CM-P00057791

Archives

Ref. TH.877

THE OVERLAP FUNCTION IN MANY-BODY THEORY

R. Jackiw ⁺

CERN - Geneva

and

Jefferson Laboratory of Physics
Harvard University, Cambridge, Mass.

A B S T R A C T

Van Hove's overlap function is studied in many-body Schrödinger theory. A formula relating the overlap function at large impact parameter to the many-body forces is established, within lowest order in these forces. The shape of the diffraction peak as a function of the forces is discussed. The inverse problem of determining the forces from the overlap function is solved. The limiting form of the overlap function for small impact parameters is determined. A simple expression for this limit is obtained when the forces are short range.

⁺ Junior Fellow, Society of Fellows.

1. INTRODUCTION

The sharp diffraction peak in differential cross-sections for elementary particle elastic scattering can be parametrized by Van Hove's overlap function ¹⁾. This parametrization is conveniently exhibited in the impact parameter representation, as developed by Cottingham and Peierls ²⁾. The purpose of this paper is to study the overlap function within the framework of many-body Schrödinger theory, in the impact parameter approximation. We relate the overlap function to the many-body forces and present a closed expression for it when the impact parameter is greater than the radius of the interaction region. Our formula is accurate to lowest order in the forces, and is derived by a method invented by Bethe ³⁾ for the study of Coulomb shadow scattering in atoms. This is done in Section 2. In Section 3 we evaluate our formula for some typical interactions, and examine the variation of the overlap function and the diffraction peak with the range of the forces. Section 4 is devoted to the solution of the inverse problem of determining the forces from the overlap function. In Section 5, we give an alternate method for determining the overlap function. With this method we rederive those results of Section 2 that are valid for large impact parameters, and also present the small impact parameter limiting form of the overlap function.

2. DERIVATION OF THE OVERLAP FUNCTION

- A. The overlap function measures the departure from elastic unitarity of the (elastic) scattering amplitude $f(k, \theta)$. In the impact parameter representation the following formulae hold by definition

$$f(k, \theta) = 2k \int_0^{\infty} h(b, k) J_0(2b \sin(\theta/2)) b db \quad (2.1a)$$

$$\sigma_{\text{elastic}} = 8\pi \int_0^{\infty} |h(b, k)|^2 b db \quad (2.1b)$$

$$\sigma_{\text{inelastic}} = 8\pi \int_0^{\infty} \frac{\rho(b, k)}{4} b db. \quad (2.1c)$$

Here $\rho(b, k)$ is the overlap function; the scattered particle carries momentum $\underline{k}$ and energy $(\hbar^2 k^2)/(2m)$ and is scattered into angle θ . The impact parameter approximation consists of identifying the kernel $h(b, k)$ with the partial wave elastic scattering amplitude $f_l(k)$:

$$h(b, k) = f_l(k) + O(k^{-2}) \quad (2.2a)$$

$$kb = l + \frac{1}{2}. \quad (2.2b)$$

The unitarity relation takes the form

$$\text{Im } h(b, k) = |h(b, k)|^2 + \frac{\rho(b, k)}{4} + O(k^2). \quad (2.3)$$

Assuming sufficiently high energies so that the phase shifts have become negligible, we have pure shadow scattering viz. $\text{Re } f(k) \approx \text{Re } h(b, k) = 0$. Then (2.3) determines the elastic scattering in terms of the overlap function

$$\begin{aligned} -i h(b, k) &= \frac{1 - \sqrt{1 - \rho(b, k)}}{2} \\ &\approx \frac{\rho(b, k)}{4} \text{ when } |\rho| \ll 1. \end{aligned} \quad (2.4)$$

Within conventional perturbation theoretic expansions of the elastic scattering amplitude, it is necessary to evaluate the second order Born approximation to determine the effect of the inelastic channels on the elastic scattering, hence to determine the overlap function. However, the second Born approximation clearly contains much information which is of no interest, as it gives an explicit form for the second order pure elastic processes, viz. the processes that would be present even in the absence of the inelastic channels.

- B. We now give a derivation of the overlap function, using Bethe's method, which concentrates on the first Born approximation to inelastic scattering, rather than on the second Born approximation to elastic scattering. As the impact parameter approximation is being made, we may conclude from (2.1c) that the overlap function is proportional to the total inelastic cross-section for a particle with impact parameter b . However, within the physical picture of the impact parameter approximation, an expression for the total inelastic scattering cross-section can be given in a simple fashion by the semi-classical method where the scattered particle is considered to be in a classical orbit $\tilde{R}(t)$ which for simplicity we take to be a straight line:

$$\underline{R}(t) = \underline{b} + \underline{v}t, \quad \underline{b} \cdot \underline{v} = 0 \quad (2.5)$$

$$\underline{v} = \hbar \underline{k} / m.$$

Let the Hamiltonian for the N particle system be $H^S(\underline{r})$,
 ($\underline{r}$ stands for the N single particle variables) with eigenstates $\Psi_n(\underline{r})$

$$H^S(\underline{r}) \Psi_n(\underline{r}) = E_n \Psi_n(\underline{r}). \quad (2.6)$$

For the interaction Hamiltonian $H^I(\underline{R}, \underline{r})$, we take

$$H^I(\underline{R}, \underline{r}) = \sum_i V_i(\underline{R} - \underline{r}_i) + V_0(\underline{R}), \quad (2.7)$$

where $V_i(\underline{R} - \underline{r}_i)$ is the interaction potential between the incident particle and the i^{th} member of the system. $V_0(\underline{R})$ represents a possible interaction of the scattered particle with an external potential, for example the interaction of a light charged particle with the nucleus in the no recoil approximation.

The transition amplitude from the ground state to the state n , to first order in H^I , is

$$T_{n0}(\underline{b}, \underline{k}) = \frac{-i}{\hbar} \int_{-\infty}^{\infty} e^{i\omega_{n0}t} dt H_{n0}^I(\underline{R}) \quad (2.8a)$$

$$\hbar \omega_{n0} = E_n - E_0$$

$$\begin{aligned} H_{n0}^I(\underline{R}) &= \int d\underline{r} \Psi_n^*(\underline{r}) H^I(\underline{R}, \underline{r}) \Psi_0(\underline{r}) \quad (2.8b) \\ &= \sum_i \int d\underline{r} \Psi_n^*(\underline{r}) V_i(\underline{R} - \underline{r}_i) \Psi_0(\underline{r}). \end{aligned}$$

We have used the orthogonality of Ψ_n to Ψ_0 to eliminate $V_0(\underline{R})$. The overlap function is therefore given by

$$\frac{\rho(b, k)}{4} = \sum_{n>0} |T_{n0}(b, k)|^2 \quad (2.9)$$

Equations (2.8) and (2.9) form the basis for a calculation of the overlap function, for all b , once a definite model has been specified. However, for large b , we can approximate in (2.8) and obtain further general results for $\rho(b, k)$. We now proceed to do this. In Section 5 we shall present general results for small b .

C. The N particle system is assumed to be localized within a sphere of radius a , viz. $|\Psi_0(\underline{r})| \ll 1$ when $|\underline{r}_i| > a$. In the following we take $V_i = V$. When $b > a$, V can be expanded to give for $H_{n0}^I(\underline{R})$

$$\begin{aligned} H_{n0}^I(\underline{R}) &= \sum_i \int d\underline{r} \Psi_n^*(\underline{r}) [V(\underline{R}) - \underline{\nabla} V(\underline{R}) \cdot \underline{r}_i] \Psi_0(\underline{r}) \\ &= \underline{F}(\underline{R}) \cdot \underline{D}_{n0} \end{aligned} \quad (2.10)$$

The force $\underline{F}$ and dipole matrix element $\underline{D}_{n0}$ have been introduced.

$$\underline{F}(\underline{R}) = -\underline{\nabla} V(\underline{R}) \quad (2.11a)$$

$$\underline{D}_{n0} = \int d\underline{r} \Psi_n^*(\underline{r}) \left(\sum_i \underline{r}_i \right) \Psi_0(\underline{r}) \quad (2.11b)$$

Taking a radial force, and setting for definiteness $\underline{b}$ to be along the z axis and $\underline{y}$ along the x axis, we have

$$T_{no}(b, k) = -\frac{ib^2}{\hbar v} \int_{-\infty}^{\infty} e^{i\omega_{no} \frac{b}{v} s} ds \times \quad (2.12a)$$

$$f(b^2 [1+s^2]) [D_{no}^z + s D_{no}^x]; \quad F(R) = f(R^2)/R.$$

An approximate evaluation of this integral, valid for large b , proceeds by neglecting the term proportional to D_{no}^x , and replacing

$$\exp \left\{ i\omega_{no} \frac{b}{v} s \right\}$$

by 0 when $\omega_{no} \frac{b}{v} \gg 1$, and by 1 when $\omega_{no} \frac{b}{v} \lesssim 1$. Then

$$T_{no}(b, k) = -\frac{ib^2 m}{\hbar^2 k} I(b) D_{no}^z \quad \omega_{no} \lesssim \frac{v}{b} \quad (2.12b)$$

$$I(b) \equiv \int_{-\infty}^{\infty} ds f(b^2 [1+s^2]) = \frac{1}{b} \int_{-\infty}^{\infty} ds f\left(b^2 + \frac{v^2}{\omega_{no}^2} s^2\right) \quad \omega_{no} \gtrsim \frac{v}{b}. \quad (2.12c)$$

The sum in (2.7) is seen to be limited by the condition $\omega_{no} \lesssim \frac{v}{b}$, which we assume to be large. However, the contribution of the highly excited states is negligible, therefore we may extend the sum over all states, and evaluate it by closure.

$$\begin{aligned} \rho(b, k) &= \frac{4m^2 b^4}{\hbar^4 k^2} I^2(b) \langle Z^2 \rangle \quad (2.13a) \\ &= 0 \quad b > \frac{v}{\omega} \end{aligned}$$

$$\langle Z^2 \rangle = \int d\mathbf{r} \Psi_0^*(\mathbf{r}) \left(\sum_i z_i \right)^2 \Psi_0(\mathbf{r}). \quad (2.13b)$$

In the above ω is an average excitation frequency for those states of the N particle system for which D_{no}^z is appreciable. It is reasonable to take $\hbar\omega$ to be a constant of order unity times $\hbar^2/2Ma^2$, with M being the mass of the individual particles in the system. Thus, we have finally our formula for the overlap function

$$\rho(b, k) = \frac{4m^2 b^4 I^2(b)}{\hbar^4 k^2} \langle Z^2 \rangle \quad a < b < c(ka)a \quad (2.14a)$$

$$= 0 \quad b > c(ka)a$$

$$c = O\left(\frac{M}{m}\right) \quad (2.14b)$$

In the following we set $c=1$ for simplicity. By assumption we are in an energy region such that $ka \gg 1$. Thus there is an appreciable range of b for which (2.14a) holds. Our approximate theory does not give $\rho(b, k)$ for $b < a$; viz. the contribution of close collisions to shadow scattering has not been properly evaluated. For orientation we shall take $\rho(b, k)$ to be given by $\rho(a, k)$ in this region. In Section 5, we return to this question.

D. The above formula for $\rho(b, k)$ exhibits several noteworthy features. The dependence on the details of the many-body system is very simple. It is contained in the parameter $\langle Z^2 \rangle$ and in the dimension of the system. It is seen that the ground state configuration determines these quantities.

The specific b dependence of $\rho(b, k)$ is set by the force $F(R)$. However, one sees that in general for large b , $\rho(b, k)$ goes rapidly to zero, while for small b $\rho(b, k)$ goes to some limit of order $\rho(a, k)$.

The k dependence of $P(b, k)$ manifests itself in two ways. Firstly, there is the over-all factor k^{-2} . Secondly, $P(b, k)$ depends on k in that the range in which $P(b, k)$ is sizeable is defined in part by k : $a < b < (ka)a$. This however need not be a general feature of $P(b, k)$ if for some dynamical reason $P(b, k)$ becomes negligible when $a < b_0 < b < (ka)a$, and b_0 is independent of k . We show below that this indeed occurs for short range forces.

E. Next we discuss the elastic shadow scattering amplitude $f_s(k, \theta)$, as given by our formulae (2.1a), (2.4) and (2.14):

$$f_s(k, \theta) = \frac{im^2 a^5 I^2(a)}{\hbar^4 k^2} \langle Z^2 \rangle \frac{J_1(2ak \sin \theta/2)}{\sin \theta/2} + \frac{i 2m^2 \langle Z^2 \rangle}{\hbar^4 k} \int_a^{(ka)a} db b^5 I^2(b) J_0(2bk \sin \frac{\theta}{2}). \quad (2.15)$$

The first term in the right-hand side of the above represents the contribution to $f_s(k, \theta)$ from $P(b, k)$, $b < a$, viz. from the close collisions. We have approximated this as discussed above.

When $2ak \sin(\theta/2) \gg 1$, i.e., when $\theta \gg (ka)^{-1}$, (2.15) is negligible. Thus we obtain the typical condition for shadow scattering: it is unimportant except in the forward direction $\theta \lesssim (ka)^{-1}$.

Very much in the forward direction, when $2(ak)^2 \sin(\theta/2) \ll 1$, i.e., $\theta \ll (ak)^{-2}$, we may replace $J_0(2bk \sin \theta/2)$ by 1, and $J_1(2ak \sin \theta/2)$ by $ak \sin \theta/2$. Then (2.15) becomes

$$\begin{aligned}
 f_s(k, \theta) \Big|_{\theta \ll (ka)^{-2}} &= \frac{i m^2 a^6 I^2(a) \langle Z^2 \rangle}{\hbar^4 k} \\
 &+ \frac{i 2 m^2 \langle Z^2 \rangle}{\hbar^4 k} \int_a^{(ka)/a} b^5 I^2(b) db \\
 &= f_s(k, 0).
 \end{aligned}
 \tag{2.16}$$

Thus the elastic shadow scattering amplitude, as given by our overlap function, is isotropic for small angles less than $(ka)^{-2}$, vanishes for angles greater than $(ka)^{-1}$, and exhibits its angular structure (if any) in the angular interval $(ka)^{-2} < \theta < (ka)^{-1}$. In this region we may replace $2 \sin \theta/2$ by θ .

3. EXAMPLES

- A. We examine the form of the overlap function and the shadow scattering amplitude for specific force laws, with special attention to the variation of these formulae with the range of the forces. We consider first an exponential potential.

$$V(\underline{R}) = \frac{g r_0}{2} e^{-R/r_0} \quad (3.1a)$$

$$f(x) = \frac{g}{2} \frac{e^{-\sqrt{x}/r_0}}{\sqrt{x}} \quad (3.1b)$$

$$I(b) = \frac{g}{b} K_0(b/r_0) \quad (3.2)$$

$$\rho(b, k) = \frac{4m^2}{\hbar^4 k^2} \langle Z^2 \rangle g^2 b^2 K_0^2(b/r_0) \quad (3.3)$$

$a < b < (ka)a.$

K_0 is the modified Bessel function of the second kind. It is seen that when the range r_0 of the force $F(\underline{R})$ decreases, $\rho(b, k)$ decreases. This reflects the physical fact that as the range decreases, the shadow effect becomes less important, since the scattered particle cannot excite the many-body system as easily.

The forward shadow scattering amplitude, which also describes the amplitude for $\theta(ka)^2$, is from (2.16)

$$f_s(k, 0) = \frac{i 2m^2 \langle Z^2 \rangle}{\hbar^4 k} \frac{g^2 r_0^4}{4} \left(\frac{a}{r_0} ka \right)^4 \log^2 \left(ka \frac{a}{r_0} \right) \quad (3.4a)$$

$r_0 > (ka)a$

$$= \frac{i 2 m^2 \langle Z^2 \rangle}{\hbar^4 k} \frac{g^2 r_0^4}{3}, (ka) a > r_0 > a \quad (3.4b)$$

$$= \frac{i 2 m^2 \langle Z^2 \rangle}{\hbar^4 k} g^2 r_0^4 \frac{\pi}{4} \left(\frac{a}{r_0} \right)^3 e^{-2 \frac{a}{r_0}} \quad (3.4c)$$

$a > r_0.$

We have given just the asymptotic expressions in the above. Equations (3.4a) and (3.4b) arise from the second term in (2.16), viz. from distant collisions; while (3.4c) arises from the first term in (2.16), viz. from close collisions. The absence of a contribution from distant collisions in the last case is obviously a consequence of the fact that the potential is short range $a > r_0$, so that distant particles cannot excite the many-body system.

B. Next we study the power law potential.

$$V(\underline{R}) = \frac{g}{(2p-2) B(p-\frac{1}{2}, \frac{1}{2}) R^{2p-2}}, p > 1 \quad (3.5a)$$

$$f(x) = \frac{g}{B(p-\frac{1}{2}, \frac{1}{2}) x^p} \quad (3.5b)$$

$$I(b) = g b^{-2p} \quad (3.6)$$

$$\rho(b, k) = \frac{4 m^2 \langle Z^2 \rangle}{\hbar^4 k^2} g^2 b^{4-4p} \quad (3.7)$$

$a < b < (ka)a.$

B is the beta function. Note the decrease of $\rho(b, k)$ with increasing p , i.e., decreasing range of the potential. The forward shadow scattering amplitude is

$$f_s(k, 0) = \frac{i 2 m^2 \langle Z^2 \rangle}{\hbar^4 k} g^2 \frac{a^{6-4p}}{6-4p} [ka]^{6-4p} \quad (3.8a)$$

$\frac{3}{2} > p > 1$

$$= \frac{i 2 m^2 \langle Z^2 \rangle}{\hbar^4 k} g^2 \log ka, \quad p = \frac{3}{2} \quad (3.8b)$$

$$= \frac{i 2 m^2 \langle Z^2 \rangle}{\hbar^4 k} g^2 a^{6-4p} \left[\frac{1}{2} + \frac{1}{4p-6} \right]_{p > \frac{3}{2}}, \quad (3.8c)$$

while for $(ka)^{-1} > \theta > (ka)^{-2}$, the shadow scattering is given by
(2.15)

$$f_s(k, \theta) = \frac{i 2 m^2 \langle Z^2 \rangle}{\hbar^4 k} g^2 a^{6-4p} \times \left[\frac{J_1(ak\theta)}{ak\theta} + \frac{1}{(ak\theta)^{6-4p}} \int_{(ak\theta)}^{(ak)^2 \theta} x^{5-4p} J_0(x) dx \right]$$

$$= \frac{i 2 m^2 \langle Z^2 \rangle}{\hbar^4 k} g^2 a^{6-4p} \left[\frac{2^{5-4p} \Gamma(3-2p)}{\Gamma(2p-2) (ka\theta)^{6-4p}} + \sqrt{\frac{2}{\pi}} \frac{\sin[(ka\theta)^2 - \pi/4]}{(ka)^{4p-3} \theta^{3/2}} \right] \quad (3.9a)$$

$\frac{9}{8} \geq p > 1$

$$= \frac{i 2 m^2 \langle Z^2 \rangle}{\hbar^4 k} g^2 a^{6-4p} \left[\frac{2^{5-4p} \Gamma(3-2p)}{\Gamma(2p-2) (ka\theta)^{6-4p}} \right] \quad (3.9b)$$

$$\frac{3}{2} > p > \frac{9}{8}$$

$$= \frac{i 2 m^2 \langle Z^2 \rangle}{\hbar^4 k} g^2 [-\log ka\theta] \quad p = 3/2 \quad (3.9c)$$

$$= \frac{i 2 m^2 \langle Z^2 \rangle}{\hbar^4 k} g^2 a^{6-4p} \left[\frac{1}{2} + \frac{1}{4p-6} \right] \quad p > 3/2 \quad (3.9d)$$

Γ is the gamma function. In the above formulae, (3.8) and (3.9), the contribution from the close collisions is present only in the case $p > \frac{3}{2}$, (3.8c) and (3.9d). There it contributes the quantity $\frac{1}{2}$ appearing in the brackets. Thus as p increases beyond $\frac{3}{2}$, so that the range decreases, $\frac{1}{2}$ dominates $(4p-6)^{-1}$, and the close collisions determine completely the shadow scattering. Note the oscillatory behaviour of the amplitude for long range forces $9/8 > p > 1$.

It is to be remembered that $p = \frac{3}{2}$ corresponds to the Coulomb interaction. The logarithmic behaviour of the scattering amplitude, (3.9c), has been observed experimentally ⁴⁾ in atomic scattering, and a detailed theoretical discussion of this has been given by Bethe ³⁾. The Coulomb interaction marks the transition from the situation where distant collisions determine the diffraction peak, to the situation where close collisions are responsible for this phenomenon (for power force laws).

- C. We have seen that for short range potentials $V(R)$ the distant collisions do not contribute to the shadow scattering amplitude. We may therefore ignore the distant collisions already in the overlap function and set $\rho(b, k) = 0$, $b > a$. Therefore, for short range potentials the region in which $\rho(b, k)$ is appreciable is independent of energy. Evidently this is a general result; however, the definition of "short range" is somewhat arbitrary. Thus for exponential potentials e^{-R/r_0} , "short range" is $r_0 \ll a$; while for power law potentials r^{-n} , "short range" is $n \gg 1$.

- D. In conclusion, note that $f_s(k,0)$ goes to zero for $k \rightarrow \infty$, except for very long range forces. Therefore at very high energy $f_s(k,0)$ will always be dominated by $\text{Re} f(k,0)$, which is energy independent as can be verified by use of the Born approximation. Thus it is a model dependent question in what energy region (if any) is the diffraction peak observable for many-body non-relativistic scattering, described by an energy independent potential. [One must also determine the relative importance of the contribution of the elastic total cross-section to $\text{Im} f(k,0)$.]

4. THE INVERSE PROBLEM

We examine the inverse problem: given the overlap function, determine the nature of the forces between the scattered particle and the constituents of the N body system. This problem is easily solved if use is made of the formula ⁵⁾

$$b^2 I(b) = \frac{\hbar^2 k}{2m} \sqrt{\frac{\rho(b, k)}{\langle Z^2 \rangle}} \quad (4.1)$$

Then (2.12c) can be viewed as an integral equation for the force. This integral equation is solved readily, since it is a recondite form of Abel's equation. The solution for $f(r^2)$ is ⁶⁾

$$f(r^2) = -\frac{1}{\pi r^2} \int_r^\infty b db \frac{\partial}{\partial b} [b^2 I(b)] \frac{1}{\sqrt{b^2 - r^2}} \quad (4.2a)$$

The potential can be got then by integration

$$V(R) = \int_R^\infty r f(r^2) dr \quad (4.2b)$$

As a formal application of the above we obtain the forces corresponding to the overlap function which is used to fit high energy πp and pp scattering data ²⁾

$$\rho(b, k) = A e^{-b^2 \mu^2 / 2} \quad (4.3)$$

A and μ^2 are energy independent constants. From (4.1), we have

$$b^2 I(b) = \frac{\hbar^2 k}{2m} \sqrt{\frac{A}{\langle Z^2 \rangle}} e^{-\frac{b^2 \mu^2}{4}} \quad (4.4a)$$

Therefore, $f(r^2)$ is

$$f(r^2) = \frac{g \mu^2 k}{2} \int_1^{\infty} dx x^{1/2} (x-1)^{-1/2} e^{-\frac{r^2 \mu^2 x}{4}} \quad (4.4b)$$

and $V(R)$ becomes

$$\begin{aligned} V(R) &= g k \int_1^{\infty} dx x^{-1/2} (x-1)^{-1/2} e^{-\frac{r^2 \mu^2 x}{4}} \\ &= g k e^{-\frac{R^2 \mu^2}{8}} K_0 \left(\frac{R^2 \mu^2}{8} \right). \end{aligned} \quad (4.4c)$$

Here g is a constant. For large R , this becomes a potential with range $4\mu^{-2}$

$$V(R) \xrightarrow{R \rightarrow \infty} \frac{2g k \pi^{1/2}}{\mu R} e^{-\frac{R^2 \mu^2}{4}}, \quad (4.5a)$$

while for small R , V possesses a logarithmic singularity

$$V(R) \xrightarrow{r \rightarrow 0} -2g k \log R \quad (4.5b)$$

Note the k dependence.

5. ALTERNATE DERIVATION, SMALL IMPACT PARAMETER LIMIT

A. The theory as presented so far does not give a good estimate for the small b behaviour of the overlap function. For long range interactions, this behaviour was seen to be irrelevant. However, for short range interactions the reverse is true. We now give an alternate derivation of the theory. The present method is sufficiently flexible to reproduce the previous results, putting a different emphasis on the various approximations. It also can be used to study the small b behaviour of the overlap function.

Equation (2.8a) may be written in a Fourier representation.

$$T_{n0}(b, k) = \frac{-i}{\hbar} \int_{-\infty}^{\infty} dt e^{i\omega_{n0}t} \left(e^{-i\mathbf{q} \cdot \mathbf{R}(t)} V(\mathbf{q}) \right) \times G_{n0}(\mathbf{q}) \frac{d^3\mathbf{q}}{(2\pi)^3} \quad (5.1)$$

We have introduced the transition form factor

$$G_{nm}(\mathbf{q}) = \int d\mathbf{r} \Psi_n^*(\mathbf{r}) \left(\sum_i e^{i\mathbf{q} \cdot \mathbf{r}_i} \right) \times \Psi_m(\mathbf{r}) = G_{nm}(q_x, q_y, q_z) \quad (5.2a)$$

and the Fourier transform of the potential

$$V(\mathbf{q}) = \int e^{i\mathbf{q} \cdot \mathbf{r}} V(\mathbf{r}) d^3\mathbf{r} = V(\sqrt{q_x^2 + q_y^2 + q_z^2}) \quad (5.2b)$$

The t integral may be performed when the t dependence of $\mathbf{R}$ is specified. Using (2.5), we have

$$\int_{-\infty}^{\infty} dt e^{i\omega_{n0}t} e^{-i\mathbf{q} \cdot \mathbf{R}(t)} = (2\pi) \delta(\omega_{n0} - v q_x) \times e^{-i\mathbf{q} \cdot \mathbf{b}} \quad (5.3)$$

Therefore from (5.1)

$$T_{no}(b, k) = \frac{-im}{\hbar^2 k} \int e^{-iq \cdot b} V \left(\sqrt{\frac{\omega_{no}^2 m^2}{\hbar^2 k^2} + q_y^2 + q_z^2} \right) \times G_{no} \left(\frac{\omega_{no} m}{\hbar k}, q_y, q_z \right) \frac{d^2 q}{(2\pi)^2} \quad (5.4)$$

B. For large b only small $q \sim \frac{1}{b}$ contribute and we expand the form factor, using

$$\nabla G_{no} \big|_{q=0} = i D_{no}. \quad (5.5)$$

For this expansion to be valid, q must be less than $1/a$, which measures the spread of the form factor; i.e., we must have $b > a$. On the other hand for very large b , the entire expression (5.4) will vanish. Therefore b must be limited from above. The only parameter setting a scale in the integral is v/ω_{no} . Thus we take $b < v/\omega_{no}$. Since we need only the large k behaviour, we may set k to infinity in the argument of V and G_{no} . Hence we have

$$T_{no}(b, k) = \frac{m}{\hbar^2 k} \int_{-\infty}^{\infty} e^{-iq_z b} V(\sqrt{q_y^2 + q_z^2}) \quad (5.6)$$

$$q_z D_{no} \frac{d^2 q}{(2\pi)^2}, \quad a < b < \frac{v}{\omega_{no}}.$$

This is just the Fourier representation of (2.12).

C. For small $b < a$, we may expand the exponential in (5.4). Taking just the constant term, and setting all internal k 's to infinity, we obtain

$$T_{no}(b, k) \approx \frac{-im}{\hbar^2 k} \int \frac{d^2 q}{(2\pi)^2} V(\sqrt{q_y^2 + q_z^2}) G_{no}(0, q_y, q_z) = T_{no}(0, k). \quad (5.7)$$

The sum (2.9) may again be evaluated by closure.

$$\rho(0, k) = \frac{4m^2}{\hbar^4 k^2} \int \frac{d^2 q d^2 q'}{(2\pi)^4} V(\sqrt{q_y^2 + q_z^2}) \quad (5.8a)$$

$$\begin{aligned} & \times V(\sqrt{q_y'^2 + q_z'^2}) \tilde{G}_{00}(0, q_y, q_z; 0, q_y', q_z') \\ \tilde{G}_{00}(q, q') &= \int d\tau \psi_0^*(\tau) \left(\sum_i e^{-i q \cdot r_i} \right) \left(\sum_j e^{i q' \cdot r_j} \right) \psi_0(\tau) \\ &= G_{00}(q) G_{00}(q') \\ &+ G_{00}^c(q, q') \end{aligned} \quad (5.8b)$$

$$G_{00}^c(q, q') = \int d\tau \psi_0^*(\tau) \left(\sum_{i \neq j} e^{i(q \cdot r_i - q' \cdot r_j)} \right) \psi_0(\tau). \quad (5.8c)$$

Note the k^{-2} dependence, which is the same as in the large b expression for $\rho(b, k)$, (2.13). It is seen that the zero b limit of the overlap function is also determined by the ground state of the target; in this instance through the complicated form factor $\tilde{G}_{00}$ which is convoluted with the potential. It is seen that G_{00}^c is related to the two-point correlation function.

- D. We may unravel this convolution if we recall that the small b limit of $\rho(b, k)$ is of interest only for short range potentials. Taking $V(r)$ to have the short range r_0 and the strength g implies that $V(q)$ will be localized in the region $q < r_0^{-1}$ and will have the strength of order gr_0^3 . Therefore, we may cut off the integrals at r_0^{-1} and replace $V(q)$ by gr_0^3 .

$$\rho(0, k) = \frac{4m^2 g^2 r_0^6}{\hbar^4 k^2 (2\pi)^4} \int_{q, q' < r_0^{-1}} d^3 q d^3 q' \delta(q_x) \delta(q'_x) \quad (5.9a)$$

$$\{ G_{00}(|q - q'|) - G_{00}(q) G_{00}(q') + G_{00}^c(q, q') \}.$$

Since r_0^{-1} is very large, the integrals can be extended to infinity, if they converge. They will converge except for those involving $G_{00}(|q - q'|)$. Here only one integration can be extended. For this extension to be valid, the form factors must be negligible for $q > r_0^{-1}$. To assure this, we assume $r_0^{-1} > a^{-1}$; i.e., "short range" shall mean $r_0 < a$. From (5.9a) it then follows that

$$\rho(0, k) = \frac{4m^2 g^2 r_0^4}{\hbar^4 k^2 (2\pi)^2} \left[\int_0^\infty q dq G_{00}(q) \left[\frac{1}{2} - r_0^2 \int_0^\infty q dq G_{00}(q) \right] \right. \\ \left. + r_0^2 \int \frac{d^3 q d^3 q'}{(2\pi)^2} G_{00}^c(q, q') \delta(q_x) \delta(q'_x) \right] \quad (5.9b)$$

The two integrals multiplying r_0^2 in the bracket in (5.9b), are of order a^{-2} and a^{-4} respectively. Since we must ignore terms $O(r_0/a)$, we drop them, obtaining the final result for the zero b limit of the overlap function for short range forces:

$$\rho(0, k) = \frac{2m^2 g^2 r_0^4}{\hbar^4 k^2 (2\pi)^2} \int_0^\infty q dq G_{00}(q) \quad (5.10a)$$

$$r_0 < a$$

In position space the above is

$$\rho(0, k) = \frac{2m^2 g^2 r_0^4}{\hbar^4 k^2 (2\pi)^2} \left\langle \frac{1}{r^2} \right\rangle r_0 < a. \quad (5.10b)$$

$$\left\langle \frac{1}{r^2} \right\rangle = \int \psi_0^*(\underline{r}) \left(\sum_i \frac{1}{r_i^2} \right) \psi_0(\underline{r}) d\tau. \quad (5.10c)$$

The result is quite simple, depending on the details of the target only through $\langle 1/r^2 \rangle$. It is seen to be of order

$$\rho(0, k) = O \frac{m^2 g^2 r_0^2}{\hbar^4 k^2} \left(\frac{r_0}{a} \right)^2 r_0 < a \quad (5.11a)$$

whereas for long range forces, from (2.13) we have

$$\rho(0, k) = O \frac{m^2 g^2 a^2}{\hbar^4 k^2} \left(\frac{a}{r_0} \right)^2 r_0 > a. \quad (5.11b)$$

ACKNOWLEDGEMENT

I am grateful to Dr. A. Kotanski for acquainting me with the overlap function. The hospitality at CERN, where this work was performed is acknowledged.

R E F E R E N C E S

- 1) L. Van Hove, Revs.Modern Phys. 36, 655 (1964).
- 2) W.N. Cottingham and R.F. Peierls, Phys.Rev. 137, B147 (1965).
- 3) H.A. Bethe, "Shadow scattering by atoms" in Preludes in Theoretical Physics, edited by A. de-Shalit, H. Feshbach and L. Van Hove, North-Holland (Amsterdam), (1966).

This method is also described in H.A. Bethe and R. Jackiw, Intermediate Quantum Mechanics, 2nd Ed., W.A. Benjamin (New York) (1968).

- 4) N.F. Mott and H.S.W. Massey, The Theory of Atomic Collisions, 3rd Ed., Oxford Univ.Press (London) (1965).
- 5) It should be remembered of course that this precise mathematical manipulation is being applied to an approximate formula. Hence, the final result may not be very accurate.
- 6) R. Omnès, Phys.Rev. 137, B653 (1965).