Z_d) W_c(\theta_c) - 0.084 \delta P_2(\theta_c) \quad (20b)$$

Polar c

$$\text{Equatorial } c \quad w_{d,p}(\theta_d) = Z_c W_d(\theta_d) + 0.084 \delta P_2(\theta_d) \quad (20c)$$

$$w_{d,e}(\theta_d) = (1-Z_c) W_d(\theta_d) - 0.084 \delta P_2(\theta_d) \quad (20d)$$

The quantity Z is $2 \int_{0.4}^{1.0} d(\cos \theta) W(\theta)$. Explicitly $Z_c = (0.6 + 0.168(3\rho_{00}^c - 1))$ and $Z_d = (0.6 + 0.168(1 - 4\rho_{33}^d))$. Since the ρ^0 decay distributions for equatorial N^* are less peaked near $|\cos \theta_c| = 1$ than those for polar N^* , the quantity δ is positive, in agreement with the forward angle result that $R_{00;33}$ is zero. The experimental values for ρ_{00}^c and ρ_{33}^d , averaged over $0 \leq -t \leq 0.6 \text{ (GeV/c)}^2$ are 0.77 ± 0.04 and 0.05 ± 0.03 .

The differential cross-section for this reaction shows a strong diffractive peak, and no anomalous behaviour near $\theta_{cm} = 0^\circ$. The individual density matrix elements are also well behaved, the diagonal elements being nearly constant at the above-mentioned values, and the off diagonal elements remaining very small. Hence it may be hoped that the forward angle predictions should be valid. To test this we assume $\langle R_{00;33} \rangle = 0$, for the momentum transfer interval used. The $\theta_c - \theta_d$ correlation is then determined by the individual decays, and the following numerical results are obtained for the four distributions

$$w_{c,p}(\theta_c) = \frac{1}{2} (0.73 + 0.94 P_2(\theta_c))$$

$$w_{c,e}(\theta_c) = \frac{1}{2} (0.27 + 0.27 P_2(\theta_c))$$

$$w_{d,p}(\theta_d) = \frac{1}{2} (0.82 + 0.73 P_2(\theta_d))$$

$$w_{d,e}(\theta_d) = \frac{1}{2} (0.18 + 0.07 P_2(\theta_d))$$

The central experimental values $\langle \rho_{00}^c \rangle = 0.77$ and $\langle \rho_{33}^d \rangle = 0.05$ have been used to obtain the numerical results. The comparison of these results with the experimental data is shown in Fig. 3. To obtain all four curves only one normalization, to the total number of events, has been made. It is clear that the assumption $\langle R_{00;33} \rangle = 0$ gives good agreement with the four experimental distributions, both in shape and relative normalization.

The success of the simple forward angle prediction for the $\theta_c - \theta_d$ correlation effect suggests that the other forward angle predictions for correlations be tested. These are that all correlations except those involving $(\varphi_c + \varphi_d)$ are small, and that the inequalities (18) hold approximately near the forward direction. Using the experimental numbers for $\langle \rho_{00}^c \rangle$ and $\langle \rho_{33}^d \rangle$ these inequalities yield

$$\begin{aligned} |Re R_{10;31}| &\leq 0.15 \\ |Re R_{1-1;3-1}| &\leq 0.06 \end{aligned}$$

It is apparent that the combination of forward angle predictions and experimental individual decays allows the coefficients of $(\varphi_c + \varphi_d)$ terms to be appreciable, particularly the quantity $R_{10;31}$. This corresponds to the $\sin 2\theta_c \sin 2\theta_d \cos(\varphi_c + \varphi_d)$ term in $I_4(\theta_c \varphi_c; \theta_d \varphi_d)$. Thus it should be possible to determine experimentally the strength of these correlations, and compare them with the forward angle predictions. The preliminary results ⁴⁾ show that the correlations not involving $(\varphi_c + \varphi_d)$ are small, while the result for $Re R_{10;31}$ is $\simeq -0.04 \pm 0.05$ for $0 \leq -t \leq 0.6 \text{ (GeV/c)}^2$. Thus the inequalities are satisfied, and the other joint density matrix elements are small, as expected.

V. - THE $\theta_c - \theta_d$ CORRELATIONS FOR THE REACTION

$$\underline{0^- + \frac{1}{2}^+ \rightarrow 2^+ + \frac{3}{2}^+ \rightarrow (0^- 0^- ; \frac{1}{2}^+ 0^-)}$$

In this reaction the joint density matrix is 20×20 ; by hermiticity and parity conservation the number of independent matrix elements is reduced to 100. The general form for the combined decay distribution is given by Eq. (4). Because of the large number of parameters we shall limit the discussion to the $\theta_c - \theta_d$ correlation. For the decay distribution of a spin 2 meson into two spinless mesons we use the form given by Högaasen et al. ¹⁵⁾. As before we obtain the $\theta_c - \theta_d$ distribution by integrating the general form over ϕ_c and ϕ_d . This gives Eq. (19a), with $W_d(\theta_d)$ again given by Eq. (19b). But in this reaction we obtain for $W_c(\theta_c)$ and $I_1(\theta_c; \theta_d)$

$$W_c(\theta_c) = \frac{1}{2} \left(1 + \left(\frac{5}{7} \right) (1 + \rho_{00}^c - 6 \rho_{22}^c) P_2(\theta_c) + \left(\frac{6}{7} \right) (5 \rho_{00}^c - 2 + 5 \rho_{22}^c) P_4(\theta_c) \right) \quad (21a)$$

$$I_1(\theta_c; \theta_d) = (\alpha P_2(\theta_c) + \beta P_4(\theta_c)) P_2(\theta_d) \quad (21b)$$

The quantities α and β can be expressed in terms of the diagonal elements $R_{mm;nn}$, namely :

$$\begin{aligned} \alpha = \left(\frac{5}{7} \right) \{ & -4((R_{22;11} + R_{22;-1-1}) - (R_{22;33} + R_{22;-3-3})) \\ & + 2((R_{11;11} + R_{11;-1-1}) - (R_{11;33} + R_{11;-3-3})) \\ & + 4 R_{00;11} - 4 R_{00;33} \\ & - (1 + \rho_{00}^c - 6 \rho_{22}^c)(1 - 4 \rho_{33}^d) \} \end{aligned} \quad (22a)$$

$$\begin{aligned}
\beta = & (6/7) \{ (R_{22;11} + R_{22;-1-1}) - (R_{22;33} + R_{22;-3-3}) \\
& - 4((R_{11;11} + R_{11;-1-1}) - (R_{11;33} + R_{11;-3-3})) \\
& + 6 R_{00;11} - 6 R_{00;33} \\
& - (5\rho_{00}^c - 2 + 5\rho_{22}^c)(1 - 4\rho_{33}^d) \} \quad (22b)
\end{aligned}$$

There are six combinations of the $R_{mm;nn}$ appearing, and from these we can form ρ_{00}^c , ρ_{22}^c , and ρ_{33}^d , as well as satisfy the trace condition, Eq. (2). Choosing $R_{00;33}$ and $(R_{22;11} + R_{22;-1-1})$ as independent, we find that α and β simplify to the following

$$\begin{aligned}
\alpha = & (10/7) (2(\rho_{00}^c \rho_{33}^d - R_{00;33}) \\
& + 6(2\rho_{22}^c \rho_{11}^d - R_{22;11} - R_{22;-1-1})) \quad (23a)
\end{aligned}$$

$$\begin{aligned}
\beta = & (60/7) (2(\rho_{00}^c \rho_{33}^d - R_{00;33}) \\
& - (2\rho_{22}^c \rho_{11}^d - R_{22;11} - R_{22;-1-1})) \quad (23b)
\end{aligned}$$

It is clear that in order that the $\theta_c - \theta_d$ distribution be uncorrelated, the quantities α and β , and hence $(\rho_{00}^c \rho_{33}^d - R_{00;33})$ and $(2\rho_{22}^c \rho_{11}^d - R_{22;11} - R_{22;-1-1})$ must vanish. Since all of the ingredients are non-negative, we obtain the inequalities

$$\begin{aligned}
-\min \{ \rho_{33}^d (1 - \rho_{00}^c), \rho_{00}^c \rho_{11}^d \} & \leq (\rho_{00}^c \rho_{33}^d - R_{00;33}) \\
& \leq \rho_{00}^c \rho_{33}^d \quad (24a)
\end{aligned}$$

$$\begin{aligned}
-\min \{ \rho_{22}^c \rho_{33}^d, \rho_{11}^d (1 - 2\rho_{22}^c) \} & \\
\leq (2\rho_{22}^c \rho_{11}^d - R_{22;11} - R_{22;-1-1}) & \leq 2\rho_{22}^c \rho_{11}^d \quad (24b)
\end{aligned}$$

These show that limits for the interference terms can be obtained from individual θ_c and θ_d distributions, but that neither the signs nor the magnitudes within the limits are determined. Thus we find again that the existence of the correlations is independent of the individual decay distributions.

Just as for reaction A, the special case of $\theta_{cm} = 0^\circ$ is of interest, since $R_{00;33}$, $R_{22;11}$ and $R_{22;-1-1}$ must vanish. We thus obtain, at $\theta_{cm} = 0^\circ$,

$$I_1(\theta_c; \theta_d) = \left\{ (10/7) (2 \rho_{00}^c \rho_{33}^d + 12 \rho_{22}^c \rho_{11}^d) P_2(\theta_c) \right. \\ \left. + (60/7) (2 \rho_{00}^c \rho_{33}^d - 2 \rho_{22}^c \rho_{11}^d) P_4(\theta_c) \right\} P_2(\theta_d) \quad (25)$$

The coefficient α is thus non negative, while β can be positive or negative. At the values $|\cos \theta_c| = |\cos \theta_d| = 1$, the $\rho_{22}^c \rho_{11}^d$ terms for $P_2(\theta_c)$ and $P_4(\theta_c)$ cancel out, and one obtains $I_1 = 20 \rho_{00}^c \rho_{33}^d$ as the necessarily non-negative contribution of the interference term. For reaction A with $\theta_{cm} = 0^\circ$ and $|\cos \theta_c| = |\cos \theta_d| = 1$, the interference term had the value $12 \rho_{00}^c \rho_{33}^d$. Thus we find a purely kinematic result that the two-body collinear production configuration ($\theta_{cm} = 0^\circ$) favours the decay into the four-body collinear final state ($|\cos \theta_c| = |\cos \theta_d| = 1$).

The experimental information on the reaction $\pi^+ p \rightarrow f^0 N^{*++} (\pi^+ \pi^- \pi^+ p)$ ^{2), 10)} is not sufficiently accurate at present to permit the use of the forward direction predictions for combined correlations. Experimentally the quantity ρ_{22}^c is found to be negative, whereas it is necessarily non-negative, being a diagonal density matrix element. This difficulty was recognized in Ref. ²⁾, and it was suggested that an S wave background was responsible for the negative ρ_{22}^c value. The lack of reasonable individual density matrix elements

prevents us from using the forward angle results to predict the possible correlation effects. However, we can make some plausible assumptions as to the behaviour of the individual density matrices, and obtain at least qualitative predictions for the combined correlation effect. The assumptions are based on an absorption model calculation we have made for the $f^0 N^*$ reaction at 8 GeV/c, assuming pion exchange. The predicted differential cross-section is not in good agreement with the data, being too large and too broad in angle. But the density matrix element predictions are reasonably good, and in particular ρ_{22}^c is predicted to be quite small near the forward direction, $\simeq 0.01$. The experimental value of ρ_{00}^c is 0.85 ± 0.1 and ρ_{33}^d is 0.04 ± 0.04 . To make our sample calculation we take $\rho_{00}^c = 0.8$, $\rho_{33}^d = 0.04$, $\rho_{22}^c = 0$ and assume that the quantity $R_{00;33} = 0$. Since we have assumed $\rho_{22}^c = 0$, the $R_{22;11}$ and $R_{22;-1-1}$ must also be zero, and the correlation terms α and β depend only on $\rho_{00}^c \rho_{33}^d - R_{00;33}$.

The joint $\theta_c - \theta_d$ distribution corresponding to these values of ρ_{00}^c , ρ_{33}^d , etc., is shown in Fig. 4a, in the form of a contour map of $W(\theta_c, \theta_d)$ in the $\cos \theta_c - \cos \theta_d$ plane. Only one quadrant is shown, the others being mirror images. In Fig. 4b, the sections of this distribution for $\cos \theta_d$ fixed at 1.0, 0.5 and 0 are shown, while in Fig. 4c the sections for fixed $\cos \theta_c$ are displayed. From these figures the nature of the correlation is clear, and it is apparent that it is more complicated than in the spin 1- spin $\frac{3}{2}$ case. If one considers the θ_c distributions as $\cos \theta_d$ varies from 0 to 1, they are seen to change uniformly, becoming progressively more peaked. This is not true of the θ_d distributions for varying $\cos \theta_c$. At $\cos \theta_c \simeq 0.5$ the θ_d distribution is nearly isotropic, while at the points $|\cos \theta_c| = 1$ and 0 the θ_d distributions are both $\propto (1 + 3\cos^2 \theta_d)^{16}$.

The gross features of the joint distribution shown here are not sensitive to small changes in the quantities ρ_{00}^c etc., so that the predictions should be in qualitative agreement with the

experimental evidence. One important consequence of these predictions is that the usual method of analysis will not be very effective for this decay. The usual classification of decay angles as polar or equatorial corresponds approximately to $P_2(\theta)$ positive or negative. For the spin 1 - spin $\frac{3}{2}$ decay the interference term is $\sim P_2(\theta_c) P_2(\theta_d)$ so the polar and equatorial distributions were augmented and diminished respectively by a nearly maximal interference term. In the spin 2 - spin $\frac{3}{2}$ reaction the interference term is $(\alpha P_2(\theta_c) + \beta P_4(\theta_c) P_2(\theta_d))$. To use the polar and equatorial choice in θ_d is still reasonable, since only $P_2(\theta_d)$ occurs. For θ_c the $P_4(\theta_c)$ term is also present, and the polar and equatorial choices are no longer so useful. From Fig. 4a, it is seen that the distributions in $\cos \theta_d$ for $0.3 \leq |\cos \theta_c| \leq 0.7$ are quite different from those for $0 \leq \cos \theta_c \leq 0.3$ and $0.7 \leq \cos \theta_c \leq 1.0$. The polar and equatorial choices are not sensitive to this feature, and thus lose much of the information. The best method to observe the correlation effect is to fit the experimental distribution in (θ_c, θ_d) using Eq. (21), and thus obtaining α and β directly.

The experimental results at 8 GeV/c confirm qualitatively the presence of correlations, but the problem of background makes the analysis quite difficult. Possible with better statistics a reasonable value for ρ_{22}^c may be obtained, and an analysis of the combined correlation effect may yield numerical values for α and β .

VI. - ABSORPTION MODEL PREDICTIONS AND SUGGESTIONS FOR
EXPERIMENTAL ANALYSES

In Refs. 2) and 3) it is found that the $\cos\theta_c - \cos\theta_d$ correlation effect in ρN^* and $K^* N^*$ production is quantitatively explained by the absorption model using single pion exchange. The model gives accurate predictions for ρ_{00}^c and ρ_{33}^d , and $R_{00;33}$ is predicted to be small, typically $\simeq 0.005$ at small production angles. From our results of Section IV, it is clear that the absorption model predictions for the $\cos\theta_c - \cos\theta_d$ correlation will be correct, since the data require $R_{00;33}$ to be small. In the peripheral model using pion exchange without absorption, ρ_{33}^d is 0, and no correlation can occur. The effect of including absorption is to produce a non-zero ρ_{33}^d , while the Adair argument keeps $R_{00;33}$ small. Hence the success of the absorption model with the $\cos\theta_c - \cos\theta_d$ correlation is a consequence of its success with the individual $\cos\theta_c$ and $\cos\theta_d$ distributions.

In order to obtain a stringent test of the absorption model predictions, the correlation terms involving $(\varphi_c + \varphi_d)$ should be measured. If the forward angle predictions are valid at small production angles, the inequalities (18) show that non-negligible values for $R_{10;31}$ and $R_{1-1;3-3}$ may occur. In the absorption model, as used in Ref. 6), all amplitudes have a common phase, and the inequalities hold as equalities. The absorption model predictions 7) for ρN^* at 8 GeV/c are that $\text{Re } R_{10;31}$ and $\text{Re } R_{1-1;3-1}$ are slowly varying near $\theta_{cm} = 0^\circ$, with the values $\text{Re } R_{10;31} \simeq -0.15$ and $\text{Re } R_{1-1;3-1} \simeq 0.05$. The preliminary experimental result for $\text{Re } R_{10;31}$ is -0.04 ± 0.05 , so that the absorption model prediction, while of the right sign, is too large in magnitude. The experimental value comes from a determination of $\langle \sin 2\theta_c \sin 2\theta_d \cos(\varphi_c + \varphi_d) \rangle$, and it is difficult to estimate how the asymmetry in the ρ^0 decay will affect the measured result. Should the measured result be correct it would show that the absorption model, while providing a good approximation to the amplitudes, is not

correct in detail. In particular the common phase assumption would be wrong, and this would be of interest to other models which assume relatively real amplitudes.

A reaction for which the study of combined correlations might prove fruitful is $\pi^+p \rightarrow \omega N^*$. If (θ_c, φ_c) are taken to be the angles of the normal to the ω decay plane as seen in the ω rest frame, the analysis of Section III holds. But the discussion of Section IV may be less relevant, since the ωN^* cross-section at 8 GeV/c has a broad distribution in production angle¹⁴⁾. Hence the forward angle predictions may not be typical of the reaction, and many combined correlations may be important. It is known that the absorption model predictions for this reaction, using ρ exchange, fail to describe the differential cross-section, and are not very good for the density matrices^{2),14)}. Since neither forward direction results nor absorption model predictions are reliable, it is difficult to estimate what the correlations will be. Their measurement will provide useful information on the production mechanism.

We wish to suggest that the measurements of combined correlation effects become a standard part of the analysis of double resonance production, and that such measurements become more quantitative in character. In the original analysis of Goldhaber et al.¹⁾, the search was performed for correlations between pairs of decay angles : (θ_c, θ_d) , (θ_c, φ_d) , (φ_c, θ_d) and (φ_c, φ_d) . The (θ_c, θ_d) effect was observed, while the others were not found. We have seen that the (θ_c, θ_d) correlation is a consequence of $R_{00;33}$ being small. From Eqs. (13b), (13c), (13d), we find that the (θ_c, φ_d) correlation depends on $(\rho_{00}^c \text{Re} \rho_{3-1}^d - \text{Re} R_{00;3-1})$, the (φ_c, θ_d) on $(\rho_{1-1}^c \rho_{11}^d - \text{Re} R_{1-1;11})$ and the (φ_c, φ_d) on $(\text{Re} R_{1-1;3-1} - \rho_{1-1}^c \text{Re} \rho_{3-1}^d)$ and $(\text{Re} R_{1-1;-13} - \rho_{1-1}^c \text{Re} \rho_{3-1}^d)$. Of these, only $\text{Re} R_{1-1;3-1}$ is not required to vanish at $\theta_{cm} = 0^\circ$, and from the inequality (18b), it is expected to be small, ≈ 0.05 . The correlations other than $\theta_c - \theta_d$ are not easily visualized, since three or four angles are involved, but they can be determined by standard analyses. The measurement of each combined correlations provides a numerical result for some $\text{Re}(R_{mm';nn'})$ and thus a test for any theoretical model.

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- 9) The $R_{mm';nn'}$ are functions of production angle θ_{cm} and energy, but for simplicity we suppress this dependence. The $R_{mm';nn'}$ are the same as the $\rho_{nn'}^{mm'}$ of Ref. 5) and the $\langle m'n' | \rho | mn \rangle$ of Ref. 6).
- 10) There is an error in Eq. (37h) of Ref. 5) ; the $\sin 2 \theta_Q$ terms should be multiplied by an additional $(1/\sqrt{2})$.

- 11) It should be noted that not all of angular functions appearing in $I_{1,2,3,4}$ are orthogonal to the angular functions occurring in the product $w_c(\theta_c, \varphi_c) \times w_d(\theta_d, \varphi_d)$. Thus while our form for the joint distribution, Eq. (7), is useful in understanding the correlation effect, the explicit form given in Ref. 5) is more practical for obtaining experimental values of the $R_{mm';nn'}$.
- 12) Three other amplitudes are related to these by parity. The notation is $f_{\lambda_c \lambda_d; \lambda_b}$ as in M. Jacob and G.C. Wick - Ann.Phys. 7, 404 (1959).
- 13) The two arguments are equivalent, one invoking conservation of angular momentum along the incident beam direction, the other using the invariance of the angle $(\varphi_c + \varphi_d)$ under rotations with the incident direction as axis.
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- 16) This result is not entirely general. It is true that if $R_{00;33} = 0$ the θ_d distribution for $|\cos \theta_c| = 1$ is $\propto (1+3\cos^2 \theta_d)$, independent of the individual density matrix. In the present case we have also chosen $\rho_{22}^c = 0$, and this gives the additional condition that at $\cos \theta_c = 0$, the θ_d distribution is $\propto (1+3\cos^2 \theta_d)$. These results follow from noting that a $|\cos \theta_c| = 1$ decay occurs only from a state with $s_z = 0$, while the $\cos \theta_c = 0$ decay can be from $s_z = 0$ or ± 2 . By choosing $\rho_{22}^c = 0$ we rule out $s_z = \pm 2$ states, and with $R_{00;33} = 0$ only the N^* states with $s_z = \pm \frac{1}{2}$ occur. Hence for the $|\cos \theta_c| = 1, 0$ decays, our assumptions require the N^* decay to be $\propto (1+3\cos^2 \theta_d)$.

FIGURE CAPTIONS

- Figure 1 The definition of the angles (θ, ϕ) made by the decay products of particle c in its rest frame.
- Figure 2 The decay configuration for the reaction $\rho^{N*} \rightarrow \pi^+ \pi^- \pi^+ p$ in the c.m. at $\theta_{cm} = 0^\circ$. The quantity $(\phi_c + \phi_d)$ is the angle between the planes and is well defined, even though ϕ_c and ϕ_d are not.
- Figure 3 The experimental distributions in $\cos \theta_c$ and $\cos \theta_d$ for the reaction $\pi^+ p \rightarrow \rho^{0N*++}$ at 8 GeV/c, together with the curves corresponding to the experimental values for ρ_{00}^c and ρ_{33}^d (0.77 and 0.05 resp.), with $R_{00;33} = 0$. The data are taken from Ref. 2).
- Figure 4 a) - The contour map of the distribution $W(\theta_c, \theta_d)$ in the $\cos \theta_c - \cos \theta_d$ plane for the reaction $0^- + \frac{1}{2}^+ \rightarrow 2 + \frac{3}{2}^- \rightarrow (0^- 0^- \frac{1}{2}^+ 0^-)$. The parameters used are $\rho_{22}^c = 0$, $\rho_{00}^c = 0.8$, $\rho_{33}^d = 0.04$, and $R_{00;33}$, $R_{22;11}$ and $R_{22;-1-1}$ are 0.
- b) - Distributions in θ_c for $\cos \theta_d = 1, 0.5$ and 0 , with the parameters as in Fig. 4a.
- c) - Distributions in θ_d for $\cos \theta_c = 1, 0.5$ and 0 , with the parameters as in Fig. 4a.

FIG. 1

FIG. 2

FIG. 3

FIG.4