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DYNAMICS AT HIGH MOMENTUM AND THE VERTEX FUNCTION
OF SPINOR ELECTRODYNAMICS

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A B S T R A C T

A modification of Feynman rules is developed such that the high energy asymptotic forms of individual Feynman diagrams or of the complete n point functions are easily obtainable. The modifications are similar to those discovered by Weinberg in connection with the dynamics at infinite momentum. The present techniques are used to obtain the high energy behaviour of the vertex of spinor electrodynamics on the mass shell. The result for large, spacelike momentum transfers is

$$\exp \left[-\frac{e^2}{16\pi^2} \log^2 \frac{|k^2|}{\mu^2} \right],$$

where μ^2 is any convenient infra-red cut-off. The off mass shell asymptotic form is also obtained and found to agree with previous results. The Bethe-Salpeter kernel for the vertex, in the relevant energy, domains is given.

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I. INTRODUCTION

In a recent investigation of dynamics at infinite momentum, Weinberg ¹⁾ has derived a new set of diagrammatic rules. These rules, to be applied in the $p = \infty$ Lorentz frame, replace the usual Feynman rules as well as the old-fashioned perturbation theory rules. They have the advantage of simplifying the set of diagrams that needs to be considered in descriptions of physical processes. The purpose of this paper is to report the results of an examination of the high energy behaviour of the vertex function (three point function) of spinor electrodynamics. In the course of this analysis we find it convenient to introduce diagrammatic rules which determine the high energy asymptotic form of the vertex, and which are strikingly similar to Weinberg's rules. It is suggested that these rules may be useful in the analysis of the high energy behaviour of the integral equations which are satisfied by the n point functions of field theory.

The high energy behaviour of the various n point functions can be studied by (renormalized) perturbation theory. The procedure, which has been utilized by various authors ²⁾, is to isolate those Feynman diagrams of the perturbative expansion that are important in the relevant energy region; estimate their high energy behaviour and sum the individual terms to obtain what is hoped to be an estimate for the entire function at high energies. On the other hand, it should be possible to analyze the integral equations which determine these functions in a fashion which yields this high energy behaviour directly. In the present work, we develop techniques for the analysis of such integral equations in the high energy region, under the approximation that the kernel of the integral equation is known. (Even with this approximation, the integral equation will not in general be solvable.) Then, instead of expanding the unknown function in powers of the coupling constant which yields the usual perturbation series, we approximate the (known) kernel in a fashion which is accurate at high energies, and which gives an easily solvable integral equation.

The kernel may be represented by a series (in general an infinite series) of diagrams whose analytic form is given by the usual Feynman rules. We find that, for purposes of determining the high energy behaviour, one may use different rules which determine an approximate kernel. This approximation to the kernel is valid in the high energy domain and leads to an easily solvable integral equation.

Specifically, we examine a Bethe-Salpeter equation for the vertex function of spinor electrodynamics, both on and off the mass shell. Existing analyses of this function in perturbation theory³⁾ provide a ready check on some of our results. We chose to examine a spinor theory, rather than a simpler scalar theory, in order to demonstrate explicitly that the complications of spin can be easily handled by the present methods.

In Section II, we define the ladder approximation to the vertex, and begin the analysis of the high energy behaviour. In Section III, we develop our modified diagrammatic methods in a general fashion. In Section IV, we return to the analysis of the ladder vertex, where we apply the modified rules to determine the high energy behaviour. In Section V, the ladder approximation is improved upon by the inclusion of the first crossed graph. An assumption is made about the behaviour of the solution when all crossed graphs are included, and the Bethe-Salpeter kernel is then determined under this assumption.

II. ANALYSIS OF THE LADDER APPROXIMATION TO THE VERTEX FUNCTION

A. $\Gamma_{\alpha\beta}^{\mu}$ The unrenormalized vertex function of spinor electrodynamics, viz. the three point function with the complete external propagators removed, satisfies the following Bethe-Salpeter type integral equation⁴⁾ (in momentum space)

$$\Gamma_{\alpha\beta}^{\mu}(p,q) = e \gamma_{\alpha\beta}^{\mu} + \int \frac{d^4 r}{(2\pi)^4} K_{\alpha\beta\gamma\delta}(p,q;r) \\ \times G_{\delta\varepsilon}(p+r) \Gamma_{\varepsilon\varepsilon'}^{\mu}(p+r,q+r) G_{\varepsilon'\gamma}(q+r). \quad (\text{II.1})$$

Here K is the Bethe-Salpeter kernel, G is the spinor propagator. Equation (II.1) may be represented graphically as in Fig. 1. Equation (II.1) is exact. The ladder approximation consists in replacing the unknown kernel $K_{\alpha\beta\gamma\delta} G_{\delta\varepsilon} \dots G_{\varepsilon'\gamma}$ by its lowest order (in e^2) value. Thus we have in the ladder approximation

$$\Gamma^{\mu}(p,q) = e \gamma^{\mu} - ie^2 \int \frac{d^4 r}{(2\pi)^4} \gamma^{\nu} [\not{p} + \not{r} - m]^{-1} \Gamma^{\mu}(p+r, q+r) \\ \times [\not{q} + \not{r} - m]^{-1} \gamma_{\nu} [r^2 - \mu^2 + i\varepsilon]^{-1}. \quad (\text{II.2})$$

The graphical representation of this is given in Fig. 2. (We have introduced a photon mass μ to avoid infra-red divergences.) We now wish to modify the kernel $\gamma^{\nu} [\not{p} + \not{r} - m]^{-1} \dots [\not{q} + \not{r} - m]^{-1} \gamma_{\nu} [r^2 - \mu^2 + i\varepsilon]^{-1}$ in a way so that the integral equation (II.2) can be easily solved for the large and spacelike $k^2 \equiv (p-q)^2$ behaviour of the renormalized vertex.

B. In order to determine this modification, let us first examine a related integral

$$I(p^2, q^2, k^2) = 2k^2 i e^2 \int \frac{d^4 r}{(2\pi)^4} \left\{ \frac{[(r-p)^2 - m^2 + i\varepsilon][(r-q)^2 - m^2 + i\varepsilon]}{[r^2 - \mu^2 + i\varepsilon]} \right\}^{-1} \quad (\text{II.3})$$

In a familiar fashion, except for the over-all factor of $2k^2$, (II.3) may be represented graphically as in Fig. 3 where the solid lines represent spin zero propagators. It is seen that (II.3) represents the first iteration of (II.2), except that the numerator of the integrand in the first iteration of (II.2), viz. $\gamma^\nu [\not{x} + \not{x} + m] \gamma^\mu [\not{x} + \not{x} + m] \gamma_\nu$, has been replaced by $2k^2$. We shall show below that I determines the asymptotic behaviour of the first iteration of (II.2).

Each of the denominators in (II.3) may be split in a natural way into two parts:

$$[p^2 - m^2 + i\varepsilon]^{-1} = \frac{1}{2\omega_p} \left[(p_0 - \omega_p + i\varepsilon)^{-1} + (-p_0 - \omega_p + i\varepsilon)^{-1} \right] \quad (\text{II.4})$$

$$\omega_p = \sqrt{p^2 + m^2} = \omega_{-p} > 0.$$

The two terms in (II.4) represent the two diagrams of old-fashioned perturbation theory which describe a particle propagating forward and backward in time.

Substituting the decomposition (II.4) into (II.3) results in eight terms, each with a denominator of the form

$$\left\{ \left[\pm(r_0 - p_0) - \omega_{p-r} + i\varepsilon \right] \left[\pm(r_0 - q_0) - \omega_{r-q} + i\varepsilon \right] \left[\pm r_0 - \nu_r + i\varepsilon \right] \right\}^{-1} \quad (\text{II.5})$$

(We use ν_r for $\sqrt{\tilde{r}^2 + \mu^2}$.) The r_0 integral may be performed by closing the contour in the upper or lower half plane and evaluating the residues. The residues arising from, say, the last factor in (II.5), $\pm r_0 - \nu_r + i\varepsilon$, are of the form

$$\left\{ \left[\pm(-\nu_r - p_0) - \omega_{p-r} \right] \left[\pm(-\nu_r - q_0) - \omega_{r-q} \right] \right\}^{-1} \quad (\text{II.6})$$

if the contour is closed in the upper half plane. In the subsequent $\tilde{r}$ integral, the two factors in the denominators in (II.6) must behave in the following fashion: a. for some values of $\tilde{r}$ both of the two factors in the denominators get very small; b. for no value of $\tilde{r}$ do both of the two factors in the denominators get very small. If terms exhibiting the first kind of behaviour are present (among the eight terms that make up I), they will dominate the terms exhibiting the second kind of behaviour. Moreover, in that case, the $\tilde{r}$ integral can be confined to that portion of $\tilde{r}$ space in which both factors are small. Therefore, for purposes of obtaining the dominant part of I, it is useful to isolate those terms which exhibit the first property. We now give a general prescription for this procedure, which serves to determine the dominant behaviour of a spin zero Feynman diagram, and which will be useful for our subsequent analysis of the integral equation (II.2).

III. MODIFIED FEYNMAN RULES

In order to facilitate the selection of the dominant terms, and to discuss the general problem, we introduce some new graphical notation. Consider a diagram (such as Fig. 3) with a closed loop, viz. with a momentum integration, over the variable r . We are interested in the behaviour of this diagram when some external momentum is large and spacelike. (We assume first that the propagators refer to spin zero particles, i.e., no numerators are present.) Let a particular direction of momentum flow be chosen so that all lines which are common to two loops carry momentum in only one direction. Thus each line in the loop now carries a label specifying the direction and amount of momentum carried. Next, let the denominator of each of the propagators in the diagram be split into two parts according to (II.4). Thus the original diagram is replaced by several "secondary" diagrams. This splitting may be represented graphically as

$$\left. \begin{array}{c} \xrightarrow{\quad r \quad} \\ \xrightarrow{\quad r \quad} \end{array} \right\} = \xrightarrow{\quad r \quad} \text{---} + \text{---} \xrightarrow{\quad r \quad} \quad (III.1)$$

$$\xrightarrow{\quad r \quad} \text{---} = \frac{1}{2\omega_r} [\tau_0 - \omega_r + i\epsilon]^{-1} \quad (III.2)$$

$$\text{---} \xrightarrow{\quad r \quad} = \frac{1}{2\omega_r} [-\tau_0 - \omega_r + i\epsilon]^{-1} \quad (III.3)$$

(We do not make a distinction between wavy lines - mass μ , and solid lines - mass m . No confusion will arise as it will always be obvious which mass is appropriate.) When the flow is from solid to dotted [as in (III.2)], we say the (secondary) line carries positive three-momentum $\underline{r}$; when it is from dotted to solid [as in (III.3)], we say the line carries negative three-momentum $\underline{r}$.

The external lines are also redrawn according to the following convention. We settle on some external four-momentum, say p , and choose a Lorentz frame in which the three-momentum $\underline{p}$ is large. (By assumption this is possible for the problem at hand.) This defines a direction $\hat{p}$. The three-momenta of the other external lines, for example $\underline{q}$, are resolved into a component parallel to $\hat{p}$: $q_{||}$, and a component perpendicular to $\hat{p}$: $\underline{q}_{\perp}$. The external lines are now redrawn as follows. An incoming line carrying positive three-momentum along $\hat{p}$ is drawn as $\rightarrow$; an outgoing line carrying positive three-momentum along $\hat{p}$ is drawn as $\leftarrow$.

Next we return to the analytical representation of these diagrams, perform the r_0 integral by contour integration, and examine the form of the three-momentum $\underline{r}$ integral. This threefold integral is decomposed into a twofold $\underline{r}_{\perp}$ and a onefold $r_{||}$ integral. The integral will contain a denominator which is made up of several factors, as in (II.6). We now assert that each factor in the denominator becomes small under certain conditions on the integration variable. Note first that three-momentum is conserved in each factor. By this we mean that if we formally associate momentum $\underline{r}$ with all the r_0 type terms and $\pm \underline{r}$ with all the ω_r type terms occurring in the factor under consideration; then, for some choice of signs, the factor can be made to vanish. The proper sign can be automatically arrived at by the statement that net momentum flow in the line must be positive [in the sense discussed below Eqs. (II.2)]. For if we have under consideration a factor represented graphically by

$$\xrightarrow{\underline{r}} \sim r_0 - \omega_r + i\varepsilon,$$

then the momentum flow is manifestly positive, according to our previous convention, and $\omega_r \sim +\underline{r}$ is the proper choice. On the other hand, if we have

$$\xleftarrow{\underline{r}} \sim -r_0 - \omega_r + i\varepsilon,$$

with manifestly negative flow, then $\omega_r \sim -\underline{r}$ is the proper choice and the net flow is again positive.

Now analytically the factors do not vanish; however, they become small in those regions of $r_{||}$ space where the net flow in the line is positive. Requiring that the net flow in all the lines of the loop be positive, places restrictions on the $r_{||}$ variable which may or may not be consistent. If it is possible to satisfy these conditions, then that secondary diagram contributes to the dominant behaviour of the entire diagram. If the conditions are not compatible, then the secondary diagram may be ignored.

A very simple diagrammatic consequence of the above rule can be given, which makes it possible to discard the irrelevant diagrams by inspection. The rule states that no secondary diagram with the structure ⁵⁾

contributes. That this diagrammatic rule follows from the statement about positive momentum flow is evident from the fact that, by momentum conservation, one of the legs meeting in the above "forbidden" vertices carries negative momentum.

Another class of secondary diagrams which does not contribute, consists of all diagrams with a closed loop where the momentum flow is always from solid to dotted (or vice versa), e.g.,

These diagrams integrate to zero since all the $i\varepsilon$ singularities lie in the same half of the complex r_0 plane.

Thus, in conclusion, the procedure for obtaining the dominant high energy, behaviour of a Feynman diagram in a spin zero theory, is to replace it by the secondary diagrams, ignoring those which are excluded by the above considerations. The remaining diagrams are represented analytically by the modified rule (III.1, 2, 3), and the relevant integration variables are confined so that net momentum flow is always positive. We discuss the extension to non-spin zero cases below.

IV. ASYMPTOTIC BEHAVIOUR OF LADDER APPROXIMATION

A. As an application of the above method, and in continuation of our analysis of the vertex, we now return to the problem of finding the asymptotic behaviour of I , Eq. (II.3). As k is large and spacelike, we may work in a Lorentz frame in which p carries a large three-momentum $\mathbf{p}$ and defines a direction $\hat{p}$, while q carries a small three-momentum, which we assume has a small positive component along $\hat{p}$. The external incoming line p is therefore drawn as $\rightarrow\text{---}\rightarrow$, while the external outgoing line q is drawn as $\text{---}\rightarrow$. The external outgoing line k carries along $\hat{p}$ the three-momentum $k_{||} = p_{||} - q_{||} > 0$; hence it is drawn $\text{---}\rightarrow$. Therefore, the diagram of Fig. 3 is decomposed into eight secondary diagrams given in Fig. 4. Diagrams 3, 4, 5 and 6 exhibit the forbidden vertex

hence they do not contribute. Diagrams 7 and 8, integrate to zero because all the singularities lie in the same half plane. Therefore, we may conclude immediately that

$$I \approx \tilde{I} = I_1 + I_2 \quad (\text{IV.1a})$$

$$\begin{aligned} I_1 &= \frac{2k^2 e^2}{(2\pi)^4} \int \frac{d^4 r}{(2\omega_{p-r})(2\omega_{r-q})(2V_r)} \left[\frac{(r_0 - V_r + i\epsilon)(r_0 - q_0 - \omega_{r-q} + i\epsilon)}{(-r_0 + p_0 - \omega_{p-r} + i\epsilon)} \right]^{-1} \\ &= \frac{2k^2 e^2}{(2\pi)^3} \int \frac{d^3 r}{(2\omega_{p-r})(2\omega_{r-q})(2V_r)} \left[\frac{(p_0 - \omega_{p-r} - V_r)(p_0 - q_0 - \omega_{p-r} - \omega_r)}{(IV.1b)} \right]^{-1} \end{aligned}$$

$$\begin{aligned} I_2 &= \frac{2k^2 e^2}{(2\pi)^4} \int \frac{d^4 r}{(2\omega_{p-r})(2\omega_{r-q})(2V_r)} \left[\frac{(r_0 - V_r + i\epsilon)(-r_0 + q_0 - \omega_{r-q} + i\epsilon)(-r_0 + p_0 - \omega_{p-r} + i\epsilon)}{(-V_r + q_0 - \omega_{r-q})(-V_r + p_0 - \omega_{p-r})} \right]^{-1} \\ &= \frac{2k^2 e^2}{(2\pi)^3} \int \frac{d^3 r}{(2\omega_{p-r})(2\omega_{r-q})(2V_r)} \left[\frac{(-V_r + q_0 - \omega_{r-q})(-V_r + p_0 - \omega_{p-r})}{(IV.1c)} \right]^{-1} \end{aligned}$$

$$I(m^2, m^2, k^2) \approx -\frac{4p_{||} m e^2}{(2\pi)^3} \int_{\mu}^{p_{||}} d\tau_{||} \int_{\mu}^{\sqrt{2m\tau_{||}}} \frac{(2\pi)\tau_{\perp} d\tau_{\perp}}{(2p_{||})(2\tau_{||})(2\tau_{||})} \left[\frac{\tau_{\perp}^2}{2\tau_{||}} m \right]^{-1}, \quad (\text{IV.6})$$

which is readily seen to integrate to (IV.3b). (In this calculation, the distinction between μ and m is immaterial, since the difference between $\log^2 \frac{|k^2|}{\mu^2}$ and $\log^2 \frac{|k^2|}{m^2}$ is of order $\log \frac{|k^2|}{m^2}$.) That this asymptotic behaviour is indeed the correct one can be verified by comparing it with the explicit evaluation of I , (II.3), in this limit ³⁾.

C. Next we specialize to an off-mass shell case, which shall prove to be of interest below. We wish the asymptotic form of I when

$$\frac{|p^2||q^2|}{\mu^2} \gg |k^2| \gg |p^2|, |q^2| \gg m^2. \quad (\text{IV.7})$$

To achieve these values, we choose a frame where

$$|k| = p_{||} \approx p_0 \gg q_0 \approx q_{||} \gg m \approx q_0 - q_{||} \gg |p_0 - p_{||}| \gg \frac{m^2}{2q_{||}} \quad (\text{IV.8})$$

$$\sqrt{2mq_{||}} \gg q_{\perp} \gg m.$$

In this frame

$$p^2 \approx 2p_{||}(p_0 - p_{||}), \quad q^2 \approx 2q_{||}m, \quad |k^2| \approx 2p_{||}m. \quad (\text{IV.9})$$

Explicit evaluation ⁴⁾ yields the result that I , in this limit, is determined solely by I_1 (i.e., again I_2 does not contribute) and the $\underline{r}$ integration can be limited to the region

$$p_{||} > r_{||} > q_{||}, \quad \sqrt{2mr_{||}} > r_{\perp} > \sqrt{2|p_0 - p_{||}|r_{||}}. \quad (\text{IV.10})$$

In this region we have

$$v_r \approx r_{||} \approx \omega_{r-q}, \quad \omega_{p-r} \approx p_{||},$$

$$p_0 - \omega_{p-r} - v_r \approx -\frac{r_{\perp}^2}{2r_{||}}, \quad p_0 - q_0 - \omega_{p-r} - \omega_{r-q} \approx -m. \quad (\text{IV.11})$$

Hence,

$$I(p^2, q^2, k^2) \approx -\frac{4|p_{||}me^2}{(2\pi)^3} \int_{q_{||}}^{p_{||}} dr_{||} \int_{\sqrt{2|p_0 - p_{||}|r_{||}}}^{\sqrt{2mr_{||}}} \frac{(2\pi)r_{\perp} dr_{\perp}}{(2p_{||})(2r_{||})(2r_{||})} \left[\frac{r_{\perp}^2}{2r_{||}} m \right]^{-1}, \quad (\text{IV.12})$$

which integrates to

$$I(p^2, q^2, k^2) = -\frac{e^2}{8\pi^2} \log \left| \frac{k^2}{p^2} \right| \log \left| \frac{k^2}{q^2} \right| + O \log \left| \frac{k^2}{m^2} \right|. \quad (\text{IV.13})$$

The correctness of this high energy estimate can be verified from an explicit evaluation of I , (II.3), in this limit ³⁾.

This completes our discussion of I . It is seen that the modification of the Feynman rules, as discussed in Section III, provides an easily analyzable expression for the asymptotic behaviour of the diagram under consideration. The application to more complicated graphs is self-evident. In Section V, we sketch another example of the application of this procedure.

D. We now examine the complete first iteration of (II.1), including the spin terms in the numerator. This is given by

$$\Lambda^\mu(p, q) = -ie^2 \int \frac{d^4 x}{(2\pi)^4} \frac{\gamma^\nu [p+x-m] \gamma^\mu [q+x-m] \gamma_\nu}{[(p+x)^2 - m^2 + i\varepsilon][(q+x)^2 - m^2 + i\varepsilon][x^2 - m^2 + i\varepsilon]}. \quad (\text{IV.14})$$

We seek the asymptotic form of this, for large spacelike k , with p and q on the mass shell, or off the mass shell as in (IV.7). We again work in a Lorentz frame where p is the largest parameter of the problem. However, it is not evident that the x integration should be cut off by $|p|$, since the region $|x| > |p|$ now contributes, due to the extra powers of x in the numerator - indeed Λ^μ is ultra-violet divergent and we must regularize the expression. However, the contribution to the regularized expression of the ultra-violet tail

of the integration leads, in a well-known fashion, to a single logarithm. But as we shall see below, there are double logarithms contributing to the asymptotic behaviour of Λ^μ , which dominate the ultraviolet contribution. Thus for present purposes we may again cut off the $\underline{r}$ integration by $|\underline{p}|$. It then follows that the dominant contribution to Λ^μ arises from that portion of the $\underline{r}$ integral, which permits the denominators to get very small. From subsection D, above, it is evident that if the numerator in (IV.14) is ignored, i.e., replaced by 1, then the integral leads to an expression $O|k^2|^{-1} \log^2|k^2|$. Hence, we must obtain a contribution $O|k^2|$ from the numerator to cancel the $|k^2|$ in the denominator. The numerator is

$$\begin{aligned} N^\mu &= \gamma^\nu [\cancel{p} - \cancel{x} + m] \gamma^\mu [\cancel{q} - \cancel{x} + m] \gamma_\nu \\ &= [(-\cancel{p} + m + \cancel{x}) \gamma^\nu + 2(p-r)^\nu] \gamma^\mu [\gamma_\nu (-\cancel{q} + m + \cancel{x}) + 2(q-r)_\nu] \end{aligned} \quad (\text{IV.15})$$

The terms $(-\cancel{p} + m)$ and $(-\cancel{q} + m)$ give zero for the mass shell case, while for the off mass shell case they cannot amount to $O|k^2|$. Therefore, the only term in N^μ which may lead to $O|k^2|$ is

$$\begin{aligned} &[\cancel{x} \gamma^\nu + 2(p-r)^\nu] \gamma^\mu [\gamma_\nu \cancel{x} + 2(q-r)_\nu] \\ &= \gamma^\mu [2\cancel{x}^2 + 4p \cdot q - 4(p+q) \cdot r] + 4[p^\mu + q^\mu - r^\mu] \cancel{x} \\ &\quad - 2\cancel{p} \gamma^\mu \cancel{x} - 2\cancel{x} \gamma^\mu \cancel{q}. \end{aligned} \quad (\text{IV.16a})$$

Again, only the first term, proportional to γ^μ can give a contribution $O|k^2|$. Furthermore, the previous analysis shows that the contour integration in r_0 picks up only the photon pole $r^2 = \mu^2$, $r_0 = \nu_r$.

$$N^\mu \approx \gamma^\mu [2\mu^2 + 4p \cdot q - 4 \{ (p_0 + q_0) \nu_r - p_{11} r_{11} - q \cdot \pi \}].$$

(IV.16b)

In the regions of $\underline{r}$ space that are important, (IV.4) or (IV.7), the term in curly brackets is much less than $p \cdot q$. Therefore

$$N^\mu \approx \gamma^\mu 4p \cdot q \approx -2k^2 \gamma^\mu. \quad (\text{IV.16c})$$

Thus the asymptotic behaviour of Λ^μ is given by $\gamma^\mu I$.

E. The first iteration to Γ^μ , whose asymptotic behaviour we have established, is determined by the kernel of the integral equation for Γ^μ . We now make the assumption that the same modifications of the kernel, which yield the high energy behaviour of the first iteration, can be made to give the high energy behaviour of Γ^μ . This is the usual assumption, since its consequence in perturbation theory is that only the leading term in each order is kept. However, it is an assumption, since it does not follow that summing the leading terms in each order of perturbation theory yields the dominant behaviour of the entire function. Therefore, making the above assumption, we modify the integral equation (II.3a) in the following way. In the first place, we replace the "spinology" by setting $\Gamma^\mu(p, q) \approx e \gamma^\mu \Gamma(p^2, q^2, k^2)$ where

$$\Gamma(p^2, q^2, k^2) = 1 + 2k^2 i e^2 \int \frac{d^4 r}{(2\pi)^4} \frac{\Gamma([p-r]^2, [q-r]^2, k^2)}{[(p-r)^2 - m^2 + i\varepsilon][(q-r)^2 - m^2 + i\varepsilon]} \times [r^2 - \mu^2 + i\varepsilon]^{-1} \quad (\text{IV.17})$$

In the second place, we may modify the kernel to the form

$$\Gamma(p^2, q^2, k^2) = 1 + \frac{2k^2 i e^2}{(2\pi)^4} \int \frac{d^4 r}{8w_{p-r} w_{r-q} v_r} \times \Gamma([p-r]^2, [q-r]^2, k^2) [(r_0 - v_r + i\varepsilon)(r_0 + p_0 - w_{p-r} + i\varepsilon) \times (r_0 - q_0 - w_{r-q} + i\varepsilon)]^{-1}, \quad (\text{IV.18})$$

where the $\underline{r}$ integration may be restricted by

$$p_{||} > r_{||} > \mu, \quad \sqrt{2m r_{||}} > r_{\perp} > \mu, \quad (\text{IV.19a})$$

for the mass shell calculation; and by

$$p_{||} > r_{||} > q_{||}, \quad \sqrt{2m r_{||}} > r_{\perp} > \sqrt{2|p_{||} - p_0| r_{||}}, \quad (\text{IV.19b})$$

for the off mass shell case discussed before [Eq. (IV.7) and following].

It is seen that these are exactly the conditions (IV.7) and (IV.8), i.e., the mass shell form factor for $-k^2 \gg m^2$ is determined by the off mass shell form factor, which we now proceed to determine.

G. We specialize (IV.21) to the case given by (IV.7), (IV.8), (IV.9) and obtain

$$\Gamma(p^2, q^2, k^2) = 1 - \frac{m e^2}{2(2\pi)^3} \int_{q_{||}}^{p_{||}} \frac{dr_{||}}{r_{||}^2} \int \frac{\sqrt{2mr_{||}} \frac{d^2 r_{\perp}}{r_{\perp}^2} \Gamma_1}{m r_{\perp}^2 / 2 r_{||} \Gamma_1 (IV.24)} \frac{1}{\sqrt{2r_{||} |p_{||} - p_0|}} \quad (IV.24)$$

It is seen that the arguments of Γ_1 satisfy the same conditions as the argument of Γ on the left-hand side of (IV.24). Hence, (IV.24) can be solved. It takes on a particularly transparent form if we set $x = p_{||} (r_{\perp}^2 / r_{||})$, $y = 2mr_{||}$. Then

$$\Gamma(p^2, q^2, k^2) = 1 - \frac{e^2}{8\pi^2} \int_{|p^2|}^{|k^2|} \frac{dx}{x} \int_{|q^2|}^{|k^2|} \frac{dy}{y} \Gamma(x, y, k^2) \quad (IV.25)$$

The solution of this, given in Appendix A, is

$$\Gamma(p^2, q^2, k^2) = J_0 \left(\sqrt{\frac{e^2}{2\pi^2} \left| \log \frac{p^2}{k^2} \right| \left| \log \frac{q^2}{k^2} \right|} \right), \quad (IV.26)$$

where J_0 is a Bessel function. Expanding in powers of e^2 , we have

$$\Gamma(p^2, q^2, k^2) = \sum_{n=0}^{\infty} e^{2n} a_n \quad (\text{IV.27a})$$

$$a_n = \frac{1}{(n!)^2} \left[- \frac{\log \left| \frac{p^2}{k^2} \right| \log \left| \frac{q^2}{k^2} \right|}{8\pi^2} \right]^n. \quad (\text{IV.27b})$$

Equation (IV.26) represents that portion of the vertex function, which arises from the leading terms in perturbation theory for the ladder approximation. We have obtained coincidentally the dominant behaviour of the $(2n)$ th order ladder graph, which is exhibited in (IV.27b). Both results are under the stated conditions, (IV.23b).

H. With this value for Γ , we may evaluate the integral (IV.22) to obtain Γ on the mass shell. The integral is evaluated in Appendix B and yields

$$\Gamma(m^2, m^2, k^2) = \cos \left(\sqrt{\frac{e^2}{8\pi^2} \log^2 \left| \frac{k^2}{m^2} \right|} \right). \quad (\text{IV.28})$$

Expanding in a power series,

$$\Gamma(m^2, m^2, k^2) = \sum_{n=0}^{\infty} e^{2n} b_n \quad (\text{IV.29a})$$

$$b_n = \frac{1}{(2n)!} \left[- \frac{\log^2 \left| \frac{k^2}{m^2} \right|}{8\pi^2} \right]^n. \quad (\text{IV.29b})$$

Equation (IV.28) represents that portion of the mass shell vertex, which arises from the leading terms in perturbation theory, for the ladder approximation. We have obtained coincidentally the dominant behaviour of the $(2n)$ th order ladder graph, which is exhibited in (IV.29b); both under the conditions $-k^2 \gg p^2 = q^2 = m^2$.

V. CONTRIBUTION FROM CROSSED DIAGRAMS

A. The results obtained here for the asymptotic form of the vertex, suffer from the defect that no crossed diagrams have been included. However, it is known from perturbation theory that crossed diagrams contribute to the vertex, expressions of the same order of magnitude as the ladder diagrams ³⁾. (Self-energy insertions and other vertex corrections, on the other hand, do not contribute in perturbation theory ³⁾; thus we may feel justified in ignoring these.) In the present Section we shall study, using the techniques of Section III, the contribution of the first crossed diagram to the Bethe-Salpeter kernel. Guided by this result we shall then assume the form for the vertex when all the crossed diagrams are included, and determine the Bethe-Salpeter kernel.

B. The integral equation for the vertex, when one crossed diagram has been included, is summarized diagrammatically in Fig. 5. The procedure for analyzing this integral equation for its high energy behaviour follows exactly that of the ladder equation. We concentrate on the first iteration of the second order kernel - the crossed graph. Ignoring first the "spinology" in the numerators of the propagators, we analyze the asymptotic behaviour of this quantity. We then include the effect of the numerators. Finally we modify the second order kernel in the fashion indicated by the analysis of the first iteration.

C. We consider the Feynman diagram of Fig. 6. We shall be interested in the same two sets of values for p , q and k as before and we shall work in the same Lorentz frames [see (IV.2), (IV.7), (IV.8)]. The external lines on Fig. 6 are redrawn, and the internal propagators are decomposed, yielding several secondary diagrams. From this set we drop all diagrams which have 1) a closed loop with circulation from solid to dotted or vice versa throughout; 2) a forbidden vertex of the form

In addition we also drop those diagrams, which although permitted by our rules, do not contribute because the integration variable is restricted to a vanishingly small interval. [Such a diagram was encountered in the analysis of the ladder equation: I_2 Eq. (IV.1c).] These diagrams have the characteristic property that the right-hand, lowest vertex, has the structure

hence they can be easily identified and excluded. After this has been done, one is left with just two diagrams; these are pictured in Fig. 7. They represent the integrals

$$\begin{aligned}
 I_1 = & \frac{-e^4}{(2\pi)^8} \int \frac{d^4 r}{4\omega_{p-r} \nu_r} \left[(\tau_0 - \omega_r + i\epsilon)(p_0 - r_0 - \omega_{p-r} + i\epsilon) \right]^{-1} \times \\
 & \int \frac{d^4 s}{16\omega_{p-r-s} \omega_{r+s-q} \omega_{s-q} \nu_s} \left[(p_0 - r_0 - s_0 - \omega_{p-r-s} + i\epsilon) \right. \quad (V.1a) \\
 & \times (-q_0 + r_0 + s_0 - \omega_{r+s-q} + i\epsilon) \\
 & \left. \times (-q_0 + s_0 - \omega_{s-q} + i\epsilon)(s_0 - \nu_s + i\epsilon) \right]^{-1}
 \end{aligned}$$

$$\begin{aligned}
I_2 = & \frac{-e^4}{(2\pi)^8} \int \frac{d^4 r}{4 w_{p-r} v_r} \left[(r_0 - w_r + i\varepsilon)(p_0 - r_0 - w_{p-r} + i\varepsilon) \right]^{-1} \\
& \times \int \frac{d^4 s}{16 w_{p-r+s} w_{r-s-q} w_{s+q} v_s} \left[(p_0 - r_0 + s_0 - w_{p-r+s} + i\varepsilon) \right. \\
& \times (-q_0 + r_0 - s_0 - w_{r-s-q} + i\varepsilon) (q_0 + s_0 - w_{s+q} + i\varepsilon) \\
& \left. \times (s_0 - v_s + i\varepsilon) \right]^{-1}
\end{aligned} \tag{V.1b}$$

We perform the r_0 and s_0 integrals by contour integration. In (V.1a) first the s_0 contour is closed in the upper half plane, and then the r_0 contour is closed in the lower half plane. In (V.1b) the s_0 and r_0 contours are closed in the upper half plane. This leaves

$$\begin{aligned}
I_1 = & \frac{e^4}{64(2\pi)^6} \int \frac{d^3 r}{w_{p-r} v_r} \left[w_{p-r} + v_r - p_0 \right]^{-1} \\
& \times \int \frac{d^3 s}{w_{p-r-s} w_{r+s-q} w_{s-q} v_s} \left[(w_{p-r-s} + w_{r+s-q} - p_0 + q_0) \right. \\
& \times (w_{p-r-s} + w_{s-q} + v_r - p_0 + q_0) (w_{p-r-s} + v_s + v_r - p_0) \left. \right]^{-1} \\
I_2 = & \frac{e^4}{64(2\pi)^6} \int \frac{d^3 r}{w_{p-r} v_r} \left[w_{p-r} + v_r - p_0 \right]^{-1} \\
& \times \int \frac{d^3 s}{w_{p-r+s} w_{r-s-q} w_{s+q} v_s} \left[(w_{p-r+s} + w_{r-s-q} - p_0 + q_0) \right. \\
& \times (w_{p-r} + w_{r-s-q} + v_s - p_0 + q_0) (w_{p-r} + w_{r-s-q} + w_{s+q} - p_0) \left. \right]^{-1}
\end{aligned} \tag{V.2a}$$

$$\tag{V.2b}$$

Explicit evaluation ⁴⁾ of these two integrals leads to the result that both on and off the mass shell I_1 and I_2 give the same contribution

$$|k^2| \gg p^2, q^2 = m^2 : I_1 = I_2 = \frac{e^4}{3 \cdot 2^6 (2\pi)^4 |k^2|^2} \log^4 \frac{|k^2|}{\mu^2} \quad (\text{V.3a})$$

$$\frac{|p^2| |q^2|}{\mu^2} \gg |k^2| \gg |p^2|, |q^2| \gg m^2 :$$

$$I_1 = I_2 = \frac{e^4}{2^6 (2\pi)^4 |k^2|^2} \log^2 \left| \frac{q^2}{k^2} \right| \log^2 \left| \frac{p^2}{k^2} \right|. \quad (\text{V.3b})$$

The relevant region of integration for I_1 and I_2 is found to be, for the mass shell case:

$$\begin{aligned} p_{||} > \tau_{||} > \mu, \quad \sqrt{2m\tau_{||}} > \tau_{\perp} > \mu \\ \tau_{||} > s_{||} > \mu, \quad \sqrt{2ms_{||}} > s_{\perp} > \mu \\ \text{and} \quad \frac{s_{\perp}^2}{2s_{||}} > \frac{\tau_{\perp}^2}{2\tau_{||}}; \end{aligned} \quad (\text{V.4a})$$

and for the off mass shell case:

$$\begin{aligned} p_{||} > \tau_{||} > q_{||}, \quad \sqrt{2m\tau_{||}} > \tau_{\perp} > \sqrt{2\tau_{||}|p_{||}-p_0|} \\ \tau_{||} > s_{||} > \mu, \quad \sqrt{2ms_{||}} > s_{\perp} > \mu \\ \text{and} \quad \frac{s_{\perp}^2}{2s_{||}} > \frac{\tau_{\perp}^2}{2\tau_{||}} \end{aligned} \quad (\text{V.4b})$$

D. Next we examine the spin terms in the numerators of the propagators which we have ignored up to now. The numerator is

$$I_1: \gamma^\nu [\cancel{p} - \cancel{x} + m] \gamma^\omega [\cancel{p} - \cancel{x} - \cancel{s} + m] \gamma^\mu [\cancel{q} - \cancel{x} - \cancel{s} + m] \gamma_\omega \times [\cancel{q} - \cancel{s} + m] \gamma_\nu \quad (V.5a)$$

$$= \gamma^\nu [\cancel{p} - \cancel{x} + m] N_1^\mu [\cancel{q} - \cancel{s} + m] \gamma_\nu$$

$$I_2 = \gamma^\nu [\cancel{p} - \cancel{x} + m] \gamma^\omega [\cancel{p} - \cancel{x} + \cancel{s} + m] \gamma^\mu [\cancel{q} - \cancel{x} + \cancel{s} + m] \times \gamma_\omega [\cancel{q} + \cancel{s} + m] \gamma_\nu$$

$$= \gamma^\nu [\cancel{p} - \cancel{x} + m] N_2^\mu [\cancel{q} + \cancel{s} + m] \gamma_\nu.$$

(V.5b)

We must have a contribution of order $\gamma^\mu |k^2|^2$ from the numerator, in the relevant region of integration, to cancel the $|k^2|^2$ in the denominator. We show that I_2 does not have such a contribution.

Examining (V.5b), we see that one power of $|k^2|$ must come from N_2 , and another from the remaining terms. We write the numerator of I_2 as

$$\left([-\cancel{p} + m + \cancel{x}] \gamma^\nu + 2(p - r)^\nu \right) N_2^\mu \left(\gamma_\nu [\cancel{q} + m - \cancel{s}] + 2(q + s)_\nu \right). \quad (V.6a)$$

On the mass shell, $-p+m$ and $-q+m$ vanish, while off the mass these two expressions cannot build up to $|k^2|$. Therefore, the $|k^2|^2$ can only come from

$$4(p-r) \cdot (q+s) N_2^\mu. \quad (\text{V.6b})$$

N_2^μ is of the same form as the numerator of the second order contribution analyzed previously. The only terms which can lead to $|k^2|$ contained in N_2^μ are

$$\gamma^\mu [2(r-s)^2 + 4p \cdot q + 4(p+q) \cdot (s-r)]. \quad (\text{V.6c})$$

By virtue of the contour integration in s_0 and r_0 we also have

$$\begin{aligned} (q-r+s)^2 &= m^2 & (p-r)^2 &= m^2 \\ r_0 &= p_0 - \omega_{p-r} & s_0 &= -q_0 + p_0 - \omega_{p-r} - \omega_{r-q-s}. \end{aligned} \quad (\text{V.6d})$$

Therefore

$$(r-s)^2 = m^2 - q^2 - 2q \cdot (s-r), \quad (\text{V.6e})$$

and (V.6c) becomes

$$\begin{aligned}
& \gamma^\mu [2m^2 - 2q^2 + 4p \cdot q + 4p \cdot (s - r)] \\
&= \gamma^\mu [2m^2 - 2q^2 + 4p_0(q_0 + s_0 - r_0) - 4p_{11}(q_{11} + s_{11} - r_{11})] \\
&\approx \gamma^\mu [2m^2 - 2q^2 - 4r_{11}(p_0 - p_{11})].
\end{aligned}$$

(V.6f)

This contains no terms of $|k^2|$ and the assertion is proved.

On the other hand, an entirely similar analysis for the numerator of I_1 gives a value $4|k^2|^2$.

Therefore, the contribution of the lowest order crossed diagram of spinor electrodynamics is ⁶⁾

$$|k^2| \gg p^2, q^2 = m^2 : \frac{e^4}{3 \cdot 2^4 (2\pi)^4} \log^4 \frac{|k^2|}{\mu^2}, \quad (\text{V.7a})$$

$$\frac{|p^2 q^2|}{\mu^2} \gg |k^2| \gg |p^2|, |q^2| \gg m^2 : \frac{e^4}{2^4 (2\pi)^4} \log^2 \left| \frac{q^2}{k^2} \right| \log^2 \left| \frac{p^2}{k^2} \right|. \quad (\text{V.7b})$$

E. We now turn to the integral equation of Fig. 5. According to what we have shown above it may be approximated by

$$\begin{aligned}
 \Gamma(p^2, q^2, k^2) &= 1 - \frac{ie^2 2|k^2|}{(2\pi)^4} \\
 &\times \int \frac{d^4 r}{4\omega_{p-r}\omega_{r-q}} \frac{\Gamma([p-r]^2, [q-r]^2, k^2)}{[p_0-r_0-\omega_{p-r}+i\epsilon][r_0-q_0-\omega_{r-q}+i\epsilon]} \\
 &\times \left\{ \frac{1}{2\nu_r[r_0-\nu_r+i\epsilon]} - \frac{ie^2 2|k^2|}{(2\pi)^4} \int \frac{d^4 s}{16\nu_s\nu_{r-s}\omega_{p-r+s}\omega_{s-q}} \right. \\
 &\quad \times \left. \left[(p_0-r_0+s_0-\omega_{p-r+s}+i\epsilon)(r_0-s_0-\nu_{r-s}+i\epsilon)(s_0-\nu_s+i\epsilon) \right. \right. \\
 &\quad \left. \left. (s_0-q_0-\omega_{s-q}+i\epsilon) \right]^{-1} \right\}.
 \end{aligned}
 \tag{V.8}$$

[We use a labelling of variables of integration which is different from that of (V.2). The reason for this is that we wish to have Γ , on the right-hand side, independent of s , so that the s_0 integral can be performed. Once this has been done, we shall transform the spatial variables to be the same as in (V.2), so that we can use the limits (V.4).]

Performing the s_0 integral leaves

To make contact with our previous analysis of the crossed term, we change variables in the double integral

$$\begin{aligned}\tilde{s}' &= \tilde{s} \\ \tilde{r}' &= \tilde{r} - \tilde{s}\end{aligned}\quad (\text{V.12})$$

and obtain finally the integral equation in the form that can be readily approximated.

$$\begin{aligned}\Gamma(p^2, q^2, k^2) &= 1 - \frac{e^2 2|k^2|}{(2\pi)^3} \int \frac{d^3 r}{8\omega_{p-r} \omega_{r-q} \nu_r [q_0 + \omega_{r-q} - \nu_r]} \\ &\times \left\{ \frac{\Gamma_1}{\omega_{p-r} + \nu_r - p_0} - \frac{\Gamma_2}{q_0 + \omega_{r-q} + \omega_{p-r} - p_0} \right\} \\ &+ \frac{e^4 4|k^2|^2}{(2\pi)^6} \int \frac{d^3 r d^3 s}{64 \omega_{p-r-s} \omega_{r+s-q} \nu_s \nu_r \omega_{p-r} \omega_{s-q} [p_0 - \nu_r - \omega_{p-r}]} \\ &\times \left\{ \frac{\Gamma_3'}{(q_0 + \omega_{s+r-q} - \nu_r - \nu_s)(\omega_{s+r-q} - \nu_r - \omega_{s-q})(p_0 - q_0 - \omega_{r+s-q} - \omega_{p-r-s})} \right. \\ &+ \frac{\Gamma_4'}{(q_0 + \omega_{s+r-q} - \nu_r - \nu_s)(q_0 + \omega_{s-q} - \nu_s)(p_0 - \nu_r - \nu_s - \omega_{p-r-s})} \\ &\left. + \frac{\Gamma_5'}{(\nu_r + \omega_{s-q} - \omega_{r+s-q})(q_0 + \omega_{s-q} - \nu_s)(p_0 - \nu_r - q_0 - \omega_{s-q} - \omega_{p-r-s})} \right\} \quad (\text{V.13})\end{aligned}$$

The term involving only the $\underline{r}$ integral is just the old ladder approximation. The term with the $\underline{r}$ and $\underline{s}$ integrals is the contribution from the crossed term in the kernel.

E. We specialize (V.13) to the mass shell with values given by (IV.2). The analysis of the ladder term can be taken over from Section IV F. For the crossed term we work in the region given by (V.4a). There the denominators may be approximated just as in the previous calculations, to give for the crossed term (we again ignore the difference between μ and m).

$$\frac{e^4 |k^2|^2}{8(2\pi)^4 p_{||}^2 m^2} \int_{\mu}^{p_{||}} \frac{d\tau_{||}}{\tau_{||}} \int_{\mu}^{\sqrt{2m\tau_{||}}} \frac{d\tau_{\perp}}{\tau_{\perp}} \int_{\mu}^{\tau_{||}} \frac{ds_{||}}{s_{||}^2} \int_{\mu}^{\sqrt{2ms_{||}}} s_{\perp} ds_{\perp} \quad (V.14a)$$

$$\times \Theta\left(\frac{s_{\perp}^2}{2s_{||}} - \frac{\tau_{\perp}^2}{2\tau_{||}}\right) \left\{ \frac{\Gamma_4}{s_{\perp}^2/2s_{||}} + \frac{\Gamma_3 - \Gamma_5}{(s_{\perp}\tau_{\perp}/\tau_{||}) - (s_{\perp}^2/2s_{||})} \right\}.$$

The term $\Gamma_3 - \Gamma_5$ does not lead to maximal logarithms as it involves a difference of logarithmic terms, hence we ignore it. Γ_4 is in this region

$$\Gamma_4 = \Gamma\left(\frac{s_{\perp}^2 p_{||}}{s_{||}}, 2m\tau_{||}, k^2\right). \quad (V.14b)$$

It can easily be verified that the argument of Γ_4 lies in the off mass shell region (IV.2) and Lorentz frame (IV.8).

Changing variables in (V.14a)

$$y = 2M r_{||}, \quad z = 2M s_{||}, \quad w = \frac{|k^2| r_{\perp}^2}{y}, \quad x = \frac{|k^2| s_{\perp}^2}{z}, \quad (\text{V.14c})$$

gives after performing the w and z integrals (apart from unimportant numerical constants in the limits)

$$\frac{e^4}{4(2\pi)^4} \int_{\mu^2}^{|k^2|} \frac{dx dy}{xy} \Theta(xy - \mu^2 |k^2|) \Gamma(x, y, k^2) \log^2 \frac{xy}{\mu^2 |k^2|}. \quad (\text{V.14d})$$

We combine this with the ladder term (IV.22), and obtain the simplified integral for Γ on the mass shell, when one crossed diagram is included in the kernel.

$$\Gamma(m^2, m^2, k^2) = 1 - \frac{e^2}{8\pi^2} \int_{\mu^2}^{|k^2|} \frac{dx dy}{xy} \Theta(xy - \mu^2 |k^2|) \Gamma(x, y, k^2) \times \left[1 - \frac{e^2}{8\pi^2} \log^2 \frac{xy}{\mu^2 |k^2|} \right]. \quad (\text{V.15})$$

It is seen that we need an estimate for Γ when

$$\frac{|p^2 q^2|}{\mu^2} \gg |k^2| \gg |p^2|, |q^2| \gg m^2.$$

(as usual $m^2 = \mu^2$). We now give this estimate.

F. We specialize (V.14) to the argument of Γ given by (IV.7) and work in the frame (IV.8). The analysis of the ladder term can be taken over from Section IV G. For the crossed term we use the restriction on the integral given by (V.4b). Again the denominators may be simplified, and the crossed term contribution is

$$\frac{e^4 |k^2|^2}{8(2\pi)^4 p_{||}^2 m^2} \int_{q_{||}}^{p_{||}} \frac{dr_{||}}{r_{||}} \int_{\sqrt{2|p_{||}-p_0|r_{||}}}^{\sqrt{2m}r_{||}} \frac{dr_{\perp}}{r_{\perp}} \int_{q_{||}}^{r_{||}} \frac{ds_{||}}{s_{||}^2} \int_{\mu}^{\sqrt{2m}s_{||}} s_{\perp} ds_{\perp} \quad (V.16a)$$

$$\times \theta\left(\frac{s_{\perp}^2}{2s_{||}} - \frac{r_{\perp}^2}{2r_{||}}\right) \left\{ \frac{\Gamma_3 - \Gamma_5}{(s_{\perp}r_{\perp}/r_{||}) - (s_{\perp}^2/2s_{||})} + \frac{\Gamma_4}{s_{\perp}^2/2s_{||}} \right\}.$$

The term $\Gamma_3 - \Gamma_5$ does not contribute. Γ_4 in the relevant region is

$$\Gamma_4 = \Gamma\left(\frac{s_{\perp}^2}{s_{||}} p_{||}, 2m r_{||}, k^2\right). \quad (V.16b)$$

Changing variables

$$y = 2m r_{||}, z = 2m s_{||}, w = \frac{|k^2| r_{\perp}^2}{y}, x = \frac{|k^2| s_{\perp}^2}{z}, \quad (V.16c)$$

and performing the z and w integrals yields

$$\frac{e^4}{4(2\pi)^4} \int_{|p^2|}^{|k^2|} \frac{dx}{x} \int_{|q^2|}^{|k^2|} \frac{dy}{y} \Gamma(x, y, |k^2|) \log \frac{x}{|p^2|} \log \frac{y}{|q^2|} \quad (V.16d)$$

Combining this with the ladder term gives the completely simplified integral equation for Γ off the mass shell.

$$\Gamma(p^2, q^2, k^2) = 1 - \frac{e^2}{8\pi^2} \int_{|p^2|}^{|k^2|} \frac{dx}{x} \int_{|q^2|}^{|k^2|} \frac{dy}{y} \Gamma(x, y, k^2) \left[1 - \frac{e^2}{8\pi^2} \log \frac{x}{|p^2|} \log \frac{y}{|q^2|} \right]. \quad (V.17)$$

Equation (V.17) can be transformed by judicious differentiation to a differential equation. However, the solution of (V.17) and (V.15) is of no particular interest since the inclusion of only the lowest order crossed diagram in the kernel is quite arbitrary. Indeed we expect that all the crossed diagrams should be included, which of course we cannot do in closed form.

Nevertheless we may use (V.17) and (V.15) for the following purpose. We obtain an expression for Γ through fourth order in e .

We can rely on this result since the diagrams we have ignored contribute only in sixth order. From this fourth order result we guess at the behaviour of the exact Γ . Then we determine what restrictions are placed on the kernel so that the exact Γ results from our integral equation.

G. Through fourth order we have

$$\Gamma(p^2, q^2, k^2) = 1 - \frac{e^2}{8\pi^2} \log \left| \frac{p^2}{k^2} \right| \log \left| \frac{q^2}{k^2} \right| + \frac{1}{2} \left(\frac{e^2}{8\pi^2} \right)^2 \log^2 \left| \frac{p^2}{k^2} \right| \log^2 \left| \frac{q^2}{k^2} \right| \quad (\text{V.18})$$

$$\Gamma(m^2, m^2, k^2) = 1 - \frac{e^2}{16\pi^2} \log^2 \left| \frac{k^2}{m^2} \right| + \frac{1}{2} \left(\frac{e^2}{16\pi^2} \right)^2 \log^4 \left| \frac{k^2}{m^2} \right|. \quad (\text{V.19})$$

Examining these expressions, it is reasonable to suppose that the complete series exponentiate, viz.

$$\Gamma(p^2, q^2, k^2) = \exp \left(- \frac{e^2}{8\pi^2} \log \left| \frac{p^2}{k^2} \right| \log \left| \frac{q^2}{k^2} \right| \right) \quad (\text{V.20})$$

$$\Gamma(m^2, m^2, k^2) = \exp \left(- \frac{e^2}{16\pi^2} \log^2 \left| \frac{k^2}{m^2} \right| \right). \quad (\text{V.21})$$

Indeed, Sudakov ³⁾ by examining individual diagrams in the off mass shell case also arrived at (V.20). Cassandro and Cini ³⁾, who examined individual diagrams on the mass shell, did not obtain (V.21). We claim their analysis to be in error.

We now take the exact integral equation for the off mass shell case to be of the form

$$\Gamma(p^2, q^2, k^2) = 1 - \frac{e^2}{8\pi^2} \int_{|p^2|}^{|k^2|} \frac{dx}{x} \int_{|q^2|}^{|k^2|} \frac{dy}{y} \Gamma(x, y, k^2) \quad (V.22)$$

$$\times K(x, y, p^2, q^2, k^2, m^2, \mu^2).$$

Further we take the mass shell Γ to be given by

$$\Gamma(m^2, m^2, k^2) = 1 - \frac{e^2}{8\pi^2} \int_{\mu^2}^{|k^2|} \frac{dx dy}{xy} \Theta(xy - \mu^2 |k^2|) \Gamma(x, y, k^2) \quad (V.23)$$

$$\times K^1(x, y, p^2, q^2, k^2, m^2, \mu^2).$$

K and K^1 are the (unknown) kernels which include all the crossed diagrams. From what we have learned above, we know that

$$K = 1 - \frac{e^2}{8\pi^2} \log \frac{x}{|p^2|} \log \frac{y}{|q^2|} + O(e^4) \quad (V.24)$$

(V.25)

$$K^1 = 1 - \frac{e^2}{8\pi^2} \log^2 \frac{xy}{\mu^2 |k^2|} + O(e^4).$$

In order to get the most specific prediction about K and K^1 , we make the assumption, consistent with (V.24) and (V.25), that

$$K(x, y, p^2, q^2, k^2, m^2, \mu^2) = K\left(\frac{e^2}{8\pi^2} \log \frac{x}{|p^2|} \log \frac{y}{|q^2|}\right) \quad (\text{V.26})$$

$$K^1(x, y, p^2, q^2, k^2, m^2, \mu^2) = K^1\left(\frac{e^2}{8\pi^2} \log^2 \frac{xy}{\mu^2 |k^2|}\right). \quad (\text{V.27})$$

Thus the equations determining K and K^1 are

$$\begin{aligned} \exp \left[-\frac{e^2}{8\pi^2} \log \left| \frac{p^2}{k^2} \right| \log \left| \frac{q^2}{k^2} \right| \right] = \\ 1 - \frac{e^2}{8\pi^2} \int_{|p^2|}^{|k^2|} \frac{dx}{x} \int_{|q^2|}^{|k^2|} \frac{dy}{y} \exp \left(-\frac{e^2}{8\pi^2} \log \frac{x}{|k^2|} \log \frac{y}{|k^2|} \right) \times \\ K \left(\frac{e^2}{8\pi^2} \log \frac{x}{|p^2|} \log \frac{y}{|q^2|} \right) \end{aligned} \quad (\text{V.28})$$

$$\exp\left[-\frac{e^2}{16\pi^2} \log^2 \frac{|k^2|}{\mu^2}\right] = 1 - \frac{e^2}{8\pi^2} \int_{\mu^2}^{|k^2|} \frac{dx dy}{xy} \Theta(xy - \mu^2 |k^2|) \quad (\text{V.29})$$

$$\times \exp\left(-\frac{e^2}{8\pi^2} \log \frac{x}{|k^2|} \log \frac{y}{|k^2|}\right) K^1\left(\frac{e^2}{8\pi^2} \log^2 \frac{xy}{\mu^2 |k^2|}\right)$$

These integral equations for K and K^1 can be put into a more manageable form by the following changes of variable. In (IV.28) we set

$$z = \sqrt{\frac{e^2}{8\pi^2}} \log \left| \frac{p^2}{q^2} \right| \log \left| \frac{q^2}{k^2} \right|$$

$$x' = \sqrt{\frac{e^2}{8\pi^2}} \log \left| \frac{k^2}{q^2} \right| \log \frac{x}{|p^2|}$$

$$y' = \sqrt{\frac{e^2}{8\pi^2}} \log \frac{x}{|p^2|} \log \frac{y}{|q^2|} \quad (\text{V.30a})$$

This gives

$$e^{-z} = 1 - \int_0^z \frac{dx}{x} \int_0^x dy K(y) \exp \frac{(y-x)(z-x)}{x}. \quad (\text{V.30b})$$

In (V.29) we set

$$\begin{aligned} z &= \sqrt{\frac{e^2}{8\pi^2}} \log \frac{|k^2|}{\mu^2} \\ x' &= \sqrt{\frac{e^2}{8\pi^2}} \log \frac{x}{\mu^2} \\ y' &= \sqrt{\frac{e^2}{8\pi^2}} \log \frac{xy}{\mu^2 |k^2|}, \end{aligned} \quad (\text{V.31a})$$

which leads to

$$e^{-z^2/2} = 1 - \int_0^z dx \int_0^x dy K^1(y^2) \exp [x-z][x-y]. \quad (\text{V.31b})$$

The integral equations are solved in Appendix A. We are unable to give a closed form for K and K^1 ; only a series solution is available. We expand

$$K(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{(n!)^2} k_n \quad (\text{V.32})$$

$$K^1(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{(2n)!} k_n^1 \quad (\text{V.33})$$

Then the expansion coefficients are determined by the recursion relations

$$(n+1)! = \sum_{m=0}^n (n-m)! k_m \quad (\text{V.34})$$

$$\frac{(2n+2)!}{2^{n+1}(n+1)!} = (2n+1)!! = \sum_{m=0}^n (n-m)! k_m^1 \quad (\text{V.35})$$

The first few terms are

$$k_0 = 1, k_1 = 1, k_2 = 3, k_3 = 13 \quad (\text{V.36})$$

$$k_0^1 = 1, k_1^1 = 2, k_2^1 = 11, k_3^1 = 84 \quad (\text{V.37})$$

VI. SUMMARY

We have presented a modification of the usual Feynman rules, such that the asymptotic high energy forms of either individual Feynman diagrams, or of complete n point functions are easily obtainable. The techniques were applied to the vertex function of spinor electrodynamics, in the ladder approximation. The crossed diagrams were included, and the form of the relevant Bethe-Salpeter kernel was obtained.

VII. ACKNOWLEDGEMENT

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A P P E N D I X A

SOLUTION OF INTEGRAL EQUATIONS

I. We solve (IV.25) by first performing the following redefinitions and changes of variable

$$\begin{aligned}
 g^2 &= \frac{e^2}{2\pi^2} \\
 a &= \log \left| \frac{p^2}{k^2} \right| \log \left| \frac{q^2}{k^2} \right| & b &= \log \left| \frac{p^2}{k^2} \right| \\
 x' &= \log \frac{x}{|k^2|} \log \left| \frac{q^2}{k^2} \right| & y' &= \log \frac{x}{|k^2|} \log \frac{y}{|k^2|}
 \end{aligned} \tag{A.1}$$

$$\Gamma(p^2, q^2, k^2) = \Gamma(|k^2| e^b, |k^2| e^{a/b}, |k^2|) \equiv \Gamma_1(a, b).$$

Then (IV.25) becomes

$$\Gamma_1(a, b) = 1 - g^2/4 \int_0^a \frac{dx}{x} \int_0^x dy \Gamma_1\left(y, \frac{xb}{a}\right) \tag{A.2}$$

Since now the kernel does not depend on b , and the solution we seek is generated by iterations, $\Gamma_1(a, b)$ may be taken to be independent of b , i.e., $\Gamma_1(a, b) = \Gamma_2(a)$

$$\Gamma_2(a) = 1 - g^2/4 \int_0^a \frac{dx}{x} \int_0^x dy \Gamma_2(y). \tag{A.3}$$

Differentiating, we now find that

$$a \Gamma_2'' + \Gamma_2' + g^2/4 \Gamma_2 = 0$$

$$\Gamma_2(0) = 1. \quad (\text{A.4})$$

The solution of this is

$$\Gamma_2(a) = J_0(\sqrt{g^2 a}), \quad (\text{A.5})$$

with J_0 being the Bessel function.

II. We solve the integral equations (V.30b) and (V.31b) by a series expansion. First we change variables. In (V.30b) we set

$$y' = y/x, \quad x' = x/z. \quad (\text{A.6})$$

This gives

$$e^{-z} = 1 - z \int_0^1 dx \int_0^1 dy K(xyz) \exp[(y-1)(1-x)z]. \quad (\text{A.7})$$

In (V.31b) we set

$$y' = y/x, \quad x' = x/z, \quad z^2 = t \quad (\text{A.8})$$

and obtain

$$e^{-t/2} = 1 - t \int_0^1 x dx \int_0^1 dy K^1(t x^2 y^2) \exp[t x(x-1)(1-y)], \quad (\text{A.9})$$

We expand the exponentials, K and K^1 in power series

$$K(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{(n!)^2} k_n \quad (\text{A.10})$$

$$K^1(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{(2n)!} k_n^1. \quad (\text{A.11})$$

The right-hand side on (A.7) and (A.9) becomes

$$1 - z \sum_{n,m} \frac{(-z)^{n+m}}{m!(n!)^2} k_n \left[\int_0^1 dy y^n (1-y)^m \right]^2 \quad (\text{A.12})$$

$$1 - t \sum_{n,m} \frac{(-t)^{n+m}}{(2n)! m!} k_n^1 \int_0^1 dx x^{2n+m+1} (1-x)^m \times \int_0^1 dy y^{2n} (1-y)^m \quad (\text{A.13})$$

The integrations are elementary and yield

$$1 - z \sum_{n,m} \frac{(-z)^{n+m} m!}{[(n+m+1)!]^2} k_n \quad (\text{A.14})$$

$$1 - t \sum_{n,m} \frac{(-t)^{n+m} m!}{(2n+2m+2)!} k_n^1. \quad (\text{A.15})$$

Rearranging the sums, (A.14) and (A.15) can be rewritten as

$$1 - z \sum_{m=0}^{\infty} \frac{(-z)^m}{(m+1)!} A_m \quad (\text{A.16a})$$

$$A_m = \frac{1}{(m+1)!} \sum_{n=0}^m (m-n)! k_n \quad (\text{A.16b})$$

$$1 - \frac{t}{2} \sum_{m=0}^{\infty} \frac{(-t/2)^m}{(m+1)!} A_m^1 \quad (\text{A.17a})$$

$$A_m^1 = \frac{(m+1)! 2^{m+1}}{(2m+2)!} \sum_{n=0}^m (m-n)! k_n^1. \quad (\text{A.17b})$$

Comparing these expressions with the expansion of the exponentials on the left-hand side of (A.7) and (A.9) we get

$$A_m^1 = 1 = A_m$$

or

$$(m+1)! = \sum_{n=0}^m (m-n)! k_n \quad (\text{A.18})$$

$$\frac{(2m+2)!}{(m+1)! 2^{m+1}} = (2m+1)!! = \sum_{n=0}^m (m-n)! k_n^1. \quad (\text{A.19})$$

A P P E N D I X BEVALUATION OF AN INTEGRAL

We need to evaluate the integral (IV.22). We change variables to $x = p_{||} \tau_{\perp}^2 / \tau_{||}$, $y = 2m \tau_{||}$,

and obtain

$$\Gamma(m^2, m^2, k^2) = 1 - \frac{e^2}{8\pi^2} \int_{\mu^2}^{|k^2|} \frac{dx}{x} \int_{\frac{\mu^2 |k^2|}{x}}^{|k^2|} \frac{dy}{y} \Gamma(x, y, k^2) \quad (\text{B.1})$$

where, according to (IV.26),

$$\Gamma(x, y, k^2) = J_0 \left(\sqrt{\frac{e^2}{2\pi^2} \log \frac{x}{|k^2|} \log \frac{y}{|k^2|}} \right). \quad (\text{B.2})$$

Redefining again by

$$x' = \sqrt{\frac{e^2}{8\pi^2}} \log \frac{x}{|k^2|}, \quad y' = \sqrt{\frac{e^2}{8\pi^2}} \log \frac{y}{|k^2|}$$

$$t = \sqrt{\frac{e^2}{8\pi^2}} \log \frac{\mu^2}{|k^2|},$$

we get

$$\Gamma(m^2, m^2, k^2) = 1 - \int_0^t dx \int_0^{t-x} dy J_0(2\sqrt{xy}). \quad (\text{B.3})$$

$$\equiv f(t).$$

Differentiating with respect to t , gives

$$f'(t) = - \int_0^t dx J_0(2\sqrt{x(t-x)}) \quad (\text{B.4a})$$

$$f(0) = 1. \quad (\text{B.4b})$$

We evaluate the right-hand side of (B.4a), by setting $y = 1 - \frac{2x}{t}$. Then

$$\begin{aligned} f'(t) &= -\frac{t}{2} \int_{-1}^1 dy J_0(t\sqrt{1-y^2}) = -t \int_0^1 dy J_0(t\sqrt{1-y^2}) \quad (\text{B.5}) \\ &= -t \int_0^{\pi/2} d\theta \sin\theta J_0(t\sin\theta) \\ &= -\sin t. \end{aligned}$$

This together with (B.4b) gives $f(t) = \cos t$.

R E F E R E N C E S

- 1) S. Weinberg, Phys.Rev. 150, 1313 (1966).
- 2) P.G. Federbush and M.J. Grisaru, Ann.Phys. 22, 263, 299 (1963);
J.C. Polkinghorne, J.Math.Phys. 4, 503 (1963);
G. Tiktopoulos, Phys.Rev. 131, 480, 2373 (1963).
- 3) The vertex off the mass shell has been studied in perturbation theory by V.V. Sudakov, Soviet Phys. JETP 3, 65 (1956). The mass shell vertex has been examined in perturbation theory by M. Cassandro and M. Cini, Nuovo Cimento 34, 1719 (1964). However, many of their numerical results are wrong.
- 4) R. Jackiw, Ph.D. thesis, Cornell, (1966) (unpublished).
- 5) The author is indebted to Professor K. Wilson for this formulation.
- 6) The off mass shell result agrees with that of Sudakov, Ref. 3). The mass shell result does not agree with that of Cassandro and Cini, Ref. 3). It does agree with a calculation of D. Yennie, who communicated his result to the author privately.

FIGURE CAPTIONS

- Figure 1 : Complete Bethe-Salpeter equation for the vertex.
- Figure 2 : Ladder approximation to the vertex.
- Figure 3 : First iteration of the ladder vertex.
- Figure 4 : Decomposition of diagram of Fig. 3 into "secondary" diagrams.
- Figure 5 : Equation for vertex with the inclusion of one crossed diagram in the kernel.
- Figure 6 : First iteration of the second order, crossed term in the kernel.
- Figure 7 : "Secondary" diagrams, which determine the asymptotic high energy form of the Feynman diagram of Fig. 6.

FIG.1

FIG.2

FIG.3

FIG.4

FIG. 5

FIG. 6

FIG. 7