The next to leading order effective potential in the 2+1 dimensional Nambu-Jona-Lasinio model at finite temperature.

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The finite temperature effective potential in the 2+1 dimensional Nambu-Jona-Lasinio model is constructed up to the next to leading order in the large $N$ expansion, where $N$ is the number of flavors in the model. The distinctive feature of the analysis is an inclusion of an additional scalar field, which allows us to circumvent the well known, and otherwise unavoidable problem with the imaginary contribution to the effective potential. In accordance with the Mermin-Wagner-Coleman theorem, applied to the dimensionally reduced subsystem of the zero Matsubara modes of the composite boson fields, the finite temperature effective potential reveals a global minimum at the zero of the composite order parameter. This allows us to conclude that the continuous global symmetry of the NJL model is not broken for any arbitrarily small, finite temperature.

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I. INTRODUCTION

The Nambu-Jona-Lasinio (NJL) model [1] plays a crucial role in our understanding of quantum field theory. Despite its non-renormalizability in 3+1 dimensions, the NJL model has already found many applications in nuclear and high energy physics [2,3].

The 2+1 dimensional NJL model has also been studied in the large $N$ limit [4,5]. This model is known to be non-renormalizable when expanded in the four-fermi coupling. Nevertheless, it is renormalizable in the $1/N$ perturbation theory [6,7]. The effective potential of this model has been studied at zero temperature up to the next to leading order in $1/N$ by carrying out the Hubbard-Stratonovich transformation [4,5]. When the fluctuations of the composite scalars are included the effective potential develops an imaginary part. At finite temperature this complicates the analysis of the symmetry breaking and restoration.

In this paper, we study the effective potential to the next to leading order in the large $N$ expansion for the three-dimensional model at zero and finite temperatures. The problem of obtaining the effective potential is an old one. There exist well developed general methods for constructing effective potentials for elementary [8–10] as well as for composite fields [11].

These methods, however, cannot be applied to some specific models in a straightforward manner. For example, should one try to construct the effective potential for a scalar theory with a double-well tree level potential, he would find at the leading order an imaginary contribution [5,12–14]. The latter is due to a tachionic vacuum at an intermediate stage in the calculations. In the scalar $\phi^4$ theory, a method for dealing with this obstacle was found long ago [15–17]. The way around the problem was the introduction of an additional scalar field.

While studying the NJL model, one usually introduces some auxiliary scalar fields that could be interpreted as composites built from fermions [4,7]. In terms of these new fields, the problem of constructing the effective potential is very much the same as in pure scalar theories except that the propagators of the composite particles are much more complicated [4,18].

The leading order effective potential in the NJL model, which plays the same role as a tree level potential in scalar theories, may also be double-well in shape. In fact, this happens as soon as the four-fermi coupling constant becomes larger than some critical value. Then, an attempt to go beyond the leading order inevitably results in an imaginary contribution to the effective potential due to the same reasons as in the pure scalar case [12].

Often there is no reason to doubt the leading order result, so this lack of a straightforward method for calculating the next to leading order corrections may seem to be of no interest. There are, however, several instances when the knowledge of the next to leading order terms becomes of prime importance. This happens, for example, when one studies the finite temperature 2+1 dimensional NJL model which enjoys a continuous global symmetry. The leading order calculation of the effective potential indicates that the symmetry remains broken for a range of non-zero

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temperatures. This result seems to contradict the predictions of the Mermin-Wagner-Coleman (MWC) theorem [19].

The argument is well known. The infrared region of the system is dominated by the zero Matsubara modes of the boson fields. The latter effectively live in a 1+1 dimensional space. Thus, by appealing to the above mentioned theorem, one concludes that the symmetry breaking is forbidden [20,21].

We find it surprising that there are no explicit calculations of the next to leading order effective potential at finite temperature in the NJL model in 2+1 dimensions. Such calculations would demonstrate explicitly that the symmetry is not broken for an arbitrarily small $T \neq 0$. In this paper we perform such calculations and show that the global minimum of the potential does indeed occur at the origin or, in other words, that the symmetry is not broken at any finite temperature. The fluctuations of the composite scalar particle, which is the NG boson at $T = 0$, play a crucial role in the symmetry restoration.

We are aware of a few papers where the finite temperature condensate and the equation of state are calculated in the 3+1 dimensional NJL model in the next to leading order of $1/N$ [22–25]. It is worthwhile to note that the problem of the imaginary contributions, which is also present in 3+1 dimensions, is usually ignored [24,25]. Obviously, without a proper cure of it, many results remain suspect. In this paper we suggest a method that resolves the imaginary part of the imaginary contributions, which is also present in 3+1 dimensions, is usually ignored [24,25].

Finally, it seems that there is no clear understanding of what happens in the 2+1 dimensional NJL model as the temperature approaches zero. Since the MWC theorem is believed to work for arbitrarily small temperatures, there is no symmetry breaking for any non-zero $T$. At $T=0$, the continuous symmetry is broken. Then, a natural question arises. How does the effective potential in the model behave as the temperature approaches zero? We answer this question below.

II. ZERO TEMPERATURE CASE

Let us start with the analysis of the NJL model in 3-dimensional Euclidean space at zero temperature. To apply the large $N$ expansion scheme, we consider the model containing $N$ identical flavors. The Lagrangian density reads

$$L^{(0)}_{NJL} = \bar{\Psi}i\gamma_{\mu}\partial_\mu \Psi - \frac{G}{2N} \left[(\bar{\Psi}\Psi)^2 + (\bar{\Psi}i\gamma_5 \Psi)^2\right], \quad (1)$$

where we have chosen four-component spinors and the $4 \times 4$ Dirac $\gamma$-matrices are all antihermitian. By introducing (“integrating in”) the auxiliary boson fields $\sigma \equiv \bar{\Psi}\Psi$ and $\pi \equiv \bar{\Psi}i\gamma_5 \Psi$, we obtain the equivalent model

$$L^{(0)}_{aux} = \bar{\Psi}(i\gamma_{\mu}\partial_\mu - \sigma - i\gamma_5 \pi)\Psi + \frac{N}{2G} (\sigma^2 + \pi^2). \quad (2)$$

The mentioned equivalence comes from the Euler-Lagrange equations of motion for the boson fields. As is easy to check, the latter Lagrangian density enjoys the same continuous chiral symmetry as that in Eq. (1). The only difference is that the integrated in boson fields are also involved in symmetry transformations.

After integrating out the fermions, we are left with the following effective action for the bosons:

$$S^{(0)}_{scal} = N \left[\frac{1}{2G} \int d^3x \left(\sigma^2 + \pi^2\right) - Tr \ln (i\gamma_5 \partial_\mu - \sigma - i\gamma_5 \pi)\right]. \quad (3)$$

This action may be expanded in inverse powers of the cutoff $\Lambda$ in the following manner,

$$S^{(0)}_{scal} \simeq \frac{N}{\pi} \int d^3x \left[-\frac{\Lambda}{2} \left(1 - \frac{1}{g}\right) \rho^2 + \frac{\rho^4}{3} + \frac{\rho^4}{8\Lambda} + O \left(\frac{\rho^6}{\Lambda^3}\right) + L^{(0)}_{kin}(\sigma, \pi)\right], \quad (4)$$

where $\rho = \sqrt{\sigma^2 + \pi^2}$ is chiral invariant, and the dimensionless coupling constant is defined as $g \equiv G\Lambda/\pi$. In the kinetic part of the action, we keep only the terms that contain up to two derivatives, i. e., terms quadratic in momenta, $L^{(0)}_{kin}(\sigma, \pi) \equiv \frac{a_{\sigma}}{2} \sum_{i=1}^{3} (\partial_\mu \sigma)^2 + (\partial_\mu \pi)^2 + \frac{a_{\pi}}{2\rho^2} \sum_{i=1}^{3} (\sigma \partial_\mu \sigma + \pi \partial_\mu \pi)^2.\quad (5)$

The straightforward calculation of the infrared asymptotic behavior of the boson propagators reveals the following expressions for the kinetic coefficients [26,27]

$$a_{\sigma}^{(0)} = \frac{1}{6\rho}, \quad a_{\pi}^{(0)} = \frac{1}{4\rho}. \quad (6)$$

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Instead of using these infrared asymptotes, we replace them with constant coefficients and argue that this would give a reasonable approximation in the whole range of momenta. The reasoning goes as follows. The composite particles in the far infrared region (large distances) should presumably behave as almost point-like objects. Their kinetic coefficients would be given by the expressions as in Eq. (6) with a specific value of $ρ = \bar{ρ}$. As for the ultraviolet region, it is known that the leading asymptotic behavior does not depend on $ρ$. Therefore, we take $a(0)$’s independent of $ρ$ as a much better approximation than that in Eq. (6). Later, when we match the zero and the finite temperature results, we will see that there is another advantage to the above approximation.

Following the standard analysis applied for scalar theories [15,17], we introduce an additional auxiliary scalar field $χ$ into our bosonic action,

$$S_{\text{scal}}^{(0)} \simeq \frac{N}{\pi} \int d^3 x \left( -\frac{\Lambda}{2} \chi^2 + \frac{\chi^2}{2} (\rho^2 - 2\Lambda \mu) - \frac{\Lambda \mu^2}{2} + \frac{\sigma^3}{3} + L_{\text{kin}}^{(0)}(\sigma, \chi) \right),$$  

(7)

where $µ = \Lambda(g - 1)/g$ is the characteristic scale of chiral symmetry breaking. Below we assume that $µ \ll \Lambda$. We keep all contributions up to the order $1/\Lambda$. However, it will be more convenient to keep exact results as they appear after integrations, without continually performing the expansions. Obviously, this will not affect any of our results since all the interesting features of the effective potential, as we shall see, occur at values of $σ$ much smaller than the cutoff $Λ$.

To construct the effective potential, we apply the standard technique [10] to the action in (7). After performing rather simple calculations up to the next to leading order of the large $N$ expansion, we obtain the following effective potential:

$$V_{\text{eff}}^{(0)}(σ, χ) \simeq \frac{N}{\pi} \left( -\frac{\Lambda}{2} \chi^2 + \frac{\chi^2}{2} (σ^2 - 2\Lambda µ) - \frac{\Lambda µ^2}{2} + \frac{σ^3}{3} \right)$$

$$+ \frac{1}{6π(\alpha_σ^{(0)})^{3/2}} \left[ (M_σ^{(0)} + a_σ^{(0)} \Lambda^2)^{3/2} - (a_σ^{(0)} \Lambda^2)^{3/2} - (M_σ^{(0)})^{3/2} \right]$$

$$+ \frac{1}{6π(\alpha_π^{(0)})^{3/2}} \left[ (M_π^{(0)} + a_π^{(0)} \Lambda^2)^{3/2} - (a_π^{(0)} \Lambda^2)^{3/2} - (M_π^{(0)})^{3/2} \right],$$  

(8)

where the “mass” parameters are

$$M_σ^{(0)} = χ + \frac{σ^2}{Λ} + 2σ$$  

(9)

and

$$M_π^{(0)} = χ + σ.$$  

(10)

These parameters define the bosonic fluctuations. Since their propagators do not have the canonical form, $M_σ^{(0)}$ and $M_π^{(0)}$ should not be confused with the sigma and pion masses.

Finally, to get the effective action for the $σ$ field, we have to eliminate the $χ$ field by the saddle point equation

$$χ + µ - \frac{σ^2}{2Λ} - \frac{1}{4NΛ(\alpha_σ^{(0)})^{3/2}} \left[ \sqrt{M_σ^{(0)} + a_σ^{(0)} \Lambda^2} - \sqrt{M_σ^{(0)}} \right]$$

$$- \frac{1}{4NΛ(\alpha_π^{(0)})^{3/2}} \left[ \sqrt{M_π^{(0)} + a_π^{(0)} \Lambda^2} - \sqrt{M_π^{(0)}} \right] = 0.$$  

(11)

We solve this equation numerically to obtain $χ = χ(σ)$. Then, we substitute it into Eq. (8) to get the effective potential as a function of $σ$. The minimum of the obtained effective potential satisfies the following gap equation,

$$χσ + σ^2 + \frac{1 + σ/Λ}{2N(\alpha_σ^{(0)})^{3/2}} \left[ \sqrt{M_σ^{(0)} + a_σ^{(0)} \Lambda^2} - \sqrt{M_σ^{(0)}} \right]$$

$$+ \frac{1}{4N(\alpha_π^{(0)})^{3/2}} \left[ \sqrt{M_π^{(0)} + a_π^{(0)} \Lambda^2} - \sqrt{M_π^{(0)}} \right] = 0.$$  

(12)

Prior to presenting the numerical results, let us analyze the saddle-point equation for the $χ$ field given in Eq. (11). It is easy to see that the left hand side of this equation is a monotonically increasing function of $χ$. We also observe
that this function takes real values only for $\chi \geq -\sigma$, and so, it is bounded below with the minimum at $\chi = -\sigma$. If this minimum value is positive, the left hand side remains positive in the whole range $\chi \geq -\sigma$, and the saddle point equation does not have a real solution. Therefore, the existence of a real solution $\chi = \chi(\sigma)$ requires the following condition to be satisfied,

$$\mu - \sigma - \frac{\sigma^2}{2\Lambda} - \frac{1}{4N\Lambda} \left( a_\sigma^{(0)} \right)^{3/2} \left( \sqrt{\frac{2\sigma^2}{\Lambda}} + \sigma + a_\sigma^{(0)} \Lambda^2 - \sqrt{2 \sigma^2 + \sigma} \right) - \frac{1}{4Na_\sigma^{(0)}} \leq 0. \quad (13)$$

This condition may be rewritten in a much more transparent way. Indeed, if we choose a large enough value of $N$ the condition in Eq. (13) does not hold for $\sigma = 0$. By combining this with the fact that the function in the left hand side of Eq. (13) decreases with increasing $\sigma$ (this is true everywhere except for a small unimportant region of order $\mu^3/(N\Lambda)^2$ around $\sigma = 0$), we conclude that the above condition is equivalent to $\sigma > \sigma_{cr}$, where

$$\sigma_{cr} \approx \mu - \frac{a_\sigma^{(0)}}{4Na_\sigma^{(0)}}, \quad (14)$$

For any value of $\sigma$ greater than $\sigma_{cr}$, we can numerically solve the saddle point equation (11). Then substituting the solution $\chi = \chi(\sigma)$ into Eq. (8), we finally obtain the effective potential as a function of $\sigma$ alone. Our numerical analysis shows that the potential is a monotonically increasing function in the region $\sigma > \sigma_{cr}$ for a very wide range of input parameters, which are the cutoff ($\Lambda$), the temperature ($T$), the number of flavors ($N$), the kinetic coefficients ($a_\sigma^{(0)}$, and $a_\pi^{(0)}$), and finally the scale $\mu = \Lambda(q - 1)/g$, which we put to 1 without loss of generality. In Figure 1, we present the plot of the effective potential in the region $\sigma > \sigma_{cr}$ obtained for a specific set of the parameters.

Obviously, the straightforward extension of our calculations to the region $0 < \sigma < \sigma_{cr}$ would result in a non-zero imaginary part in the effective potential. Such a situation looks rather disappointing. It has been claimed, though, that the Maxwell construction has to be used for the region $0 < \sigma < \sigma_{cr}$ [28]. The latter solution of the problem presumably corresponds to the flat potential in the “forbidden” range of $\sigma$ as is plotted in Figure 1. We would like also to mention Ref. [29] where a strong argument in support of the flat potential was given. This behavior of the potential is interpreted as a signal for the phase with the broken symmetry.

### III. NON ZERO TEMPERATURE

Now, let us study the NJL model at finite temperature. The partition function is given by the following path integral

$$Z = \int \left( \prod_{x,\tau} d\bar{\Psi} d\Psi \right) e^{-S_{NJL}}, \quad (15)$$

where the functional measure is defined in a way consistent with the standard anti-periodic boundary conditions for the fermion field. The Euclidean “action” that appears in the partition function is given by

$$S_{NJL} = \int_0^\beta d\tau \int d^2x \left( \bar{\Psi} i\gamma_\mu \partial_\mu \Psi - \frac{G}{2N} \left[ (\bar{\Psi}\Psi)^2 + (\bar{\Psi}i\gamma_5 \Psi)^2 \right] \right), \quad (16)$$

where $\beta \equiv 1/T$ is the inverse temperature.

As in the case of zero temperature, we introduce two auxiliary scalar fields, so that the partition function is given by a path integral over both the fermion and the boson fields. The corresponding action reads

$$S_{aux} = \int_0^\beta d\tau \int d^2x \left[ \bar{\Psi} (i\gamma_\mu \partial_\mu - \sigma - i\gamma_5 \pi) \Psi + \frac{N}{2G} \left( \sigma^2 + \pi^2 \right) \right]. \quad (17)$$

We are interested in the effective action for the $\sigma$-field. In order to obtain it, first, we calculate the expression for the partition function in terms of the scalar fields alone by performing the Gaussian integral over the fermions. As a result, we are left with the following “exact” action for the scalars,
\[ S_{\text{scal}} = N \left[ \frac{1}{2G} \int_0^\beta \int d\tau \int d^2x \left( \sigma^2 + \pi^2 \right) - Tr \ln(i\gamma^\mu\partial_\mu - \sigma - i\gamma_5\pi) \right], \quad (18) \]

and the partition function is given by the integral over the scalar fields,

\[ Z = \int \left( \prod_{x,\tau} d\sigma d\pi \right) e^{-S_{\text{scal}}}. \quad (19) \]

Following closely our zero temperature calculation, we approximate the action in Eq. (18) by a truncated derivative expansion (which in momentum space corresponds to the expansion in powers of momenta up to the second order). The approximate action is

\[ S_{\text{scal}} \simeq \frac{N}{\pi} \int_0^\beta d\tau \int d^2x \left[ -\frac{\Lambda}{2} \left( 1 - \frac{1}{g} \right) \rho^2 + \frac{\rho^3}{3} + \frac{\rho^4}{8\Lambda} + O \left( \frac{\rho^6}{\Lambda^3} \right) \right] + 2T \int_0^\rho dx \ln \left( 1 + \exp \left( -\frac{x}{T} \right) \right) + L_{\text{kin}}(\sigma, \pi), \quad (20) \]

where we use the same notation as in the previous section. The kinetic term looks somewhat different from that at zero temperature,

\[ L_{\text{kin}}(\sigma, \pi) \simeq a_\sigma + \frac{1}{2} \sum_{i=1}^2 \left[ (\partial_\tau \sigma)^2 + (\partial_\pi \pi)^2 \right] + b_\pi \left[ (\partial_\tau \sigma)^2 + (\partial_\pi \pi)^2 \right] + a_\sigma a_\pi \sum_{i=1}^2 (\sigma \partial_\tau \sigma + \pi \partial_\pi \pi)^2 + b_\pi b_\pi \frac{\sigma \partial_\sigma \sigma + \pi \partial_\pi \pi)^2}{2\rho^2}, \quad (21) \]

where the coefficient \( a \)'s and \( b \)'s are now not equal. This is due to the heat bath which always breaks Lorentz invariance. In particular, the infrared asymptotic behavior of the kinetic term leads to the following coefficients,

\[ a_\sigma = \frac{1}{6\rho} \left( \tanh \left( \frac{\rho}{2T} \right) + \frac{\rho}{4T \cosh^2(\rho/2T)} \right), \quad (22) \]
\[ b_\sigma = \frac{1}{6\rho} \left( \tanh \left( \frac{\rho}{2T} \right) - \frac{\rho}{2T \cosh^2(\rho/2T)} + \frac{\rho^2 \tanh(\rho/2T)}{2T^2 \cosh^2(\rho/2T)} \right), \quad (23) \]
\[ a_\pi = \frac{1}{4\rho} \tanh \left( \frac{\rho}{2T} \right), \quad \text{and} \quad (24) \]
\[ b_\pi = \frac{1}{4\rho} \left( \tanh \left( \frac{\rho}{2T} \right) - \frac{\rho}{2T \cosh^2(\rho/2T)} \right). \quad (25) \]

As in the zero temperature case, we do not use these expressions later in the calculation, but switch to the same approximation as before. Namely, we assume that the \( a \)'s and \( b \)'s are independent of \( \rho \). Later we give arguments to substantiate this claim.

After “integrating in” the auxiliary \( \chi \) field, we arrive at the following action,

\[ S_{\text{scal}} \simeq \frac{N}{\pi} \int_0^\beta d\tau \int d^2x \left[ -\frac{\Lambda}{2} \chi^2 + \frac{\chi^2}{2} (\rho^2 - 2\Lambda \mu) - \frac{\Lambda \mu^2}{2} + \frac{\rho^3}{3} \right] + 2T \int_0^\rho dx \ln \left( 1 + \exp \left( -\frac{x}{T} \right) \right) + L_{\text{kin}}(\sigma, \pi). \quad (26) \]

This is used for the construction of the effective potential. As in the case of zero temperature, we obtain the result of interest in the next to leading order of the large \( N \) expansion:

\[ V_{\text{eff}}(\sigma, \chi) \simeq \frac{N}{\pi} \left[ -\frac{\Lambda}{2} \chi^2 + \frac{\chi^2}{2} (\sigma^2 - 2\Lambda \mu) - \frac{\Lambda \mu^2}{2} + \frac{\sigma^3}{3} + 2T \int_0^\sigma dx \ln \left( 1 + \exp \left( -\frac{x}{T} \right) \right) \right] + T \sum_{n=-\infty}^{\infty} \int \frac{d^2p}{(2\pi)^2} \ln(M_\pi + a_\pi p^2 + b_\pi \Omega^2) + T \sum_{n=-\infty}^{\infty} \int \frac{d^2p}{(2\pi)^2} \ln(M_\pi + a_\pi p^2 + b_\pi \Omega^2) \], \quad (27)
where we define
\[ M_\sigma = \chi + \frac{\sigma^2}{2\Lambda} + \frac{2\sigma}{1 + \exp(-\sigma/T)} + 2T \ln \left[ 1 + \exp(-\sigma/T) \right] \quad \text{and} \quad \frac{M_\sigma}{\Lambda} = \chi + \sigma + 2T \ln \left[ 1 + \exp(-\sigma/T) \right]. \] (28)

After carrying out the summation over Matsubara frequencies and the integration over momenta (with the sharp cutoff at \( \Lambda \)), we arrive at the result
\[ V_{eff}(\sigma, \chi) \approx \frac{N}{\pi} \left[ -\frac{\Lambda}{2} \frac{\chi^2}{2} + \frac{\chi}{2} (\sigma^2 - 2\Lambda \mu) - \frac{\Lambda \mu^2}{2} + \frac{\sigma^3}{3} + 2T \int_0^\sigma dx \ln \left( 1 + \exp \left( -\frac{x}{T} \right) \right) \right] \\
+ \frac{1}{6\pi a_\sigma \sqrt{b_\sigma}} \left[ (M_\sigma + a_\sigma \Lambda^2)^{3/2} - (a_\sigma \Lambda^2)^{3/2} - (M_\sigma)^{3/2} \right] \\
+ \frac{1}{6\pi a_\pi \sqrt{b_\pi}} \left[ (M_\pi + a_\pi \Lambda^2)^{3/2} - (a_\pi \Lambda^2)^{3/2} - (M_\pi)^{3/2} \right] \\
- \frac{T}{2\pi a_\sigma} \int_0^\chi dx \ln \left( 1 - \exp \left( -\frac{\sqrt{x}}{T \sqrt{b_\sigma}} \right) \right) - \frac{T}{2\pi a_\pi} \int_0^\pi dx \ln \left( 1 - \exp \left( -\frac{\sqrt{x}}{T \sqrt{b_\pi}} \right) \right). \] (30)

Despite the fact that the calculations were simplified due to the nontrivial assumption on the form of the kinetic part of the boson action, we argue that the result should be reliable. Indeed, the essential details of both the infrared and the ultraviolet regions seem to be captured if the kinetic coefficients are chosen to be independent of \( \sigma \).

The only property that we require of the boson propagators in the ultraviolet region is a soft dependence on the momentum \( b_\sigma \)'s and \( b_\pi \)'s on the value of the field \( \sigma \). Fortunately, there is no reason to believe that a mild dependence of the propagators on \( \sigma \) in this region could change any of our qualitative results.

Now, let us mention that the naïve limit \( T \to 0 \) of our finite temperature potential given in Eq. (30) coincides with the effective potential obtained in the previous section for \( T = 0 \) if we assume that \( a_\sigma, b_\sigma \to a_\sigma(0) \) and \( a_\pi, b_\pi \to a_\pi(0) \).

To eliminate the auxiliary field \( \chi \), we have to solve the saddle point equation, as was done in the case of zero temperature. This saddle point equation is obtained by taking the derivative of the effective potential (30) with respect to \( \chi \):
\[ \chi + \mu - \frac{\sigma^2}{2\Lambda} - \frac{1}{4N\Lambda a_\sigma \sqrt{b_\sigma}} \sqrt{M_\sigma + a_\sigma \Lambda^2 - \sqrt{M_\sigma}} + \frac{T}{2N\Lambda a_\sigma} \ln \left( 1 - \exp \left( -\frac{\sqrt{M_\sigma}}{T \sqrt{b_\sigma}} \right) \right) \]
\[ + \frac{T}{2N\Lambda a_\pi} \ln \left( 1 - \exp \left( -\frac{\sqrt{M_\pi}}{T \sqrt{b_\pi}} \right) \right) = 0. \] (31)

Let us analyze this equation. As in the case of \( T = 0 \), the left hand side of Eq. (31) is a monotonically increasing function of \( \chi \). Unlike the former, though, the left hand side of eq. (31) is not bounded below, but approaches negative infinity as \( \chi \to -\sigma - 2T \ln[1 + \exp(-\sigma/T)] \), so that \( M_\sigma \to 0 \). This simple property has far reaching consequences. The most important among them is that the saddle point equation has a real solution for all values of \( \sigma \). This is quite remarkable if we notice that Eq. (31) in the "naïve" limit \( T \to 0 \) coincides with its counterpart that was obtained for \( T = 0 \).

The saddle point equation for \( T = 0 \), as we saw, does not have a solution for \( 0 < \sigma < \sigma_{cr} \), while the finite temperature equation allows a real solution for all values of \( \sigma \). How could this happen? The answer is quite simple, and involves the way the limit \( T \to 0 \) is taken. In the "naïve" limit, we assume that \( M_\sigma \) and \( M_\pi \) become much larger than the value of \( T \) as we approach zero temperature. On the other hand, at any small but finite temperature, we find that the solution to the saddle point equation requires that \( M_\sigma \sim T^2 b_\pi \exp(-Nkf(\sigma)/T) \), where \( f(\sigma) \) is a positive function of \( \sigma \) for \( 0 < \sigma < \sigma_{cr}(T) \).
Therefore, we observe that the $T = 0$ quantum field theory corresponds to the “naïve” limit when $T \to 0$ is taken before solving the saddle point equation. We get a very different result by taking the limit in the other order, i.e., solving the saddle point equation first and then taking the value of $T$ to zero.

The finite temperature gap equation is given by

$$
\chi \sigma + \sigma^2 + \frac{\partial \sigma (M_\sigma)}{4 N a_\sigma \sqrt{\beta_\sigma}} \left[ \sqrt{M_\sigma + a_\sigma \Lambda_\sigma^2} - \sqrt{M_\sigma} \right] \right] + \frac{\partial \sigma (M_\sigma)}{4 N a_\sigma \sqrt{\beta_\sigma}} \left[ \sqrt{M_\sigma^2 + a_\sigma \Lambda_\sigma^2 - \sqrt{M_\sigma}} \right] \right] = 0,
$$

and admits the trivial solution $\sigma = 0$. For $T = 0$, this was not the case. Actually, the existence of the trivial solution is a necessary condition for concluding that the symmetry is restored, but not a sufficient condition. Below, while presenting our numerical results, we will see that this $\sigma = 0$ solution is the only solution to the gap equation, and that it corresponds to the global minimum of the effective potential.

Our numerical analysis begins with a solution of the saddle point equation (31). While solving it, we notice that there are two qualitatively different regions of the field $\sigma$. For small enough values of the field $\sigma < \sigma_{cr}(T)$, where the $T = 0$ equation would not have a solution, we can give a very good analytical estimate for the finite temperature solution,

$$
\chi = -\sigma - 2T \ln \left[ 1 + \exp(\sigma/T) \right] + \tilde{M}_\sigma,
$$

where

$$
\tilde{M}_\sigma \sim T^2 b_\sigma \exp \left( -\frac{\Lambda_\sigma}{T} f(\sigma) \right),
$$

and where $f(\sigma)$ is a positive function of $\sigma$ for $0 < \sigma < \sigma_{cr}(T)$. After $\sigma$ reaches the value $\sigma_{cr}(T)$, the solution $\chi(\sigma)$ sharply changes its behavior and starts to follow closely the solution found for $T = 0$. The results of our numerical solution to the saddle point equation for four different temperatures are plotted in Figure 2.

By substituting the solution $\chi = \chi(\sigma)$ of the saddle point equation into Eq. (30), we obtain the effective potential as a function of $\sigma$. The corresponding plots for the four different temperatures are given in Figure 3. As we see, the effective potential reveals the behavior which was expected: its global minimum lies at the origin. The latter means that the chiral symmetry is not broken.

By comparing the zero and finite but small temperature effective potentials, we find that they follow very closely each other in the range of large enough values of the field $\sigma > \sigma_{cr}$.

Another property of the obtained effective potential which is worth noting is its convexity. This is a feature that one expects for the effective potential based on general arguments [29–31]. To see that our effective potential is indeed convex, we have to show that its derivative is non-negative in the whole range of field $\sigma$. For this purpose, in Figure 4, we plot the left hand side of the gap equation (32). Up to an unessential factor, it is equal to the derivative of the potential as a function of $\sigma$. As we see, the latter is everywhere non-negative.

Before concluding this section, we have to note that all the qualitative results, such as the existence of one global minimum at $\sigma = 0$, the convex character of the potential, and the two-branched character of the solution to the saddle point equation, persist for a rather wide choice of the input parameters.

### IV. CONCLUSION

In this paper we have studied the effective potential in the three-dimensional NJL model at finite and zero temperatures up to the next to leading order in the large $N$ expansion. The distinctive feature of our analysis is the introduction of an additional scalar field. This allows us to circumvent the well known, and otherwise unavoidable, problem of the imaginary contributions to the finite temperature effective potential.

In agreement with the MWC theorem, the analysis of the effective potential in the three-dimensional NJL model at finite temperature does not reveal breaking of the continuous chiral symmetry. This remains true for an arbitrarily small value of the temperature. The leading order effective potential that indicates symmetry breaking changes drastically in the next to leading order. The essential role is played by the pion-like particle, which would have become the NG boson were the symmetry broken.

We present the numerical results that clarify and confirm our analytical findings in Figures 1-4. The zero temperature effective potential is given in Figure 1. Since it is not defined in the range of fields $0 < \sigma < \sigma_{cr}$, we extended it
with the Maxwell construction to the forbidden region. More rigorous arguments in support of such an extension can be found in [29]. Figures 2 through 4 present the results at finite temperature.

By comparing our analytical and numerical results for both $T = 0$ and $T \neq 0$, we come to the conclusion that the zero temperature field theory can be defined in two different ways. One definition comes from the standard $T = 0$ quantum field theory. The other is obtained by the limiting procedure $T \to 0$ from the Matsubara imaginary time formalism.

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FIG. 1. The effective potential at zero temperature extended by the Maxwell construction in the region of small values of the field. The results given in units of $\mu$: $V(x) = V_{\text{eff}}(x)/\mu^3, x = \sigma/\mu$. The input parameters are $\Lambda = 10\mu$, $a_s^{(0)} = a_\psi^{(0)} = 5/\mu$, and $N = 50$. 
FIG. 2. The finite temperature solution to the saddle point equation given in units of $\mu$: $k(x) = \chi(x)/\mu$, $x = \sigma/\mu$. The input parameters are $\Lambda = 10\mu$, $a_\sigma^{(0)} = a_\pi^{(0)} = 5/\mu$, and $N = 50$. 
FIG. 3. The effective potential at non-zero temperatures given in units of $\mu$: $V(x) = V_{\text{eff}}(x)/\mu^3$, $x = \sigma/\mu$. The input parameters are $\Lambda = 10\mu$, $a^{(0)} = a^{(0)} = 5/\mu$, and $N = 50$. 

$V(x)$

$T=0.01$

$T=0.2$

$T=0.4$

$T=0.7$
FIG. 4. The left hand side of the gap equation (which is proportional to the derivative of the effective potential) as the function of $x = \sigma / \mu$. The input parameters are $\Lambda = 10\mu$, $a_\sigma^{(0)} = a_\mu^{(0)} = 5/\mu$, and $N = 50$. 