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CAN CURRENT ALGEBRA DETERMINE THE LOW ENERGY
 $\pi - \pi$ SCATTERING AMPLITUDE ?

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ABSTRACT

Using current algebra techniques, we calculate the low energy $\pi - \pi$ scattering amplitude, taking into account unitarity and analyticity. Four solutions have been found and the "smoothest" one gives Weinberg's scattering lengths.

In a previous article ¹⁾, hereafter quoted as I, we had tried to incorporate unitarity in Weinberg's original calculation of the $\pi-\pi$ scattering lengths through current algebra techniques ²⁾. This was achieved by expanding the amplitude in powers of k_s, k_t and k_u given in terms of the usual Mandelstam variables by :

$$k_s = \frac{1}{2} \sqrt{4-s} \quad , \quad k_t = \frac{1}{2} \sqrt{4-t} \quad , \quad k_u = \frac{1}{2} \sqrt{4-u} \quad (1)$$

The expression used in I [Eq. (13) in I] contained terms of the form $k_s k_t$, $k_s k_u$ and $k_t k_u$. However, one can show that such terms must be absent from the general expansion. A simple way of proving it is the following ³⁾ .:

Let M be the $\pi-\pi$ amplitude and let us expand $\text{Im} M$ in Legendre series :

$$\text{Im} M = \text{Im} M_0 + \sum_{\ell=1}^{\infty} \text{Im} M_{\ell} P_{\ell} \left(1 + \frac{t}{2q^2} \right) \quad (2)$$

where q is the modulus of the three momentum in the c.m.s. The right-hand side of (2) converges inside the large Lehmann ellipse for fixed t . In particular, when $q \rightarrow 0$, with t fixed, $\cos\theta \rightarrow \infty$ and the large Lehmann ellipse expands and covers the whole $\cos\theta$ plane. For small q $\text{Im} M_{\ell}$ behaves like ⁴⁾ $q^{4\ell+1}$ and $P_{\ell} \sim t^{\ell} q^{-2\ell}$. It is therefore obvious that the coefficient of q does not depend on t . The t dependence starts at q^3 ($\ell = 1$).

Based on this result, we can write the most general expansion for the off-the-mass-shell amplitude including terms of second order in k_s, k_t, k_u ⁵⁾

$$\begin{aligned} M = & \delta_{ab} \delta_{cd} \left[(A-2E) + B(k_s+k_u) + C k_t + (D+E) k_t^2 + E(k_s^2+k_u^2) \right] + \\ & + \delta_{ad} \delta_{cb} \left[(A-2E) + B(k_s+k_t) + C k_u + (D+E) k_u^2 + E(k_s^2+k_t^2) \right] + \\ & + \delta_{ac} \delta_{bd} \left[(A-2E) + B(k_t+k_u) + C k_s + (D+E) k_s^2 + E(k_u^2+k_t^2) \right] \end{aligned} \quad (3)$$

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Expansion (3) contains five parameters and we can determine them by using the following conditions :

- 1) s wave unitarity for the $I=0$ and $I=2$ parts of the amplitude at threshold;
- 2) Adler's self-consistency condition ⁶⁾;
- 3) the current algebra commutator :

$$\delta(x_0 - y_0) [A_0^a(x), A_\mu^b(y)] = 2i \epsilon_{abc} V_\mu^c(x) \delta^4(x-y) \quad (4)$$

- 4) the " σ -term" commutator :

$$\delta(x_0 - y_0) [A_0^a(x), \partial_\mu A_\mu^b(y)] = \delta_{ab} \sigma(x) \delta^4(x-y) \quad (5)$$

Finally, we use, as an extra piece of information, the results obtained by making full use of unitarity and crossing, combined with the rigorous analyticity properties of the $\pi - \pi$ scattering amplitude ⁷⁾. In this way we shall be able to check the general shape of our solutions and, possibly, reject the bad ones.

Let F^{00} be the on-the-mass-shell $\pi^0 \pi^0 \rightarrow \pi^0 \pi^0$ amplitude and F_0^{00} its s wave projection. The following results have been proved by Martin :

- a) $F^{00}(s, t)$ has a minimum at the symmetric point $s = t = u = 4/3$. The existence of an extremum at that point follows from crossing symmetry alone, therefore the only extra information that we can get from this property is the sign of the second derivative :

$$\frac{\partial^2 F^{00}(s, t)}{\partial^2 s} \bigg|_{s=t=\frac{4}{3}} > 0 \quad (6)$$

b) $F_0^{00}(s)$ and its derivative satisfy :

$$F_0^{00}(s) < F_0^{00}(4) \quad 0 \leq s < 4 \quad (7.a)$$

$$F_0^{00}(0) \geq F_0^{00}\left(2\left[1 + \frac{1}{\sqrt{3}}\right]\right) \quad (7.b)$$

$$\frac{dF_0^{00}(s)}{ds} > 0 \quad 1.7 \leq s \leq 4 \quad (7.c)$$

$$\frac{dF_0^{00}(s)}{ds} < 0 \quad 0 \leq s \leq 1.29 \quad (7.d)$$

Conditions (1) to (4) form a system of five equations with five unknowns, the parameters $A, \dots, E$. This system has four independent solutions with scattering lengths :

$$\alpha_0 = 0.21 \quad \alpha_2 = -0.053 \quad (8.a)$$

$$\alpha_0 = 2.10 \quad \alpha_2 = -0.025 \quad (8.b)$$

$$\alpha_0 = 0.77 \quad \alpha_2 = 2.30 \quad (8.c)$$

$$\alpha_0 = 1.50 \quad \alpha_2 = 2.30 \quad (8.d)$$

All these solutions satisfy the inequalities (6), and (7.a) to (7.d). It is worth noticing that solution (8.a) is, with a very good precision, Weinberg's original one.

In order to be able to discriminate among these four solutions, we attempt to calculate the next order in the expansion (3). It is easy to verify that, for the on-the-mass-shell amplitude, the only terms of third order are those proportional to k_t^3 and $(k_s^3 + k_u^3)$ and permutations.

When we go off-the-mass-shell, we can add two other terms, namely $k_t(k_s^2+k_u^2)$ and $[k_s(k_t^2+k_u^2)+k_u(k_t^2+k_s^2)]$. Thus, by going off-the-mass-shell, we add three new terms in our expansion [the above-mentioned two terms of third order and the term proportional to $(k_s^2+k_u^2)$ in second order] and, on the other hand, we gain three equations, namely the conditions 2), 3) and 4).

Therefore, no extra information can be gained by going off-the-mass-shell and using current algebra. So, in what follows, we shall restrict ourselves to the on-the-mass-shell amplitude. We write M as :

$$M = \delta_{ab}\delta_{cd} \left[A + B(k_s+k_u) + Ck_t + Dk_t^2 + Fk_t^3 + G(k_s^3+k_u^3) \right] + \text{permutations} \quad (9)$$

Since (9) includes terms of the third order, we can apply unitarity up to that order. We obtain, in this way, the following equations :

$$2B+3C = -\frac{1}{32\pi} \left(5A+8B+2C+2D+2F+8G \right)^2 \quad (10)$$

$$2B+3C-6F-4G = -\frac{1}{16\pi} \left[(2B+3C)^2 + 2 \left(5A+8B+2C+2D+2F+8G \right) \left(2B+\frac{C}{2}-2D+\frac{3F}{2}+6G \right) \right] \quad (11)$$

$$B = -\frac{1}{16\pi} (A+B+C+D+F+G)^2 \quad (12)$$

$$B-2G = -\frac{1}{8\pi} \left[B^2 + (A+B+C+D+F+G) \left(\frac{B}{2} + \frac{C}{2} + D + \frac{3F}{2} + \frac{3G}{2} \right) \right] \quad (13)$$

Equations (10) and (11) come from the $I=0$ s wave unitarity in first and third order in q , respectively, and Eqs. (12) and (13) are the corresponding ones for the $I=2$ s wave.

Our purpose is to check whether by including the higher order unitarity corrections, represented by Eqs. (11) and (13), we modify substantially the results obtained in the second order calculation. If an expansion of the form (9) makes sense, one does not expect, however, such modifications to be important at threshold, since the unitarity has already been taken into account there. Therefore, we shall use as input to our third order calculation the values of the s wave scattering lengths obtained previously, namely the solutions (8.a) to (8.d). In this way we obtain for the six parameters $A, \dots, G$ of Eq. (9) a system of six equations, i.e., the four unitarity relations (10) to (13) and the two equations expressing the s wave scattering lengths given by (8.a) to (8.d).

The results are included in Table I ⁸⁾. We see that only solution (a) meets the requirements of a "smooth" solution, namely, one in which the values of the parameters obtained in second and third order calculations are practically the same. Furthermore, we must notice that the solution (b) must, in any case, be disregarded because it violates the inequality (7.c) when the third order terms are included. Therefore, we arrive at the conclusion, already reached by several authors using different methods ⁹⁾, that unless one is willing to accept rapidly varying amplitudes, current algebra, combined with unitarity and crossing, uniquely determines the $\pi-\pi$ scattering amplitude at low energies. The values obtained in this way for the s wave scattering lengths are the ones predicted by Weinberg.

Since our third order calculation satisfies unitarity in order q^3 , we can try to compute, besides the s wave scattering lengths, the next order terms of the amplitude. One usually uses for this purpose the effective range formula. However, in our case, this formula has a very slow convergence. The reason is that $q \cot \tilde{\delta}_0^I$ has poles wherever the amplitude vanishes below threshold and, in our calculation, we have explicitly introduced a zero by making use of Adler's self-consistency condition ¹⁰⁾. Therefore we choose a function directly related to the phase shifts by :

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$$f(q^2) = \frac{s_1 \delta_0^I}{q} = \alpha_I + b_I q^2 + \dots \quad (14)$$

The values of the b_I 's obtained in the third order calculation, together with the p wave scattering lengths, are given in Table II. We again see that the "smoothest" solution is the first one which gives Weinberg's s wave scattering lengths.

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TABLE I : Values of the parameters of the power series expansion.
Columns 2 and 3 contain the results of the second and third order calculation, respectively.

Solution	A		B	C	D		F	G
	2	3			2	3		
(a)	9.2	8.4	-0.14	-0.14	-10.4	-14.0	4.0	0.36
(b)	70.0	68.5	-0.032	-142.0	71.0	136.0	-64.0	0.75
(c)	475.0	842.0	-263.0	156.0	-253.0	-717.0	281.0	-183.0
(d)	496.0	806.0	-260.0	97.0	-218.0	-540.0	168.0	-155.0

TABLE II : Low energy parameters of the $\pi\pi$ scattering amplitude.

a_0	a_2	b_0	b_2	a_1
0.21	-0.053	0.25	-0.053	0.03
2.1	-0.025	3.5	-0.029	0.11
0.77	2.3	3.1	4.6	-0.63
1.5	2.3	2.0	4.7	-0.41

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The range of validity of (7.c) has been extended in the region $1.7 \leq s \leq 2$ by A.D. Common.
- 8) The parameters B and C are always identical in both the second and third order calculations because they are determined by the threshold unitarity equations which are common in both cases.

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- 9) See, for example, T. Akiba and K. Kang, Ref. 2), and A. Donnachie, Ref. 2).
- 10) This bad convergence is particularly clear in the case of the $I=2$ s wave where the scattering length turns out to be very small indicating that the zero of the amplitude is not very far. In fact, if we write $q \cot \delta_0^2 = 1/a_2 + r_2/2 q^2 + \dots$, we find that $1/a_2 \sim -20$ and $r_2 \approx 38$.