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REGGEIZATION OF HELICITY AMPLITUDES

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A B S T R A C T

A method of Reggeizing helicity amplitudes is investigated. The usual P-type functions $d_{\lambda,\mu}^j(z)$ which occur in the Jacob and Wick helicity formalism are replaced by the Q-type functions $e_{\lambda,\mu}^j(z)$ in terms of which Mandelstam's method of shifting the Sommerfeld-Watson contour into the left half-plane can be applied in analogy to the spinless case. The "non-sense channel" contributions are studied in detail especially with respect to the fixed singularities of the partial wave amplitudes. The modification of the formulae due to possible branch cuts in the j plane are given, and the behaviour of the Regge pole contributions to the helicity amplitudes is discussed when a trajectory passes through a "nonsense" value of angular momentum.

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Some years ago, Calogero, Charap and Squires ¹⁾ derived a generalized Froissart-Gribov formula for the partial wave amplitude for a two-body reaction involving particles with spin. Starting from the helicity representation ²⁾ of the scattering amplitude and taking out kinematical singularities which are introduced by an expansion in terms of d^j functions, the authors assumed a fixed s dispersion relation in $\cos\theta$ for a helicity amplitude free of kinematical singularities ^{*)} and showed that analytic continuation in j is possible, provided signature is introduced appropriately. The question of allowing a Sommerfeld-Watson transformation was discussed and answered positively [in the case of potential scattering for particles with spin the proof was given in Born approximation by Charap and Squires ⁵⁾]. Independently of these investigations the problem of Reggeizing helicity amplitudes was discussed at the same time by Gell-Mann, Goldberger, Low, Marx and Zachariasen ⁶⁾ who started from parity conserving helicity amplitudes free of kinematical singularities which were then Reggeized in analogy to the spinless case. In this paper we do not introduce parity conserving helicity amplitudes. Since we have in mind the application to situations where cuts in the angular momentum plane might contribute, and since the exchange of two or more Regge poles cannot be associated with the exchange of a system having definite parity ⁷⁾, whereas signature still has a meaning in this case ⁸⁾ we shall follow here more closely the presentation given in Ref. ¹⁾. Our treatment differs, however, from the one given in Ref. ¹⁾ in the respect that we introduce Q -type functions $e^j_{\lambda,\mu}(z)$ instead of the usual P -type functions $d^j_{\lambda,\mu}(z)$. The functions $e^j_{\lambda,\mu}(z)$ which for physical j are expressible in terms of Jacobi polynomials of the second kind in the same way as the $d^j_{\lambda,\mu}(z)$ are expressible in terms of Jacobi polynomials of the first kind were investigated extensively by Andrews and Gunson ⁹⁾ ^{**)} . The use of these

*) Compare also the work of Hara ³⁾ and Wang ⁴⁾ in this connection.

**) The $e^j_{\lambda,\mu}(z)$ (cf., the Appendix) should not be confused with the $e^{j\pm}_{\lambda,\mu}(z)$ of Ref. ⁶⁾ which are of P -type.

functions enables us to carry through the Reggeization procedure for a general two-body reaction involving particles with spin in complete analogy to Mandelstam's method ¹⁰⁾ for the spinless case. The introduction of the functions $e^j_{\lambda,\mu}(z)$ and the direct Reggeization of the helicity amplitudes in terms of them (with all complications due to the spins of the external particles being present) allows also a better insight into the difficulties due to the "nonsense channels". We do not discuss in this paper in great detail all questions for justifying the performance of a Sommerfeld-Watson transformation on the amplitude since this has already been done in Ref. ¹⁾ and in earlier papers by the same authors ^{5), 11)}.

In Section 2 we review shortly the work of Calogero, Charap and Squires, introduce our notation and formulate the problem in terms of the Q -type functions $e^j_{\lambda,\mu}(z)$ of Andrews and Gunson. In Section 3 the Sommerfeld-Watson transformation is carried out, and the cancellation of various terms is discussed which arise when the integration contour is shifted through the "sense-nonsense" (sn) and "nonsense-nonsense" (nn) regions into the left half-plane. Our discussion lacks rigour in the respect that we do not give a general proof for the required symmetry properties of the actual physical partial wave amplitudes continued down to these "nonsensical" j values on the real axis symmetrical to $j = -\frac{1}{2}$. But we use in a heuristic way similar to the discussion in Ref. ⁶⁾ the generalized Froissart-Gribov formula for deriving these relations. We mention briefly how such a proof could in principle be obtained from the N/D equations.

The generalized Froissart-Gribov formula leads, when continued below $\lambda_{\max}$ (maximum helicity) to singularities at fixed values of j . These singularities of the partial wave amplitudes at "nonsense" values of angular momentum have been discussed by Azimov ¹²⁾. They correspond to the Gribov-Pomeranchuk singularities ¹³⁾ in the spinless case. The problem of fixed singularities in the j plane has recently attracted renewed attention ^{14), 15), 16)}. It was shown in Refs. ¹⁴⁾ and ¹⁵⁾ that in the presence of cuts in the j plane ^{17), 18)} the

Gribov-Pomeranchuk mechanism leading to an accumulation of poles at integer or half-integer j values below $\lambda_{\max}$ does not operate ^{*)}. The authors showed that the analytically continued, signed, partial wave amplitudes $F_{\lambda_c \lambda_d; \lambda_a \lambda_b}^{\pm}(s, j)$ - although having no essential singularities - do have simple poles in the "nonsense-nonsense" region and one-over-square-root singularities in the "sense-nonsense" region at "wrong" signature points, due to the fact that the left-hand discontinuity in s of $F_{\lambda_c \lambda_d; \lambda_a \lambda_b}^{\pm}(s, j)$ has such singularities at these j values. It is shown in Section 3 that these singularities of the partial wave amplitudes are not present in the full amplitude continued in the angular momentum plane. This result is due to the fact that the product $F_{\lambda_c \lambda_d; \lambda_a \lambda_b}^{\pm}(s, j) e^{\xi_{-\lambda, \mu}^{\pm}(-z, -j-1)}$ ^{**)} occurring in the expansion of the helicity amplitudes is regular at sn points and has simple poles of opposite residues for opposite signature at nn points with $(j-\lambda)$ integer. Hence, there will be no "Kronecker-delta" contribution picked up in the expressions for the helicity amplitudes when the Sommerfeld-Watson contour in the j plane is shifted from the physical points (ss) to the left across integer values of $(j-\lambda)$ and j in the "nonsense" region. This resolves the difficulty connected with the asymptotic behaviour of the scattering amplitude, which one encounters if essential singularities of the partial wave amplitude existed.

In Section 4, we shortly discuss contributions coming from Regge cuts. Section 5 deals with the behaviour of individual Regge pole terms in helicity amplitudes when the trajectory $\alpha_{\pm}(s)$ passes through a "nonsense" point. We review shortly the new dip mechanism introduced by Mandelstam and Wang ¹⁵⁾ which automatically gives finite and non-zero contributions for "sense-nonsense" transitions but still requires the introduction of a ghost killing factor when the trajectory goes for $s = s_1$ through a "nonsense-nonsense" point with $(\alpha_{\pm}(s_1) - \lambda)$ integer.

^{*)} The singularities occur at integer j for integer helicities, and at half-odd integer j for half-odd integer helicities.

^{**)} The function $e^{\xi_{-\lambda, \mu}^{\pm}(-z, -j-1)}$ is a linear combination of rotation functions of the second kind which will be defined below.

The results obtained are not completely new, but we hope that due to the consistent use of the e^j functions instead of the d^j functions the formulae are more transparent and more easy to apply, especially in cases where Regge cuts could contribute. Finally, in the Appendix we collect a number of formulae for the functions $d^j_{\lambda,\mu}(z)$ and $e^j_{\lambda,\mu}(z)$ which are frequently used.

2. - THE GENERALIZED FROISSART-GRIBOV FORMULA

In the Jacob and Wick helicity formalism ²⁾ the scattering amplitude for a general two-body process $a+b \rightarrow c+d$ with helicities λ_a, λ_b in the initial, and helicities λ_c, λ_d in the final state is given by

$$f_{\lambda_c \lambda_d; \lambda_a \lambda_b}(s, z) = \sum_{j=\lambda_{\min}}^{\infty} (2j+1) F_{\lambda_c \lambda_d; \lambda_a \lambda_b}^j(s) d_{\lambda, \mu}^j(z) \quad (1)$$

with $\lambda = \lambda_a - \lambda_b$; $\mu = \lambda_c - \lambda_d$, and $z = \cos \theta$ *).

Taking out the factors $[\sqrt{1-z}]^a$ and $[\sqrt{1+z}]^b$, ($a = |\lambda - \mu|$, $b = |\lambda + \mu|$) introduced by the d^j functions, it is commonly assumed that the quantities

$$\bar{f}_{\lambda_c \lambda_d; \lambda_a \lambda_b}(s, z) = [\sqrt{1-z}]^{-a} [\sqrt{1+z}]^{-b} f_{\lambda_c \lambda_d; \lambda_a \lambda_b}(s, z) \quad (2)$$

*) Expressed in terms of s and t , with $s = (p_a + p_b)^2$ and $t = (p_a - p_c)^2$ z is given by : $z = (1/(4sp')) [2st + s^2 - s \sum_k m_k^2 + (m_a^2 - m_b^2)(m_c^2 - m_d^2)]$ where $p = (1/(2\sqrt{s})) \sqrt{[s - (m_a + m_b)^2][s - (m_a - m_b)^2]}$ and $p' = (1/(2\sqrt{s})) \times \sqrt{[s - (m_c + m_d)^2][s - (m_c - m_d)^2]}$ are the initial and final momenta in the s channel, respectively, and m_k ($k = a, b, c, d$) the masses of the corresponding particles.

which have an expansion in terms of $P_{j-\lambda_{\max}}^{(a,b)}(z)$ [see Eq. (A.1) in the Appendix] have only dynamical singularities $(1,3), (4)$. Assuming, therefore, $\bar{F}_{\lambda_c \lambda_d; \lambda_a \lambda_b}(s, z)$ to obey a dispersion relation in z at fixed s

$$\bar{F}_{\lambda_c \lambda_d; \lambda_a \lambda_b}(s, z) = \frac{1}{\pi} \int_{z_0(s)}^{\infty} \frac{S_{\lambda_c \lambda_d; \lambda_a \lambda_b}^d(s, z')}{z' - z} dz' + \frac{1}{\pi} \int_{z_0'(s)}^{\infty} \frac{S_{\lambda_c \lambda_d; \lambda_a \lambda_b}^e(s, z')}{z' + z} dz' \quad (3)$$

Calogero, Charap and Squires derived the following generalized Froissart-Gribov formula :

$$\begin{aligned} F_{\lambda_c \lambda_d; \lambda_a \lambda_b}^{\pm}(s, j) &= \text{sign}(\lambda, \mu) 2^{-\lambda_{\max}} \left[\frac{\Gamma(j + \lambda_{\max} + 1) \Gamma(j - \lambda_{\max} + 1)}{\Gamma(j + \lambda_{\min} + 1) \Gamma(j - \lambda_{\min} + 1)} \right]^{1/2} \\ &\cdot \left\{ \frac{1}{\pi} \int_{z_0(s)}^{\infty} S_{\lambda_c \lambda_d; \lambda_a \lambda_b}^d(s, z) (z-1)^a (z+1)^b Q_{j-\lambda_{\max}}^{(a,b)}(z) dz \right. \\ &\left. \pm (-1)^{\lambda_{\max}-\vartheta} \frac{1}{\pi} \int_{z_0'(s)}^{\infty} S_{\lambda_c \lambda_d; \lambda_a \lambda_b}^e(s, z) (z-1)^b (z+1)^a Q_{j-\lambda_{\max}}^{(b,a)}(z) dz \right\} \quad (4) \end{aligned}$$

In order to make possible the continuation in j , signature has been introduced in Eq. (4) through the replacement $(-1)^{j-\vartheta} \rightarrow \pm$ ($\vartheta = 0$ for λ, μ integer; $\vartheta = \frac{1}{2}$ for λ, μ half-odd integer). For $j = 0, 2, 4, \dots$ and $j = 1/2, 5/2, 9/2, \dots$ $F_{\lambda_c \lambda_d; \lambda_a \lambda_b}^+(s, j)$ coincides with the physical amplitude $F_{\lambda_c \lambda_d; \lambda_a \lambda_b}^j(s)$, whereas for $j = 1, 3, 5, \dots$ and $j = 3/2, 7/2, 11/2, \dots$ $F_{\lambda_c \lambda_d; \lambda_a \lambda_b}^-(s, j)$ coincides with $F_{\lambda_c \lambda_d; \lambda_a \lambda_b}^j(s)$. Equation (1) can now be written :

$$F_{\lambda_c \lambda_d; \lambda_a \lambda_b}(s, z) = \sum_{j=\lambda_{\max}}^{\infty} (2j+1) \left\{ F_{\lambda_c \lambda_d; \lambda_a \lambda_b}^+(s, j) d_{\lambda, \mu}^+(z, j) + F_{\lambda_c \lambda_d; \lambda_a \lambda_b}^-(s, j) d_{\lambda, \mu}^-(z, j) \right\} \quad (5)$$

*) A factor $(-1)^{j-\lambda_{\max}}$ occurred in the transformation of the second integral in Eq. (4) using (still for physical j) the formula $P_n^{(a,b)}(-z) = (-1)^n P_n^{(b,a)}(z)$. $\lambda_{\min}$ and $\lambda_{\max}$ are defined in the Appendix.

with

$$d_{\lambda, \mu}^{\pm}(z, j) = \frac{1}{2} \left[d_{\lambda, \mu}^j(z) \pm (-1)^{\lambda-j} d_{\lambda, -\mu}^j(-z) \right] \quad (6)$$

It was shown in the work of Charap and Squires that Eq. (5) allows a Sommerfeld-Watson transformation. Expressing Eq. (4) in terms of hypergeometric functions using Eq. (A.8), one easily sees that the representation (4) converges for $\text{Re } j \geq \lambda_{\max}$ if the dispersion relation needs no subtraction as implied in Eq. (3). If the $\rho^{e, d}$ behave like $z^{\alpha - \lambda_{\max}}$ with $\alpha - \lambda_{\max} > 0$ for large z , Eq. (3) would of course need subtractions, but Eq. (4) would converge independently of s for $\text{Re } j > \alpha$ defining an analytic function in this region *).

After this résumé of essentially known results, we introduce the functions $e_{\lambda, \mu}^j(z)$ of the Appendix to write Eq. (4) :

$$F_{\lambda_c \lambda_d; \lambda_a \lambda_b}^{\pm}(s, j) = \frac{1}{2p(s)p'(s)} \left\{ \frac{1}{\pi} \int_{t_0}^{\infty} A_{\lambda_c \lambda_d; \lambda_a \lambda_b}^{\pm}(s, t) [\sqrt{1-z}]^a [\sqrt{1+z}]^b e_{\lambda, \mu}^j(z) dt \right. \\ \left. \pm (-1)^{\lambda-j} \frac{1}{\pi} \int_{u_0}^{\infty} A_{\lambda_c \lambda_d; \lambda_a \lambda_b}^{\pm}(s, u) [\sqrt{1-z}]^a [\sqrt{1+z}]^b e_{\lambda, -\mu}^j(-z) du \right\} \quad (7)$$

Here we have introduced the absorptive parts of $\bar{f}_{\lambda_c \lambda_d; \lambda_a \lambda_b}(s, z)$ in the t and u channel, respectively, through the identifications :

*) The assumption here is that there exists a dispersion relation (3) with a finite number of subtractions independent of s in the same way as for the spinless case [cf., Ref. 19). But see also the remarks below in Section 3.]

$$A_{\lambda_c \lambda_d, \lambda_a \lambda_b}^t(s, t) = \int_{\lambda_c \lambda_d, \lambda_a \lambda_b}^d(s, z(s, t))$$

and

$$A_{\lambda_c \lambda_d, \lambda_a \lambda_b}^u(s, u) = \int_{\lambda_c \lambda_d, \lambda_a \lambda_b}^e(s, -z(s, \sum_k m_k^2 - s - u))$$

In the first integral of Eq. (7) z is a function of s and t , whereas in the second integral z is a function of s and u . It follows from Eq. (A.10) that the function $g_{\lambda, \mu}^j(z) = [\sqrt{1-z}]^a \times [\sqrt{1+z}]^b e_{\lambda, \mu}^j(z)$ has, for $(j - \lambda_{\max})$ integer, only the cut in z extending from $z = -1$ to $z = +1$ with the discontinuity given by $h_{\lambda, \mu}^j(z) = [\sqrt{1-z}]^a [\sqrt{1+z}]^b d_{\lambda, \mu}^j(z)$ *):

$$\text{Disc } g_{\lambda, \mu}^j(z) = -\frac{i}{2} h_{\lambda, \mu}^j(z) \quad (8)$$

Hence, in computing the discontinuity of $F_{\lambda_c \lambda_d, \lambda_a \lambda_b}^\pm(s, j)$ in s on the left-hand cut for equal masses, essentially the same argument applies as for the spinless case treated in Ref. 19). Also here the left-hand discontinuity in s is given by an integral over a finite t (or u) interval, showing that from Eqs. (7) a valid representation follows for this discontinuity even for those j values for which the Froissart-Gribov formula itself does not converge.

For later convenience we give here the formulae for the left-hand discontinuity of the partial wave amplitudes for $(j - \lambda)$ integer and equal masses ($m_k = m$, $k = a, b, c, d$)

*) $\text{Disc } f(z) = \frac{1}{2i} [f(z+i\epsilon) - f(z-i\epsilon)]$.

For

$$4m^2 - u_0 - t_0 \leq S \leq \begin{cases} 4m^2 - t_0 & \text{for the } t \text{ integral} \\ 4m^2 - u_0 & \text{for the } u \text{ integral} \end{cases}$$

$$\text{Disc } F_{\lambda_c \lambda_d; \lambda_a \lambda_b}^{\pm}(s, j) = \frac{1}{4m^2 - s} \left\{ \int_{t_0}^{4m^2 - s} A_{\lambda_c \lambda_d; \lambda_a \lambda_b}^t(s, t) \cdot h_{\lambda, \mu}^j(z) dt \pm (-1)^{\lambda - \nu} \int_{u_0}^{4m^2 - s} A_{\lambda_c \lambda_d; \lambda_a \lambda_b}^u(s, u) h_{\lambda, \mu}^j(-z) du \right\} \quad (9)$$

for $S \leq 4m^2 - u_0 - t_0$.

$$\begin{aligned} \text{Disc } F_{\lambda_c \lambda_d; \lambda_a \lambda_b}^{\pm}(s, j) = & \frac{2}{\pi(4m^2 - s)} \left\{ \int_{t_0}^{4m^2 - s - u_0} S_{\lambda_c \lambda_d; \lambda_a \lambda_b}^{tu}(t, 4m^2 - t - s) q_{\lambda, \mu}^j(z - i\varepsilon) dt \right. \\ & \left. \pm (-1)^{\lambda - \nu} \int_{u_0}^{4m^2 - s - t_0} S_{\lambda_c \lambda_d; \lambda_a \lambda_b}^{tu}(4m^2 - s - u, u) q_{\lambda, \mu}^j(-z + i\varepsilon) du \right\} \\ & + \frac{1}{4m^2 - s} \left\{ \int_{t_0}^{4m^2 - s} A_{\lambda_c \lambda_d; \lambda_a \lambda_b}^t(s - i\varepsilon, t) h_{\lambda, \mu}^j(z) dt \right. \\ & \left. \pm (-1)^{\lambda - \nu} \int_{u_0}^{4m^2 - s} A_{\lambda_c \lambda_d; \lambda_a \lambda_b}^u(s + i\varepsilon, u) h_{\lambda, \mu}^j(-z) du \right\} \quad (9') \end{aligned}$$

The function $S_{\lambda_c \lambda_d; \lambda_a \lambda_b}^{tu}(t, u)$ is the third double spectral function of $\bar{F}_{\lambda_c \lambda_d; \lambda_a \lambda_b}(s, z)$. The first bracket in the expression (9') contains the irregular part in j of $\text{Disc } F^{\pm}$, which can be expressed for equal thresholds $t_0 = u_0$ and with $\tilde{z} = z(s, t) - i\varepsilon$ compactly as

$$\begin{aligned} [\text{Disc } F_{\lambda_c \lambda_d; \lambda_a \lambda_b}^{\pm}(s, j)]_{\text{irreg.}} &= \frac{4}{\pi(4m^2 - s)} \int_{t_0}^{4m^2 - t_0 - s} S_{\lambda_c \lambda_d; \lambda_a \lambda_b}^{tu}(t, 4m^2 - s - t) \\ &\quad \cdot [\sqrt{1 - z}]^a [\sqrt{1 + z}]^b e_{\lambda, \mu}^{\pm}(\tilde{z}, j) dt \quad (10) \end{aligned}$$

Here we have introduced the functions $e_{\lambda,\mu}^{\pm}(z,j)$ defined by

$$e_{\lambda,\mu}^{\pm}(z,j) = \frac{1}{2} \left[e_{\lambda,\mu}^j(z) \pm (-1)^{\lambda-\mu} e_{\lambda,-\mu}^j(-z) \right] \quad (11)$$

Using the relation

$$e_{\lambda,-\mu}^j(-z) = -e^{\eta i \pi (j-\lambda)} e_{\lambda,\mu}^j(z) ; \quad \eta = \text{sign } \text{Im } z \quad (12)$$

which follows from Eq. (A.11), the Eq.(11) can more conveniently be written

$$e_{\lambda,\mu}^{\pm}(z,j) = \frac{1}{2} \left[1 \mp e^{\eta i \pi (j-\mu)} \right] e_{\lambda,\mu}^j(z) \quad (11')$$

Representing now Eq. (5) in the usual way as a contour integral with the curve C encircling in the clockwise direction the points $j - \lambda_{\max} = 0, 1, 2, \dots$ on the real axis (see the figure) one obtains

$$f_{\lambda_c \lambda_d; \lambda_a \lambda_b}(s, z) = -\frac{1}{2i} \int_C dj (2j+1) \frac{1}{\sin \pi(j-\lambda)} \cdot \left\{ \bar{F}_{\lambda_c \lambda_d; \lambda_a \lambda_b}^+(s, j) d_{\lambda, -\mu}^+(-z, j) + \bar{F}_{\lambda_c \lambda_d; \lambda_a \lambda_b}^-(s, j) d_{\lambda, -\mu}^-(-z, j) \right\} \quad (13)$$

In the same way as in Ref. ¹⁰⁾ we now add and subtract in Eq. (13) the sum

$$S_{\ell=0}^{\infty} = -(-1)^{\lambda_{\max}-\lambda} \frac{1}{\pi} \sum_{\ell=0}^{\infty} (-1)^{\ell} [2(\ell + \lambda_{\max}) + 2] \left\{ \bar{F}_{\lambda_c \lambda_d; \lambda_a \lambda_b}^+(s, \lambda_{\max} + \ell + \frac{1}{2}) \cdot e_{\lambda, -\mu}^+(-z, \lambda_{\max} + \ell + \frac{1}{2}) + \bar{F}_{\lambda_c \lambda_d; \lambda_a \lambda_b}^-(s, \lambda_{\max} + \ell + \frac{1}{2}) e_{\lambda, -\mu}^-(-z, \lambda_{\max} + \ell + \frac{1}{2}) \right\} \quad (14)$$

Eq. (13) can then be written :

$$\begin{aligned} f_{\lambda_c \lambda_d; \lambda_a \lambda_b}(s, z) = & -\frac{1}{2\pi i} \int_C dj (2j+1) \left\{ F_{\lambda_c \lambda_d; \lambda_a \lambda_b}^+(s, j) \left[\frac{\pi d_{\lambda, \mu}^+(-z, j)}{\sin \pi(j-\lambda)} \right. \right. \\ & \left. \left. - \frac{e_{\lambda, \mu}^+(-z, j)}{\cos \pi(j-\lambda)} \right] + F_{\lambda_c \lambda_d; \lambda_a \lambda_b}^- \left[\frac{\pi d_{\lambda, \mu}^-(-z, j)}{\sin \pi(j-\lambda)} - \frac{e_{\lambda, \mu}^-(-z, j)}{\cos \pi(j-\lambda)} \right] + \sum_{\ell=0}^{\infty} \right\} \end{aligned} \quad (15)$$

With Eq. (A.12) the quantities in square brackets are

$$\frac{\pi d_{\lambda, \mu}^{\pm}(-z, j)}{\sin \pi(j-\lambda)} - \frac{e_{\lambda, \mu}^{\pm}(-z, j)}{\cos \pi(j-\lambda)} = - \frac{e_{-\lambda, \mu}^{\pm}(-z, -j-1)}{\cos \pi(j-\lambda)} \quad (16)$$

where

$$\varepsilon_{\pm} = \begin{cases} \pm & \text{for } \lambda, \mu \text{ integer} \\ \mp & \text{for } \lambda, \mu \text{ half-odd integer} \end{cases}$$

In analogy to Eq. (11') the function $e_{-\lambda, \mu}^{\varepsilon_{\pm}}(-z, -j-1)$ can be represented as

$$e_{-\lambda, \mu}^{\varepsilon_{\pm}}(-z, -j-1) = \frac{1}{2} [1 \pm e^{i\pi(j-\lambda)}] e_{-\lambda, \mu}^{\varepsilon_{\pm}}(-z) \quad (17)$$

3. - THE SOMMERFELD-WATSON TRANSFORMATION

In Eq. (15) we now open up the contour C until it coincides with the path C' of the figure, avoiding possible branch points and crossing the real axis at $\lambda_{\max} - \varepsilon$. Calling $\beta_{\lambda_c \lambda_d; \lambda_a \lambda_b}^{i, \pm}(s)$ the residues of $F_{\lambda_c \lambda_d; \lambda_a \lambda_b}^{\pm}(s, j)$ at the i th Regge pole, $\alpha_{\pm}^i(s)$, which we meet for $\text{Re } \alpha_{\pm}^i(s) > \lambda_{\max}$, one obtains from Eq. (15) :

$$f_{\lambda_c \lambda_d; \lambda_a \lambda_b}(s, z) = \frac{1}{2\pi i} \int_{C'} \frac{dj(2j+1)}{\cos \pi(j-\lambda)} \left\{ F_{\lambda_c \lambda_d; \lambda_a \lambda_b}^+(s, j) e_{-\lambda, \mu}^{\varepsilon+}(-z, -j-1) \right. \\ \left. + F_{\lambda_c \lambda_d; \lambda_a \lambda_b}^-(s, j) e_{-\lambda, \mu}^{\varepsilon-}(-z, -j-1) \right\} \\ + \sum_{\text{Re } \alpha_{\pm}^i > \lambda_{\max}} (2\alpha_{\pm}^i + 1) \beta_{\lambda_c \lambda_d; \lambda_a \lambda_b}^{i, \pm} \frac{e_{-\lambda, \mu}^{\varepsilon_{\pm}}(-z, -\alpha_{\pm}^i - 1)}{\cos \pi(\alpha_{\pm}^i - \lambda)} + \sum_{l=0}^{\infty} \quad (18)$$

We have discarded in Eq. (18) the semicircles at infinity which were shown in Ref. ¹⁾ to give no contribution for $j > \lambda_{\max} - 1$. The same result can be derived here in an analogous way using the asymptotic form of $e_{\lambda, \mu}^j(z)$ for large $|j|$ quoted in Ref. ⁹⁾.

Shifting now the contour C' through the "sense-nonsense" and "nonsense-nonsense" regions symmetrical to $j = -\frac{1}{2}$ into the left half-plane to a position C'' in the figure crossing the real axis at $j = -M$ where $-\lambda_{\max} - N - \frac{1}{2} < -M < -\lambda_{\max} - N + \frac{1}{2}$ with N being a positive integer, we get the following contributions :

- a) additional Regge poles with $\text{Re } \alpha_{\pm}^i > -M$ (possible contributions from branch lines are still buried in the integral along C'' and will be treated later) ;

b) contributions coming from the singularities of the integrand on the real axis due to :

- i) the vanishing of the denominator at half-odd integer values of $(j-\lambda)$,
- ii) the singularities of the functions $e_{\lambda, \mu}^{\pm}(-z, -j-1)$ at integer $(j-\lambda)$ in the region $-\lambda_{\max} \leq j \leq \lambda_{\max}^{-1}$ *, and
- iii) "Kronecker-delta" singularities of the partial wave amplitudes.

First, we would like to comment on point iii) the problem of fixed singularities on the real j axis of $F_{\lambda_c \lambda_d; \lambda_a \lambda_b}^{\pm}(s, j)$ which follow from the Froissart-Gribov formula. It has been mentioned by Mandelstam and Wang that the expression (10) has singularities at "nonsense" values of j of the "wrong" signature, which means that the singularities of the functions $e_{\lambda, \mu}^j(z)$ at integer $j-\lambda$ (which are of one-over-square-root type for $j \in sn$, and of simple pole type for $j \in nn$) will, due to the factor $\frac{1}{2}[1+(-1)^{j-v}]$ in Eq. (11'), only survive for $j=1, 3, 5, \dots$ or $j=3/2, 7/2, 11/2, \dots$ in the case of positive signature partial waves, and for $j=0, 2, 4, \dots$ or $j=1/2, 5/2, 9/2, \dots$ in the case of negative signature partial waves. It was concluded in Ref. 15) that the physical partial wave amplitude will have these singularities when the left-hand discontinuity has them. It then results that, for $\mathcal{G}^{tu} \neq 0$ the location and type of the fixed singularities of the signed partial wave amplitudes for a general two-body process at a "nonsense" value of j is given by the function $e_{\lambda, \mu}^{\pm}(\tilde{z}, j)$, provided that the partial wave amplitudes have only those fixed singularities in j coming from the left-hand cut (**).

We now turn to the question whether these singularities of the $F_{\lambda_c \lambda_d; \lambda_a \lambda_b}^{\pm}(s, j)$ contribute to the helicity amplitudes in Eq. (18) when the contour C' of integration is shifted across

*) The functions $e_{\lambda, \mu}^{-j-1}(z)$ are analytic in j for $j < -\lambda_{\max}$ as can be seen from Eq. (A.10).

**) It is assumed here that the right-hand cut does not give rise to additional fixed j singularities of the $F_{\lambda_c \lambda_d; \lambda_a \lambda_b}^{\pm}(s, j)$, i.e., at "right" signature points. I am indebted to Professor D. Amati for this remark.

the "nonsense" values of j of the "wrong" signature. To answer this question one has to determine the singularities for $(j-\lambda)$ integer of the product $e_{\lambda,\mu}^{\pm}(\tilde{z}, j) \cdot e_{-\lambda,\mu}^{\pm}(-z, -j-1)$ which, using Eqs. (11') and (17) can be written for integer $(j-\lambda)$:

$$e_{\lambda,\mu}^{\pm}(\tilde{z}, j) e_{-\lambda,\mu}^{\pm}(-z, -j-1) = \frac{1}{4} [1 \mp (-1)^{j-\nu}] [1 \pm (-1)^{j-\nu}] e_{\lambda,\mu}^j(\tilde{z}) e_{-\lambda,\mu}^{-j-1}(-z) \quad (19)$$

Because of the presence of two signature factors this expression is completely symmetrical between "wrong" and "right" signature, giving thus always an additional zero due to the product of the first two brackets. The product $e_{\lambda,\mu}^j(\tilde{z}) e_{-\lambda,\mu}^{-j-1}(-z)$ has simple poles at all positive and negative integer values of $(j-\lambda)$ except in the nn regions symmetrical to $j = -\frac{1}{2}$ ($-\lambda_{\min} \leq j \leq \lambda_{\min}-1$) where it has double poles. From this it follows that the signed product (19) is regular everywhere on the real j axis except for simple poles at integer $(j-\lambda)$ with j in the region $-\lambda_{\min} \leq j \leq \lambda_{\min}-1$. Adjusting now the phases such that $\text{sign Im } \tilde{z} = -\text{sign Im } z$, which can always be fulfilled, the residues at these poles of the product (19) are equal and opposite for opposite signature :

$$\text{Res}_{\substack{(j-\lambda) \text{ integer} \\ j \in \text{nn}}} \left\{ e_{\lambda,\mu}^{\pm}(\tilde{z}, j) e_{-\lambda,\mu}^{\pm}(-z, -j-1) \right\} = \mp \frac{i\pi}{16} (-1)^{j-\nu} \cdot [(-1)^{j-\lambda} d_{\lambda,\mu}^j(\tilde{z}) - d_{\lambda,\mu}^j(-\tilde{z})] [(-1)^{j-\lambda} d_{\lambda,\mu}^j(-z) - d_{\lambda,\mu}^j(z)] \quad (20)$$

From this discussion, it then follows that one obtains no contributions to the full amplitude $f_{\lambda_c \lambda_d; \lambda_a \lambda_b}(s, z)$ in Eq. (18) when the contour C' is shifted through the sn region to the left, and that there are picked up cancelling contributions between the terms with signature + and signature - in the nn region.

Having treated the points ii and iii we come to point i mentioned above, i.e., the contributions occurring due to the zeros of the denominator in the integral of Eq. (18) at $j = \lambda_{\max}^{-\frac{1}{2}-k}$; $k=0,1,2,\dots$. It can easily be shown that between $j = \lambda_{\max}^{-\frac{1}{2}}$ and $j = -\lambda_{\max}^{-\frac{1}{2}}$ the contributions cancel pairwise. For λ, μ integer and j half-odd integer, the pole at $j = -\frac{1}{2}$ is cancelled by the factor $(2j+1)$. For $j \leq -\lambda_{\max}^{-\frac{3}{2}}$ the poles cancel the first $N-1$ terms of the sum in Eq. (18). For this cancellation to occur one needs the relation

$$F_{\lambda_c \lambda_d; \lambda_a \lambda_b}^{\pm}(s, j) = (-1)^{\lambda-\mu} F_{\lambda_c \lambda_d; \lambda_a \lambda_b}^{\pm}(s, -j-1) \quad (21)$$

at $(j-\lambda)$ half-odd integer for physical as well as "nonsensical" j values. The proof of Eq. (21) follows at once from Eq. (7) using Eq. (A.12) which reads for $(j-\lambda)$ half-odd integer

$$e_{\lambda, \mu}^j(z) = e_{-\lambda, -\mu}^{-j-1}(z) = (-1)^{\lambda-\mu} e_{\lambda, \mu}^{-j-1}(z)$$

This "proof" needs a comment. Although the Froissart-Gribov continuation must not represent the physical partial wave amplitude in the "nonsense channel" region, we have used Eq. (7) to derive the required relation between $F_{\lambda_c \lambda_d; \lambda_a \lambda_b}^{\pm}(s, j)$ and $F_{\lambda_c \lambda_d; \lambda_a \lambda_b}^{\pm}(s, -j-1)$, assuming that it holds true also for the physical partial wave amplitude continued to the relevant points on the real axis. To motivate this assumption without actually solving the N/D equations in the presence of "nonsense channels" we would argue as follows: from the above remark it follows that the generalized Froissart-Gribov formula correctly gives the discontinuity in s on the left-hand cut. Hence, the relation we want to establish is correct for the input (the "potential") in the N/D equations. Provided now that the kernel in the integral equation for the N function is of Fredholm type, the same relation is also true for

the N function and, hence, for the D function and the physical amplitude. The fact that the kernel is required to be of Fredholm type implies restrictions on the high energy behaviour of the absorptive parts $A^{\pm, u}_{\lambda_c \lambda_d; \lambda_a \lambda_b}$. It follows from Eqs. (9) and (9') that one needs the assumption of Eq. (3) to be correct without subtractions in order to establish the mentioned relation for the physical partial wave amplitudes.

After this discussion Eq. (18) assumes the form

$$f_{\lambda_c \lambda_d; \lambda_a \lambda_b}(s, z) = \frac{1}{2\pi i} \int_{C''} dj (2j+1) \frac{1}{\cos \pi(j-\lambda)} \left\{ F^+_{\lambda_c \lambda_d; \lambda_a \lambda_b}(s, j) e^{\varepsilon_+}_{-\lambda, \mu}(-z, -j-1) + F^-_{\lambda_c \lambda_d; \lambda_a \lambda_b}(s, j) e^{\varepsilon_-}_{-\lambda, \mu}(-z, -j-1) \right\} \\ + \sum_{\substack{i \\ \text{Re } \alpha_{\pm}^i > -M}} (2\alpha_{\pm}^i + 1) \beta^{\varepsilon_{\pm}}_{\lambda_c \lambda_d; \lambda_a \lambda_b}(s) \frac{e^{\varepsilon_{\pm}}_{-\lambda, \mu}(-z, -\alpha_{\pm}^i - 1)}{\cos \pi(\alpha_{\pm}^i - \lambda)} + \sum_{\ell=N-1}^{\infty} \quad (22)$$

If there were no cuts present, the integral along C'' would represent the usual background integral taken from $-M-i\infty$ to $-M+i\infty$ behaving for large z like z^{-M} , as can be seen from Eq. (A.10). The final sum $S_{\ell=N-1}^{\infty}$ in Eq. (22) goes like $z^{-(N+\lambda_{\max}+\frac{1}{2})}$ for large z , whereas the dominant contribution comes as usually from the rightmost Regge pole of the second sum in Eq. (22) behaving like z^{α} asymptotically. In the common application of the Regge model where $f_{\lambda_c \lambda_d; \lambda_a \lambda_b}(s, z)$ represents the amplitude in the crossed channel, and z being large for high energies in the original channel, one would then in the usual way conclude that the background integral and the final sum in Eq. (22) are negligible. In the presence of branch cuts in the j plane, the situation is different, and there are still contributions in the integral along C'' which are not negligible for large z and hence should be taken into account.

4. - REGGE CUT CONTRIBUTIONS

Up to this point we have only mentioned the possible existence of branch lines in the j plane ^{17),18),20),21),8)} without considering them in more detail. We would like now to add a few remarks concerning these cuts. We are aiming at a formalism which can be used in a phenomenological application to general two-body processes also in cases where Regge cuts could have some influence. In order to arrive at such a formula, we have to make some simplifying assumptions concerning the position of these branch lines relative to each other and to the real axis. One such assumption is already implicit in the above discussion leading to Eq. (22), namely, that there are no branch lines crossing the real axis in the interval $-M \leq j \leq \lambda_{\max}$ considered there. Eventually, in pushing the contour further into the left half-plane, this will probably no longer be true, giving thus an upper limit to M . On the other hand, two branch lines could cross and thereby cause additional difficulties resulting in a complicated sheet structure of the j plane. Since at present very little is known about these cuts in general (apart from the position and energy dependence of the branch points), we assume here without justification that such crossings do not occur even when the energy is varied in some region and the branch points start to move about in the j plane dragging the cut behind. For application, this assumption is perhaps not too restrictive, since in an actual calculation one would probably only be able to include one branch line reflecting the higher order correction (two Reggeon exchange in the crossed channel) to the usual Regge pole model. For this pragmatical reason, we regard the contour C'' in Eq. (22) to contain the contribution of only a single branch cut (as shown in the fig.). We call $\Delta_{\lambda_a \lambda_b}^{\pm}(s, j)$ the discontinuity (with signature $\pm$) across this cut $B_{\pm}^{\pm}$ [oriented from the branch point $\alpha_{\pm}^c(s)$ towards the left to the line $j = -M$] and write the Eq. (22) finally :

$$\begin{aligned}
f_{\lambda, \lambda_a, \lambda_b}(s, z) &= \frac{1}{2\pi i} \int_{-M-i\infty}^{*-M+i\infty} dj (2j+1) \frac{1}{\cos \pi(j-\lambda)} \left\{ F_{\lambda, \lambda_a, \lambda_b}^+(s, j) \right. \\
&\quad \left. \cdot e_{-\lambda, \mu}^{\varepsilon_+}(-z, -j-1) + F_{\lambda, \lambda_a, \lambda_b}^-(s, j) e_{-\lambda, \mu}^{\varepsilon_-}(-z, -j-1) \right\} \\
&+ \sum_{\substack{i \\ \operatorname{Re} \alpha_{\pm}^i > -M}} (2\alpha_{\pm}^i + 1) \beta_{\lambda, \lambda_a, \lambda_b}^{i, \pm}(s) \frac{e_{-\lambda, \mu}^{\varepsilon_{\pm}}(-z, -\alpha_{\pm}^i - 1)}{\cos \pi(\alpha_{\pm}^i - \lambda)} \\
&+ \frac{1}{2\pi i} \int_{B^{(\pm)}} dj (2j+1) \Delta_{\lambda, \lambda_a, \lambda_b}^{\pm}(s, j) \frac{e_{-\lambda, \mu}^{\varepsilon_{\pm}}(-z, -j-1)}{\cos \pi(j-\lambda)} + S_{\ell=N-1}^{\infty}
\end{aligned} \tag{23}$$

The final sum in Eq. (23) and the background integral, which now is a principal value integral along the line $j = -M$, have for large z the behaviour mentioned at the end of the last Section. The dominant contributions in Eq. (23) come from the poles and the cut behaving asymptotically like $z^{\alpha_{\pm}^i}$ and $z^{\alpha_{\pm}^c}/\log z$, respectively.

The Eq. (23) could be rewritten introducing the functions

$$\mathcal{D}_{\lambda, \mu}^{\pm}(z, j) = \frac{1}{2} \left[\mathcal{Q}_{\lambda, \mu}^j(z) \pm (-1)^{\lambda-j} \mathcal{Q}_{\lambda, -\mu}^j(-z) \right] \tag{24}$$

With the help of Eq. (A.14) one obtains :

$$\mathcal{D}_{\lambda, \mu}^{\pm}(z, j) = -\frac{1}{\pi} \log \pi(j-\lambda) e_{-\lambda, -\mu}^{\varepsilon_{\pm}}(z, -j-1) \tag{25}$$

From Carlson's theorem and the properties of the functions $D_{\lambda, \mu}^j(z)$ discussed in the Appendix, it follows that our method of Reggeization is equivalent to the one of Ref. ⁶⁾, where the functions $E_{\lambda, \mu}^j(z)$ ^{*)} have been defined by replacing in the expansion of the functions $[V1-z]^{-a}[V1+z]^{-b} \cdot d_{\lambda, \mu}^j(z)$ in terms of $P_j(z)$ and their derivatives with respect to z all $P_j(z)$ by the corresponding $\mathcal{P}_j(z) = -\frac{1}{\pi} \text{tg} \pi j \cdot Q_{-j-1}(z)$ and their derivatives.

In concluding this Section, we would like to add a few remarks concerning possible additional relations among helicity amplitudes at certain energy values. Such constraints have first been discussed by Goldberger, Grisaru, MacDowell and Wong ²²⁾ in their paper on nucleon-nucleon scattering. The origin of these relations is the following: when the five scalar amplitudes G_i , which obey a Mandelstam representation, are expressed in terms of kinematical singularity free helicity amplitudes, additional poles are introduced which can only be avoided when a certain linear combination of helicity amplitudes vanishes at this particular energy [cf., Eq. (4.33a) of Ref. ²²⁾]. As was shown by Volkov and Gribov ²³⁾, this GGMW relation implies for the partial wave amplitudes a correlation of the singularities in the j plane for different helicities. It is clear that for the general spin case all these constraints between helicity amplitudes are of special importance in connection with Reggeization. In a recent paper by Abers and Teplitz ²⁴⁾ the generalized GGMW conditions for equal mass scattering have been investigated in detail. It was shown that they follow directly from the crossing relations for the helicity amplitudes. The situation for the unequal mass case is more complicated ^{**)} and we refer to the last two chapters of Ref. ²⁴⁾ for a more detailed investigation.

^{*)} We leave out the label $\pm$ referring to parity in Ref. ⁶⁾.

^{**)} I thank Dr. H. Högaasen for a discussion on the constraints in this case.

5. - THE BEHAVIOUR OF REGGE POLE TERMS WHEN THE TRAJECTORY
PASSES THROUGH A "NONSENSE" POINT

In this Section we would like to study the behaviour of an individual Regge pole term

$$(2\alpha_{\pm}(s)+1) \beta_{\lambda_c \lambda_d; \lambda_a \lambda_b}^{\pm}(s) \frac{e_{-\lambda, \mu}^{\pm}(-z, -\alpha_{\pm}(s)-1)}{\cos \pi(\alpha_{\pm}(s)-\lambda)} \quad (26)$$

when the trajectory $\alpha_{\pm}(s)$ approaches for some value s_1 a real point in the "nonsense" region with $(\alpha_{\pm}(s_1)-\lambda)$ integer. Mandelstam and Wang have shown that, if $\alpha_{\pm}(s_1)$ is a "nonsense" point of "wrong" signature, the residue functions $\beta_{\lambda_c \lambda_d; \lambda_a \lambda_b}^{\pm}(s)$ will behave near s_1 in the following way

$$\beta_{\lambda_c \lambda_d; \lambda_a \lambda_b}^{\pm}(s) \sim \begin{cases} C_1 & \text{for } \alpha_{\pm}(s_1) \in SS \\ C_2 (s-s_1)^{-1/2} & \text{for } \alpha_{\pm}(s_1) \in SN \\ C_3 (s-s_1)^{-1} & \text{for } \alpha_{\pm}(s_1) \in NN \end{cases} \quad (27)$$

The multiplying function $e_{-\lambda, \mu}^{\pm}(-z, -\alpha_{\pm}(s)-1)/(\cos \pi(\alpha_{\pm}(s)-\lambda))$ in the Regge pole term (26) has the following behaviour for s near s_1 , $(\alpha_{\pm}(s_1)-\lambda)$ integer, and $\alpha_{\pm}(s_1)$ being a "nonsense" point of "wrong" signature :

$$\frac{e_{-\lambda, \mu}^{\pm}(-z, -\alpha_{\pm}(s)-1)}{\cos \pi(\alpha_{\pm}(s)-\lambda)} \sim \begin{cases} a_1 & \text{for } \alpha_{\pm}(s_1) \in SS \\ a_2 (s-s_1)^{1/2} & \text{for } \alpha_{\pm}(s_1) \in SN \\ a_3 & \text{for } \alpha_{\pm}(s_1) \in NN \end{cases} \quad (28)$$

Hence, a Regge pole term in the helicity amplitude will behave for s near s_1 and $\alpha_{\pm}(s_1) \neq -\frac{1}{2}$ *) as

$$(2\alpha_{\pm}(s)+1) \beta_{\lambda_c \lambda_d; \lambda_a \lambda_b}^{\pm}(s) \frac{e_{-\lambda, \mu}^{\pm}(-z, -\alpha_{\pm}(s)-1)}{\cos \pi(\alpha_{\pm}(s)-\lambda)} \sim \begin{cases} b_1 & \text{for } \alpha_{\pm}(s_1) \in ss \\ b_2 & \text{for } \alpha_{\pm}(s_1) \in sn \\ b_3 (s-s_1)^{-1} & \text{for } \alpha_{\pm}(s_1) \in nn \end{cases} \quad (29)$$

The behaviour when $\alpha_{\pm}(s)$ approaches a real point of "right" signature in the "nonsense" region with $(\alpha_{\pm}(s_1)-\lambda)$ integer will also be given by Eq. (29). In this case the $\beta_{\lambda_c \lambda_d; \lambda_a \lambda_b}^{\pm}(s)$ are not singular at $s=s_1$ and behave like :

$$\beta_{\lambda_c \lambda_d; \lambda_a \lambda_b}^{\pm}(s) \sim \begin{cases} C_1' (s-s_1) & \text{for } \alpha_{\pm}(s_1) \in ss \\ C_2' (s-s_1)^{1/2} & \text{for } \alpha_{\pm}(s_1) \in sn \\ C_3' & \text{for } \alpha_{\pm}(s_1) \in nn \end{cases} \quad (30)$$

but the multiplying functions are singular at s_1 behaving like

$$\frac{e_{-\lambda, \mu}^{\pm}(-z, -\alpha_{\pm}(s)-1)}{\cos \pi(\alpha_{\pm}(s)-\lambda)} \sim \begin{cases} a_1' (s-s_1)^{-1} & \text{for } \alpha_{\pm}(s_1) \in ss \\ a_2' (s-s_1)^{-1/2} & \text{for } \alpha_{\pm}(s_1) \in sn \\ a_3' (s-s_1)^{-1} & \text{for } \alpha_{\pm}(s_1) \in nn \end{cases} \quad (31)$$

*) For $\alpha_{\pm}(s_1) = -\frac{1}{2}$ the first factor in Eq. (29) develops an additional zero.

The Eq. (29) shows that the helicity amplitude develops a singularity only when a trajectory crosses a real point in the nn region, whereas the contribution is finite and non-vanishing when the trajectory goes through a sn point with $(\alpha_{\pm}(s) - \lambda)$ integer and "right" or "wrong" signature ^{*}). This is the new dip mechanism discussed in Ref. ¹⁵⁾. It makes ghost killing unnecessary in the "sense-nonsense" region, but still requires ghost killing factors in the "nonsense-nonsense" region for "wrong" as well as "right" signature.

As pointed out by Mandelstam and Wang this now allows the Pomeranchuk trajectory to contribute to forward Compton scattering and vector meson photoproduction and eliminates the difficulties encountered by Mur ²⁵⁾ and more recently by Shepard ²⁶⁾.

6. - CONCLUSION

The rotation functions of the second kind, $e^j_{\lambda, \mu}(z)$, which enter the generalized Froissart-Gribov formula, were introduced in the Jacob and Wick expansion of the helicity amplitudes. The trick applied by Mandelstam in potential scattering of spinless particles was used to write the multiplying function of the partial wave amplitudes $F^{\pm}_{\lambda_c \lambda_d; \lambda_a \lambda_b}(s, j)$ as a linear combination $e^{\pm}_{\lambda, \mu}(-z, -j-1)$ of rotation functions of the second kind with angular momentum $-j-1$. The contributions were discussed which arise when the Sommerfeld-Watson contour is shifted away from the physical points through the "sense-nonsense" and "nonsense-nonsense" regions symmetrical to $j = -\frac{1}{2}$ into the left half-plane. It was shown that the fixed singularities of the

^{*}) Compare the footnote ^{**)} on p. 12 about the assumptions made to reach this conclusion.

partial wave amplitudes found by Jones and Teplitz and by Mandelstam and Wang do not show up in the helicity amplitudes due to the properties of the $e^{\pm}$ functions which multiply them, and that the remaining contributions cancel pairwise. After some remarks concerning cuts in the j plane the behaviour of an individual Regge pole term was investigated for the case when the trajectory passes through a ss , sn or nn value of angular momentum with $(\alpha_{\pm} - \lambda)$ integer. The consequences with respect to the ghost killing - and dip mechanism were stated. This part, which mostly relied on the work of Mandelstam and Wang, has been included for completeness.

After completion of the first version of the manuscript we heard of the results obtained by Jones and Teplitz and by Mandelstam and Wang, which made a revision of parts of our presentation necessary. We are quite aware of the fact that the paper in its present form now touches a number of subjects already treated in Ref. ¹⁵⁾. In spite of this we thought that the publication of a more detailed investigation would perhaps still be interesting for the general discussion of the present problems in the Regge theory.

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A P P E N D I X

In this Appendix we would like to collect some definitions and relations for the functions $d_{\lambda,\mu}^j(z)$ and $e_{\lambda,\mu}^j(z)$, the rotation functions of the first and second kind. Frequent use is made of the results obtained in Ref. ⁹⁾. We first consider the more familiar functions $d_{\lambda,\mu}^j(z)$.

In order to have a representation valid in all four regions A,B,C and D [see Ref. ⁹⁾ and the Table below] we define :

$$\begin{aligned} a &= |\lambda - \mu| & \lambda_{\max} &= \text{Max}(|\lambda|, |\mu|) \\ b &= |\lambda + \mu| & \lambda_{\min} &= \text{Min}(|\lambda|, |\mu|) \end{aligned}$$

The function $d_{\lambda,\mu}^j(z)$ is then given by :

$$\begin{aligned} d_{\lambda,\mu}^j(z) &= \text{sign}(\lambda, \mu) \, 2^{-\frac{1}{2}(a+b)} \left[\frac{\Gamma(j+\lambda_{\max}+1) \Gamma(j-\lambda_{\max}+1)}{\Gamma(j+\lambda_{\min}+1) \Gamma(j-\lambda_{\min}+1)} \right]^{1/2} \\ &\quad \left[\sqrt{1-z} \right]^a \left[\sqrt{1+z} \right]^b P_{j-\lambda_{\max}}^{(a,b)}(z) \end{aligned} \tag{A.1}$$

The factor $\text{sign}(\lambda, \mu)$ which reflects the relations

$$d_{\lambda,\mu}^j(z) = (-1)^{\lambda-\mu} d_{\mu,\lambda}^j(z) = (-1)^{\lambda-\mu} d_{-\lambda,-\mu}^j(z) = d_{-\mu,-\lambda}^j(z)$$

can be read off from the Table where the four regions are defined.

Region	A	B	C	D
$\lambda - \mu$	≥ 0	≤ 0	≤ 0	≥ 0
$\lambda + \mu$	≥ 0	≥ 0	≤ 0	≤ 0
$\text{sign}(\lambda, \mu)$	1	$(-1)^{\lambda - \mu}$	$(-1)^{\lambda - \mu}$	1

In Eq. (A.1), the functions $P_{j-\lambda_{\max}}^{(a,b)}(z)$ are the Jacobi polynomials of the first kind which are related to hypergeometric functions in the following way :

$$P_{j-\lambda_{\max}}^{(a,b)}(z) = \frac{\Gamma(j-\lambda_{\max}+a+1)}{\Gamma(j-\lambda_{\max}+1)} \frac{1}{\Gamma(a+1)} F(-j+\lambda_{\max}, j+\lambda_{\max}+1; a+1; \frac{1-z}{2}) \quad (\text{A.2})$$

For unphysical j we frequently continue to use Eq. (A.1) with the understanding that the $P_{j-\lambda_{\max}}^{(a,b)}(z)$ are to be replaced by the expression (A.2) which defines the continuation in j . With $\lambda_{\max} = \frac{1}{2}(a+b)$ and $\lambda_{\min} = \frac{1}{2}|a-b|$, Eq. (A.1) can be written

$$d_{\lambda, \mu}^j(z) = \text{sign}(\lambda, \mu) z^{-\frac{1}{2}(a+b)} \left[\frac{\Gamma(j+\frac{1}{2}(a+b)+1) \Gamma(j+\frac{1}{2}(a-b)+1)}{\Gamma(j-\frac{1}{2}(a+b)+1) \Gamma(j-\frac{1}{2}(a-b)+1)} \right]^{1/2} \quad (\text{A.3})$$

$$[\sqrt{1-z}]^a [\sqrt{1+z}]^b \frac{1}{\Gamma(a+1)} F(-j+\frac{1}{2}(a+b), j+\frac{1}{2}(a+b)+1; a+1; \frac{1-z}{2})$$

The d^j functions have cuts in z from $z=-1$ to $-\infty$ coming from the hypergeometric function [or the function $P_{j-\lambda_{\max}}^{(a,b)}(z)$ in the representation (A.1)]. They have additional cuts running outwards from $z=-1$ and $z=+1$ due to the factors $\sqrt{1+z}$ and $\sqrt{1-z}$, respectively. The $d_{\lambda, \mu}^j(z)$ obey the following relations [Eqs. (2.2), (5.1) and (5.2) of Ref.⁹]

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The contours C , C' and C'' in the j plane drawn for λ and μ integer.