CM-P00057782

GENERALIZED UNITARITY AND LOW ENERGY THEOREMS

D. Horn
Tel Aviv University
and
CERN - Geneva

and

M. Jacob
CERN - Geneva

A B S T R A C T

We discuss the question how and when one has to include low energy theorems within the framework of S matrix theory together with generalized unitarity. We show on explicit examples how the low energy theorems emerge when a photon is involved and why they are not implied if we replace the photon by a massive vector meson.

I. - GENERALIZED UNITARITY AND POLE TERMS

In the framework of S matrix theory, it is generally assumed that the singularities occurring in scattering amplitudes can be traced and related to exchanged systems. Above thresholds, these singularities are obtained from unitarity applied in the different channels. Below threshold unitarity is generalized using Landau's rules and Cutkosky's prescription ¹⁾. It is assumed that the knowledge of these singularities, together with the asymptotic behaviour, fully define the physical amplitude.

The most simple singularities met are pole terms which are associated with one particle exchange. They are most useful since their residues factorize and information can thus simply be conveyed from one physical amplitude to another. The above mentioned principles lead us to expect a pole whenever it is possible to separate the particles participating in a reaction into two groups both of which have the same quantum numbers as an existing particle. This property should, however, hold without restriction only for spinless external particles. If any of these particles have spin different from zero, we are faced with the possibility of nonsense transition ²⁾. At the position of the pole, the exchange particle, which is physical, cannot carry an angular momentum greater than its spin. It will therefore lead to the existence of a pole only in certain spin configurations. In the following we will deal with two-body scattering amplitudes. In that case, the exchange of a particle of spin J will lead to a pole only in the helicity amplitudes which satisfy

$$|\lambda_c - \lambda_a| \leq J \quad \text{and} \quad |\lambda_b - \lambda_d| \leq J \quad (1)$$

where λ_a and λ_b (λ_c and λ_d) are the helicities of the initial (final) pair of particles. This is called a sense-sense transition ²⁾.

In any other case, the transition is forbidden and no pole will occur. It would therefore seem that no simple information can be obtained for nonsense transitions in the framework of S matrix theory. This is an uneasy situation since perturbation theory does not separate sense from nonsense transitions. In the case of sense, it gives the pole term. In the case of nonsense, it provides a non-singular term which is expressed in terms of the coupling constants and the masses involved. As a matter of fact this is known to provide low energy theorems ³⁾ whenever it is possible to prove that, in certain limits, only the Born terms contribute to the amplitude. It may then seem desirable to see how such low-energy theorems may arise in the framework of S matrix theory. This is interesting since it is often tempting to consider only some of the helicity amplitudes and analyze their properties with emphasis on nearby singularities which can be obtained from generalized unitarity. This is the case in processes involving photons that have only transverse polarization states (helicity ± 1). The interest in particular helicity amplitudes can also arise due to special asymptotic properties.

We discuss two particular examples and see that in such cases where a photon is involved in the reaction, the poles that should not exist according to generalized unitarity impose, via the gauge condition, the low energy theorems. This also ensures the existence of the Born terms in the nonsense amplitudes. However, the situation is different if the photon is replaced by a vector particle of non-zero mass. Then, even if it is coupled to a conserved vector current, additional freedom prevails and a low energy theorem cannot be proved.

II. - BORN TERMS

We use the name Born terms to designate the exchange of a particle that is the same as one of the incoming or outgoing particles that participate in the process in question. As already mentioned, such Born terms can give rise either to poles or to non-singular contributions depending on the particular amplitude in question. We deal with examples where all possible Born terms have to be considered together. This is due to the fact that we deal with interactions with a conserved vector current. We thus encounter situations in which Born terms in the s , t and u channels appear simultaneously.

The two examples that we study here are isovector Compton scattering on pions and pion photoproduction off nucleons. In both cases the photon interacts through a conserved vector current and, therefore, all Born terms have to be considered together. Thus a t channel Born term is related to the contribution of the s and u pole terms. It turns out that, although some of these terms occur in sense channel while others occur in nonsense channels, they are all interconnected by the Gauge condition. This connection necessitates the inclusion of the Born contributions to the nonsense amplitudes as low-energy theorems. Then we try to apply the same principles to ρ production in pion-nucleon collision, assuming that the ρ is coupled to a conserved vector current ⁴⁾. This results again in similar gauge conditions to those of the previous case. However, it turns out that the mass of the ρ plays a crucial role and that the conclusion about the low energy theorem can be arrived at only in the limit of zero mass. This is related to the fact that the universality property of the vector meson coupling can be defined without ambiguity only in the zero mass limit.

III. - ISOVECTOR COMPTON SCATTERING ON PIONS

As a first example we consider isovector Compton scattering on pions. We use the notation and results of Ref. 5). Let us define the gauge invariance scattering amplitude in the form

$$T_{\mu\nu} = A \left\{ (p_1 - p_2)_\mu (p_1 - p_2)_\nu - k_{1\mu} k_{2\nu} + \frac{2(s - \mu^2)(u - \mu^2)}{t} g_{\mu\nu} + \right. \\ \left. + \frac{u - s}{t} [(p_1 - p_2)_\mu k_{2\nu} - (p_1 - p_2)_\nu k_{1\mu}] \right\} + B \left\{ g_{\mu\nu} - \frac{2k_{1\mu} k_{2\nu}}{t} \right\} \quad (2)$$

where p_1 and p_2 are the incoming momenta of the pions and k_1 and k_2 are the incoming momenta of the isovector photons. The identification with the helicity amplitudes is readily given by

$$f_{++}^s = -f_{+-}^t = \left((s - \mu^2)^2 + st \right) \frac{2A}{t} \\ f_{+-}^s = f_{++}^t = 2\mu^2 A + B \quad (3)$$

In the t channel we find a possible singularity at $t = 0$ due to the exchange of an isovector photons. However, that diagram cannot be taken alone due to the lack of gauge invariance. It will not lead to a contribution of the kind (2). In order to achieve gauge invariance one has to add the pion s and u Born terms although they correspond to nonsense transitions. This is an example where we are forced to use the larger definition of generalized unitarity. Taking all three diagrams into account, we find for the $I = 1$ exchange amplitudes the following form at $t = 0$

$$A = -\frac{e^2}{s - \mu^2} + \frac{e^2}{u - \mu^2} \quad ; \quad B = 0 \quad (4)$$

This result leads to the following limits

$$\begin{aligned} f_{+-}^S &\xrightarrow[t \rightarrow 0]{} - \frac{4\mu^2 e^2}{s - \mu^2} \\ f_{++}^S &\xrightarrow[t \rightarrow 0]{} - \frac{4(s - \mu^2) e^2}{t} \end{aligned} \quad (5)$$

In this particular case the forward non-flip amplitude is singular and the flip amplitude is finite due to the existence of the infinite range force. This is a somewhat peculiar behaviour. However, it does not really affect the formalism. In particular note that we are considering the above-mentioned scattering amplitude only to second order in the electromagnetic coupling e . Therefore the pole at $t = 0$ is really a secluded pole and not a branch point.

In the case of $I = 1$ exchange we are therefore forced to use generalized unitarity in its larger sense, supplying us with low energy and low momentum transfer theorems. The question now arises whether this is the case also in the symmetric t channel combinations of the amplitudes. Although only nonsense Born terms exist there (in the s and u channels) one can nevertheless invoke isospin arguments to show that, also there, generalized unitarity must be taken in its broader sense. The reasoning goes as follows: suppose that only the t channel $I = 1$ amplitude A has the Born poles. Then, by isospin crossing they would appear in all the three possible isotopic spin combinations in the s channel. However, that would lead to poles where no particle are present. Therefore we have to assume that this is not the case and the only possible meaningful way out is to consider simultaneously all Born terms in all possible channels corresponding to their physical isospin values. Using now the optical theorem one finds the low t behaviour of the various isospin amplitudes ⁵⁾

$$\begin{aligned} \text{Im } A^I &= \pi e^2 \delta\left(\mu\nu - \frac{t}{2}\right) + \frac{t}{\mu\nu} \nabla_{tot}^I(\nu) + O(t^2) \\ &\pm (\nu \leftrightarrow -\nu) \end{aligned} \quad (6)$$

where ν is defined by

$$\nu = \frac{s-u}{2\mu} = \frac{1}{\mu} \left(s - \mu^2 + \frac{t}{2} \right) = \frac{1}{\mu} \left(-u + \mu^2 - \frac{t}{2} \right)$$

and σ_{tot}^I corresponds to the total photoproduction cross-section (to order e^2) for the I th isospin combination in the t channel ($I=0,1,2$). The $I=0$ and 2 amplitudes are symmetric under the $\nu \leftrightarrow -\nu$ exchange, while $I=1$ is antisymmetric. We thus arrived at the expected low momentum and low energy theorems. For a general discussion of the latter in pion and nucleon Compton effect we refer the reader to the article of Abarbanel and Goldberger⁶⁾.

IV. - PION PHOTOPRODUCTION OFF NUCLEONS

We proceed here along slightly different lines and start with the fact that the photoproduction amplitude can be most generally written as

$$\varepsilon^\mu T_\mu = \sum_{i=1}^8 B_i(s, t, k^2) N_i \quad (7)$$

where

$$\begin{aligned} N_1 &= i\gamma_5 \cancel{\not{k}} & N_5 &= \gamma_5 \cancel{\not{k}} \\ N_2 &= i\gamma_5 \varepsilon \cdot (p - p') & N_6 &= \gamma_5 \cancel{\not{k}} \varepsilon \cdot \frac{p - p'}{2} \\ N_3 &= -2i\gamma_5 \varepsilon \cdot q & N_7 &= \gamma_5 \cancel{\not{k}} \varepsilon \cdot k \\ N_4 &= 2i\gamma_5 \varepsilon \cdot k & N_8 &= -\gamma_5 \cancel{\not{k}} \varepsilon \cdot q \end{aligned}$$

We have chosen crossed channel variables and k and q are the incoming momenta of the photon and π meson when p and p' stand for the outgoing momenta of the nucleon and anti-nucleon.

We have followed the choice and notation of Ball ⁷⁾, who has shown that the invariant amplitudes thus defined are free of kinematical singularities. Not all of them are physical amplitudes: B_4 and B_7 do not affect the cross-section. However, they enter into the gauge invariance relations ⁷⁾.

$$\begin{aligned} k^2 \left(\frac{B_1}{2} + B_4 \right) + \frac{1}{2} B_2 \left(s + \frac{t - 2m^2 - \mu^2 - k^2}{2} \right) - \frac{1}{2} (t - \mu^2 - k^2) B_3 &= 0 \\ B_5 - k^2 B_7 + \frac{1}{2} B_6 \left(s + \frac{t - 2m^2 - \mu^2 - k^2}{2} \right) - \frac{1}{2} (t - \mu^2 - k^2) B_8 &= 0 \end{aligned} \quad (8)$$

The physical amplitudes for photoproduction have the photons transversely polarized (helicity ± 1) and, therefore, pion exchange would correspond to a nonsense transition. It is readily checked that such a pion Born term can contribute only to B_3 and B_4 which do not affect the physical helicity amplitudes. Nevertheless the gauge invariance condition (8) shows that, at the position of the pion pole in B_3 ($t = \mu^2$) and for $k^2 = 0$, the amplitude B_2 is fully determined (for all s) in terms of the pion pole residue. For π^+ photoproduction one finds, for instance, that the pion pole residue, as given in the Table, are $-\sqrt{2}$, e.g., in B_3 and $\sqrt{2}/2$ e.g., in B_4 , where g is the pion-nucleon coupling constant. At $t = \mu^2$, we find therefore that B_2 is given by

$$B_2 = - \frac{\sqrt{2} \, e g}{s - m^2} \quad (9)$$

To understand how it happens that the pion pole generates, through the gauge condition, the form of B_2 we have to recall the well-known fact from perturbation theory into account simultaneously. Indeed it turns out that the form (9) is nothing but the contribution of the nucleon electric Born term. Thus the two possible Born terms satisfy together the condition (8) ⁷⁾. The point that interests us here is that the pion pole that contributes to a nonsense transition, is necessarily connected to the sense nucleon Born term. Similarly, the way that led us to Eq. (9) we could have started from the nucleon Born term and deduced the right constant term in the B_3 amplitude at $s = m^2$, that would be nothing else but the pion Born term.

The amplitude B_2 contributes to two of the nonsense t channel helicity amplitudes (as defined for pion exchange). These contributions read

$$f_{1/2, 1/2; 1} = - f_{-1/2, -1/2; 1} = \frac{i B_2}{\sqrt{2}} \frac{\sqrt{t(t/4 - m^2)}}{m} \sin \theta_t$$

where we write as $f_{\lambda\lambda';\mu}$ the contribution to the amplitude with nucleon, anti-nucleon and photon helicities λ, λ' and μ respectively⁸⁾. The scattering angle in the crossed t channel is denoted by θ_t . One has

$$\sin \theta_t = \frac{2}{t - \mu^2} \left(\frac{st(2m^2 - \mu^2 - s - t) - tm^2(m^2 - \mu^2) - m^2\mu^4}{t - 4m^2} \right)^{1/2}$$

and it follows that

$$\lim_{t \rightarrow \mu^2} f_{1/2, 1/2; 1} = \frac{\mu^2 e g}{m(t - \mu^2)} \quad (11)$$

This contribution of B_2 to the amplitude is therefore fully determined at $t = \mu^2$ even though the pion pole term is absent from B_2 and, in the sense of generalized unitarity, from the physical helicity amplitudes.

These results may be viewed as low t theorems which are valid near the pion (nonsense) point. They may also be considered in terms of a low s theorem near the pole at $s = m^2$. We saw already that, at $t = \mu^2$, B_2 is fully determined and has just the pole term at $s = m^2$. Alternatively, at $s = m^2$, B_3 is fully determined as having the constant value given by the pion Born term.

Opposite contributions to the two amplitudes $f_{1/2, 1/2; 1}$ and $f_{-1/2, -1/2; 1}$ could be associated with the exchange of a Regge trajectory with positive signature and negative parity and G parity. They may thus be associated with the contribution of the pion trajectory which is then fully determined at $t = \mu^2$, even though the exchanged trajectory, which chooses sense, does not lead to a pion pole in the physical amplitude. This is a complementary approach to the one followed in Ref.⁸⁾, in dealing with the Reggeized pion contribution^{8), 9)}.

We encountered here a behaviour that is characteristic of reactions that involve a photon as an external particle¹⁰⁾. Note that the exchange of any particle that is not one of the external ones can be written in a gauge invariant form so that it satisfied by itself Eq. (8). However, once the exchanged particle is one of the external ones (that we call here a Born term) its coupling is no longer free and it will not satisfy all gauge conditions by itself if other Born terms may exist. When all Born terms are taken into account they together satisfy the Gauge conditions and may provide low energy theorems in the s or t channels.

V. - VECTOR MESON PRODUCTION

A similar analysis can be carried out for the reaction $\pi N \rightarrow \rho N$, provided the ρ meson couples to a conserved vector current ^{4), 11)}. This assumption, which seems to be reasonable, brings in gauge invariance as a new dynamical concept ^{4), 11)}. The constraints are the same as given by Eqs. (8) with k^2 now replaced by m_ρ^2 . Contrary to the situation in photoproduction, the existence of a nucleon Born term does not imply and determine the pion Born term and vice versa. Nevertheless, let us first assume for the discussion that both exist. Their contributions to the various amplitudes are given in the Table. One finds that, when introduced in Eqs. (8), these singular contributions all cancel out against one another and leave only a constant term $\frac{1}{2}(-g\gamma_{\rho NN} + g\gamma_{\rho\pi\pi})$. If only pion and nucleon intermediate states were allowed this would lead to the conclusion that $\gamma_{\rho NN} = \gamma_{\rho\pi\pi}$ which is conventionally called universality of ρ coupling. However, we have to allow for regular contributions at $t = \mu^2$ and $s = m^2$. Denoting by b_1, b_2, b_3 and b_4 these contributions to the various invariant amplitudes, condition (8) reduces to

$$\begin{aligned} \frac{m_\rho^2}{2} \left(b_1 + 2b_4 - \frac{1}{2}b_2 + b_3 \right) + \frac{1}{2} \left(\gamma_{\rho\pi\pi} - \gamma_{\rho NN} \right) g + \\ + \frac{1}{2} (t - \mu^2) \left(\frac{1}{2}b_2 - b_3 \right) + \frac{1}{2} (s - m^2) b_2 = 0 \end{aligned} \quad (12)$$

At the particular point $s = m^2$ and $t = \mu^2$, we find

$$g \left(\gamma_{\rho NN} - \gamma_{\rho\pi\pi} \right) = m_\rho^2 \left(b_1 + 2b_4 - \frac{1}{2}b_2 + b_3 \right) \quad (13)$$

Universality in its usual sense is thus required only in the limit of zero ρ mass, in much the same way as it can be defined in field theory. Otherwise we have only a condition that links together the Born terms on one hand with a remainder on the other hand. There

is no reason why these two parts should separately satisfy a gauge invariance property. In the case of photoproduction the zero photon mass decouples the Born terms from the rest and leads to universality. A low energy theorem then results for the case where a zero mass meson is involved.

	Nucleon Born term contribution	Pion Born term contribution
B_1	$g \frac{G'}{2} \left(\frac{1}{s-m^2} + \frac{1}{m^2-u} \right)$	0
B_2	$-g \frac{G'}{2} \left(\frac{1}{s-m^2} + \frac{1}{m^2-u} \right)$	0
B_3	$-\frac{gG'}{4} \left(\frac{1}{s-m^2} - \frac{1}{m^2-u} \right)$	$-\frac{Gg}{t-\mu^2}$
B_4	$-\frac{gG'}{4} \left(\frac{1}{s-m^2} - \frac{1}{m^2-u} \right)$	$\frac{1}{2} \frac{Gg}{t-\mu^2}$

g is the pion-nucleon coupling constant ;

G and G' are the ρ pion $\gamma_{\rho\pi\pi}$ and ρ nucleon $\gamma_{\rho NN}$ coupling constants. They would be both equal to e for isovector photon coupling and we then get an extra minus sign in front of the u pole term for isoscalar coupling. We take $\sqrt{2}g$ instead of g for π^+ photoproduction.

REFERENCES

- 1) See, e.g., G.F. Chew "S matrix theory of strong interactions" - Benjamin, New York (1962).
- 2) M. Gell-Mann - Proceedings of the International Conference on High Energy Physics, CERN, p. 533 (1962).
- 3) F.E. Low - Phys.Rev. 96, 1428 (1954) ;
M. Gell-Mann and M.L. Goldberger - Phys.Rev. 96, 1433 (1954).
- 4) J.J. Sakurai - Annals of Physics 11, 1 (1960).
- 5) D. Horn - Caltech Internal Report 68-131 (1967).
- 6) H.D.I. Abarbanel and M.L. Goldberger - Princeton Preprint (1967).
- 7) J.S. Ball - Phys.Rev. 124, 2014 (1961).
- 8) J.S. Ball and M. Jacob - CERN Preprint TH. 825 (1967) ; Nuovo Cimento (to be published).
- 9) G. Zweig - Nuovo Cimento 32, 689 (1964) ;
N. Dombey - Nuovo Cimento 31, 1025 (1964).
- 10) We have used only a mass shell gauge invariance property, which results from Lorentz invariance for zero mass particles.
S. Weinberg - Phys.Rev. 135B, 1049 (1964) ;
D. Zwanziger - Phys.Rev. 133B, 1036 (1964).
- 11) M. Gell-Mann - Phys.Rev. 125, 1067 (1962).