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INTERNAL CO-ORDINATES AND EXPLICIT REPRESENTATIONS
OF THE POINCARÉ GROUP

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A B S T R A C T

Using a certain co-ordinate space (which is an extension of the Minkowsky space) as a carrier space we derive explicit positive mass representations of the Poincaré group. Besides that our procedure illustrates the peculiarity of the Dirac equation, it is intended as a first step in trying to develop an elementary particle theory where the spin plays a more pronounced role in the interaction process.

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Since the pioneering work by Wigner ¹⁾ on the unitary, irreducible representations (u.i.r.) of the Poincaré group (P) in 1939, many authors have played variations on the theme ²⁾. This article is going to add another and, as far as the author knows, new explicit form for these representations. What could be the reason to write it? The answer lies in the following circumstance. Thus while Wigner gave a complete description of the u.i.r. of P up to unitary equivalence it is only the Dirac representation and the representation defined by the electromagnetic field which have so far proved valuable as a basis for an interaction theory namely quantum electrodynamics ³⁾. But in strong or weak interactions, the Dirac equation has not proved very useful. Therefore, one may hope to get a more suitable basis for a theory of these interactions by looking for other explicit realizations of the Wigner representations. As Ref. ²⁾ shows there exist many ways of making the u.i.r. of P explicit by choosing different spaces related to the group space of P as a carrier space for the wave functions ⁴⁾. Thus the real problem is to find a principle which allows the selection of a specific carrier space. In Ref. ⁵⁾, we have proposed such a carrier space. The reason why we say that this space is selected is due to the fact that it is defined operationally starting from the properties of the classical electromagnetic field. As pointed out in Ref. ⁵⁾, our space can be identified with an eight-dimensional coset space of P.

Because of the way in which the carrier space $\mathcal{G}$ has been introduced there is a natural way in which the transformations of P act on $\mathcal{G}$. By defining a linear space of functions on $\mathcal{G}$ one then induces a linear representation of P also in a natural way. This representation can be made irreducible. On the irreducible spaces a positive sesquilinear form is then introduced so that the transformations of P preserve its values. By completing the linear space in the norm defined by this sesquilinear form one finally arrives at a unitary irreducible representation in a Hilbert space.

We shall often find it convenient to alternate between the description of $\mathcal{G}$ as a physically defined co-ordinate space and as a coset space of P .

In Section 2, the properties of the carrier space are reviewed and the transformations of P are defined. In Section 3, we define irreducible representations characterized by mass (>0), spin and two extra parameters. In Section 4 the different possible scalar products are introduced and the u.i.r. are defined. For special values of the extra parameters one finds the conventional realizations by means of spinors of the homogeneous Lorentz group $\mathcal{L}$. In Section 5 we extend P by the discrete transformations.

2. - THE CARRIER SPACE AND THE ACTION OF P

In this Section we shall define the carrier space $\mathcal{G}$ in two different ways. In the first case we are led by physical considerations, in the second case by purely group theoretical elaborations. Since we have already at some length discussed the physical motivation to introduce $\mathcal{G}$ and the way it is done in Ref. 5), we shall here only repeat the main results. $\mathcal{G}$ is the topological product of two four-dimensional spaces, the first of which is identified with the Minkowsky space and whose points are denoted $x^\mu \equiv (t, \vec{x})$. A point in the second factor is denoted $(\vec{e}, \vec{h})$ where $\vec{e}$ and $\vec{h}$ are two three-dimensional vectors satisfying $\vec{e}^2 = \vec{h}^2$, $\vec{e} \cdot \vec{h} = 0$. It can be thought of as an electromagnetic null field and sometimes we use the notation $F^{\mu\nu}$, where

$$F^{\mu\nu} = \begin{bmatrix} 0 & e_1 & e_2 & e_3 \\ -e_1 & 0 & h_3 & -h_2 \\ -e_2 & -h_3 & 0 & h_1 \\ -e_3 & h_2 & -h_1 & 0 \end{bmatrix} \quad (1)$$

Mostly we shall however use the following parametrization in terms of three Euler angles (φ, θ, ψ) and a scale parameter s

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = e^s \begin{bmatrix} -s\varphi s\psi + c\theta c\varphi c\psi \\ c\varphi s\psi + c\theta s\varphi c\psi \\ -s\theta c\psi \end{bmatrix}$$

$$\begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix} = e^s \begin{bmatrix} -s\varphi c\psi - c\theta c\varphi s\psi \\ c\varphi c\psi - c\theta s\varphi s\psi \\ s\theta s\psi \end{bmatrix}$$

$$\begin{bmatrix} (\vec{e} \times \vec{h})_1 \\ (\vec{e} \times \vec{h})_2 \\ (\vec{e} \times \vec{h})_3 \end{bmatrix} = e^{2s} \begin{bmatrix} s\theta c\varphi \\ s\theta s\varphi \\ c\theta \end{bmatrix} \quad (2)$$

Let an element of the restricted Poincaré group P_0 be denoted $g \equiv (b, \Lambda)$ where b is the translation and $\Lambda \in \mathcal{L}$ is the homogeneous Lorentz transformation. We can think of b as a four-vector b^μ and Λ as a 4×4 matrix Λ^μ_ν . Then the element (b, Λ) acts on $\mathcal{G}$ in the following fashion

$$\begin{aligned} (x^\mu, F^{\mu\nu}) &\xrightarrow{b} (x'^\mu, F'^{\mu\nu}) = (x^\mu + b^\mu, F^{\mu\nu}) \\ (x^\mu, F^{\mu\nu}) &\xrightarrow{\Lambda} (x'^\mu, F'^{\mu\nu}) = (\Lambda^\mu_\alpha x^\alpha, \Lambda^\rho_\alpha \Lambda^\sigma_\nu F^{\alpha\beta}) \end{aligned} \quad (3)$$

From these equations it is a simple matter to calculate the explicit transformation formulae in terms of the independent co-ordinates $(x^\mu, \varphi, \theta, \psi, s)$. We postpone this for the moment and turn instead to the mathematical characterization of $\mathcal{G}$.

For this we first introduce a suitable realization of the group P_0 . The element (b, Λ) of P_0 can be realized as a 5×5 matrix ⁶⁾

$$(b, \Lambda) \rightarrow \begin{pmatrix} & & & & \\ & \Lambda & & & \\ & & & & \\ & & & & \\ - & - & 0 & - & - \\ & & & & 1 \end{pmatrix} \quad (4)$$

and the multiplication law

$$(b_1, \Lambda_1)(b_2, \Lambda_2) = (b_1 + \Lambda_1 b_2, \Lambda_1 \Lambda_2) \quad (5)$$

follows from the ordinary matrix multiplication. Starting from this realization we now introduce a parametrization of P_0 by means of the Iwasawa decomposition of L ⁷⁾. To that end we first define the Lie algebra in terms of 5×5 matrices as follows. Put

$$\begin{aligned} M_{01} &= \begin{pmatrix} 0 & -1 & & & \\ -1 & 0 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix} & M_{02} &= \begin{pmatrix} 0 & & -1 & & \\ & 0 & & & \\ -1 & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix} \\ M_{03} &= \begin{pmatrix} 0 & & & -1 & \\ & 0 & & & \\ & & 0 & & \\ -1 & & & 0 & \\ & & & & 0 \end{pmatrix} & M_{12} &= \begin{pmatrix} 0 & & -1 & & \\ & 0 & -1 & & \\ 1 & 0 & & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix} \\ M_{23} &= \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & -1 & \\ & & 1 & 0 & \\ & & & & 0 \end{pmatrix} & M_{31} &= \begin{pmatrix} 0 & & & & \\ & 0 & & 1 & \\ & & 0 & & \\ -1 & & & 0 & \\ & & & & 0 \end{pmatrix} \\ P_0 &= \begin{pmatrix} & & & & 1 \\ & & & & 0 \\ & & & & 0 \\ & & & & 0 \\ & & & & 0 \end{pmatrix} & P_1 &= \begin{pmatrix} & & & & 0 \\ & & & & 1 \\ & & & & 0 \\ & & & & 0 \\ & & & & 0 \end{pmatrix} \\ P_2 &= \begin{pmatrix} & & & & 0 \\ & & & & 0 \\ & & & & 1 \\ & & & & 0 \\ & & & & 0 \end{pmatrix} & P_3 &= \begin{pmatrix} & & & & 0 \\ & & & & 0 \\ & & & & 0 \\ & & & & 1 \\ & & & & 0 \end{pmatrix} \end{aligned} \quad (6)$$

One then easily verifies the commutation relations

$$\begin{aligned}
 [M_{\mu\nu}, M_{\rho\sigma}] &= g_{\nu\rho} M_{\mu\sigma} + g_{\mu\sigma} M_{\nu\rho} \\
 &\quad - g_{\mu\rho} M_{\nu\sigma} - g_{\nu\sigma} M_{\mu\rho} \\
 [M_{\mu\nu}, P_\sigma] &= g_{\nu\sigma} P_\mu - g_{\mu\sigma} P_\nu
 \end{aligned}
 \tag{7}$$

The Iwasawa decomposition of the Lie algebra $\underline{\mathcal{L}}$ of the homogeneous Lorentz group is now stated as follows. The generators M_{12}, M_{23}, M_{31} define the maximal compact subalgebra $\underline{k}$, M_{03} defines a maximal Abelian subalgebra $\underline{a}$ among the "accelerations" M_{0i} and $M_{01} + M_{31}$ and $M_{02} - M_{23}$ generate a nilpotent subalgebra $\underline{n}$. Then

$$\underline{\mathcal{L}} = \underline{k} + \underline{a} + \underline{n} \quad (\text{direct vector sum}) \tag{8}$$

and furthermore $\underline{a} + \underline{n}$ is a solvable algebra with $\underline{n}$ as an ideal. A corresponding factorization of the group also holds

$$\mathcal{L} = K A N \tag{9}$$

where K , A and N are the subgroups corresponding to $\underline{k}$, $\underline{a}$ and $\underline{n}$. We now introduce parameters in $\mathcal{L}$ which are adapted to this decomposition. Thus we write for the arbitrary element Λ of $\mathcal{L}$

$$\Lambda = e^{\varphi M_{12}} e^{\theta M_{31}} e^{\psi M_{12}} e^{-\lambda M_{03}} e^{\pm(M_{01} + M_{31})} e^{\kappa(M_{02} - M_{23})} \tag{10}$$

where the generators $M_{\mu\nu}$ are the matrices of Eq. (6). Correspondingly we get a parametrization of the element of P_0

$$(b, 1) \rightarrow b \cdot 1 =$$

$$\left[\begin{array}{cc} \cosh s + e^{\frac{u^2}{2} + \frac{t^2}{2}} & -te^{\frac{u^2}{2} + \frac{t^2}{2}} \\ k_{13} \sinh s + k_{13} e^{\frac{u^2}{2} + \frac{t^2}{2}} - k_{11}t - k_{12}u & k_{11} - tk_{13}e^{\frac{u^2}{2} + \frac{t^2}{2}} \\ k_{23} \sinh s + k_{23} e^{\frac{u^2}{2} + \frac{t^2}{2}} - k_{21}t - k_{22}u & k_{21} - tk_{23}e^{\frac{u^2}{2} + \frac{t^2}{2}} \\ k_{33} \sinh s + k_{33} e^{\frac{u^2}{2} + \frac{t^2}{2}} - k_{31}t - k_{32}u & k_{31} - tk_{33}e^{\frac{u^2}{2} + \frac{t^2}{2}} \\ 0 & 0 \end{array} \right]$$

$$\left[\begin{array}{cc} -ue^{\frac{u^2}{2} + \frac{t^2}{2}} & \sinh s - e^{\frac{u^2}{2} + \frac{t^2}{2}} \\ k_{12} - uk_{13}e^{\frac{u^2}{2} + \frac{t^2}{2}} & k_{13} \cosh s - k_{13} e^{\frac{u^2}{2} + \frac{t^2}{2}} + k_{11}t + k_{12}u \\ k_{22} - uk_{23}e^{\frac{u^2}{2} + \frac{t^2}{2}} & k_{23} \cosh s - k_{23} e^{\frac{u^2}{2} + \frac{t^2}{2}} + k_{21}t + k_{22}u \\ k_{32} - uk_{33}e^{\frac{u^2}{2} + \frac{t^2}{2}} & k_{33} \cosh s - k_{33} e^{\frac{u^2}{2} + \frac{t^2}{2}} + k_{31}t + k_{32}u \\ 0 & 0 \end{array} \right] \begin{array}{l} b^0 \\ b^1 \\ b^2 \\ b^3 \\ 1 \end{array}$$

(11)

where

$$k_{ij} = \left[\begin{array}{cc} c\varphi c\theta c\psi - s\varphi s\psi; & -c\varphi c\theta s\psi - s\varphi c\psi \\ s\varphi c\theta c\psi + c\varphi s\psi; & -s\varphi c\theta s\psi + c\varphi c\psi \\ -s\theta c\psi & s\theta s\psi \\ & c\theta \end{array} \right] \quad (12)$$

The ranges of the parameters are

$$\begin{aligned}
 -\infty < b^\mu, \lambda, t, u < \infty \\
 0 \leq \theta \leq \pi \\
 0 \leq \varphi + \psi < 4\pi \\
 -2\pi \leq \varphi - \psi < 2\pi
 \end{aligned}
 \tag{13}$$

Notice that we have chosen ranges for φ and ψ corresponding to the group space of $SU(2)$ instead of $SO(3)$. This trick enables us to get the representations of the covering groups of P_0 instead of P_0 itself.

From the explicit parametrization (11) it is now possible to calculate the effect of a left multiplication by an arbitrary group element. Evidently a translation b' just changes the parameters b according to

$$b^\mu \xrightarrow{b'} b^\mu + b'^\mu \tag{14}$$

A homogeneous Lorentz transformation will change also the other parameters. As an example we calculate the effect of a pure Lorentz transformation $\exp \xi M_{03}$. Then

$$\begin{aligned}
 b^0 &\rightarrow b'^0 = b^0 \cosh \xi - b^3 \sinh \xi \\
 b^1 &\rightarrow b'^1 = b^1 \\
 b^2 &\rightarrow b'^2 = b^2 \\
 b^3 &\rightarrow b'^3 = b^3 \cosh \xi - b^0 \sinh \xi \\
 e^\lambda &\rightarrow e^{\lambda'} = e^\lambda [\cosh \xi - c\theta \sinh \xi] \\
 c\theta &\rightarrow c\theta' = \frac{c\theta \cosh \xi - \sinh \xi}{\cosh \xi - c\theta \sinh \xi} \\
 \varphi &\rightarrow \varphi' = \varphi \\
 \psi &\rightarrow \psi' = \psi
 \end{aligned}$$

(15)

The transformations of t and u are not listed since we shall anyhow not be interested in them.

The carrier space $\mathcal{C}$ introduced earlier is now identified with the coset space P_0/N in such a way that the parameters b^μ are identified with x^μ and the variables φ, θ, ψ and s with those of Eq. (2). This is possible since the two spaces have the same topology and the Poincaré group acts in the same way on P_0/N as on $\mathcal{C}$. For the pure Lorentz transformation the explicit formulae are those of Eq. (15). It is a simple matter to generalize this to an arbitrary transformation of P_0 .

Define now a linear space of functions $f(y)$ over $\mathcal{C}$ and a transformation

$$T_g f(y) = f(g^{-1} \cdot y) \quad (16)$$

where $g \in P_0$, $y \in \mathcal{C}$ and $g^{-1} \cdot y$ is the action of g by left multiplication. Assuming that f is differentiable the infinitesimal transformations are realized as differential operators. In this way one finds using the same notation for the infinitesimal operator as for the corresponding element of the Lie algebra

$$\begin{aligned} P_\mu &= - \frac{\partial}{\partial x^\mu} \\ M_{\mu\nu} &= x_\mu \frac{\partial}{\partial x^\nu} - x_\nu \frac{\partial}{\partial x^\mu} + S_{\mu\nu} \end{aligned} \quad (17)$$

where

$$S_{23} = c\varphi \frac{c\theta}{s\theta} \frac{\partial}{\partial \varphi} + s\varphi \frac{\partial}{\partial \theta} - \frac{c\varphi}{s\theta} \frac{\partial}{\partial \psi}$$

$$S_{31} = s\varphi \frac{c\theta}{s\theta} \frac{\partial}{\partial \varphi} - c\varphi \frac{\partial}{\partial \theta} - \frac{s\varphi}{s\theta} \frac{\partial}{\partial \psi}$$

$$S_{12} = -\frac{\partial}{\partial \varphi}$$

$$S_{01} = -\frac{s\varphi}{s\theta} \frac{\partial}{\partial \varphi} + c\varphi c\theta \frac{\partial}{\partial \theta} + s\varphi \frac{c\theta}{s\theta} \frac{\partial}{\partial \varphi} + c\varphi s\theta \frac{\partial}{\partial \lambda}$$

$$S_{02} = \frac{c\varphi}{s\theta} \frac{\partial}{\partial \varphi} + s\varphi c\theta \frac{\partial}{\partial \theta} - c\varphi \frac{c\theta}{s\theta} \frac{\partial}{\partial \varphi} + s\varphi s\theta \frac{\partial}{\partial \lambda}$$

$$S_{03} = -s\theta \frac{\partial}{\partial \theta} + c\theta \frac{\partial}{\partial \lambda}$$

(18)

These expressions have been given already in Ref. 5) with some minor changes of sign. [The factor $(\partial/\partial s) - 1$ there should be replaced by $\partial/\partial s$ which only introduces a trivial change in the definition of ψ later on.] Notice that the transformation of the variables never depends on the values of φ and s as can be seen either from Eq. (15) or Eqs. (18).

3. - LINEAR IRREDUCIBLE REPRESENTATION OF P_0

In Eq. (16) we have as a matter of fact introduced a linear representation of P_0 in a function space over $\mathcal{C}$. The topological properties of this space cannot be discussed until we have introduced a scalar product in Section 4. For the moment the functions $f(y)$ are differentiable. That Eq. (16) defines a representation follows from

$$\begin{aligned}
T_{g_2} T_{g_1} f(y) &= T_{g_2} f(g_1^{-1} \cdot y) = \\
&= f(g_1^{-1} g_2^{-1} \cdot y) = f((g_2 g_1)^{-1} \cdot y) = \\
&= T_{g_2 g_1} f(y).
\end{aligned}$$

(19)

While the question of unitarity of this representation also cannot be discussed until a scalar product has been defined the requirement of irreducibility can be imposed and will lead to certain constraints on the functions f . As already noted in connection with Eqs. (17) and (18), the differential operators $\partial/\partial y$ and $\partial/\partial s$ commute with every generator. Consequently $f(y)$ can be chosen to be eigenfunction of $\partial/\partial y$ and $\partial/\partial s$. Furthermore, of course, the two Casimir operators $P_\mu P^\mu$ and $W_\mu W^\mu$ with $W_\mu = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} M^{\nu\rho} P^\sigma$ commute with the Lie algebra. Thus we require as irreducibility conditions

$$\begin{aligned}
\frac{\partial f}{\partial y} &= i k f \\
\frac{\partial f}{\partial s} &= (i\nu - 1) f \\
P_\mu P^\mu f &= -m^2 f \\
W_\mu W^\mu f &= m^2 j(j+1) f.
\end{aligned}
\tag{20}$$

But then the number of restrictions on the function space is four so that there are left four degrees of freedom of $\mathcal{G}$. This is the expected number in an irreducible positive mass representation of P_0 . (Cf., the variables $\vec{p}, f$ in Wigner's terminology.) In Eq. (20), k has to be integer or half-integer since f has to be continuous on $\mathcal{G}$. The eigenvalue $(i\nu - 1)$ is arbitrary complex. The last two eigenvalues obviously imply that we are looking for positive mass representations with spin j . To satisfy the first three of the Eqs. (20), we put

$$\begin{aligned}
f(x^\mu, \varphi, \theta, \psi, s) &= \\
&= e^{(i\nu - 1)s} e^{ik\psi} \int \frac{d^3 p}{p_0} \hat{f}(\vec{p}, \varphi, \theta) e^{i p_\mu x^\mu}
\end{aligned}
\tag{21}$$

$p_0 = \sqrt{m^2 + \vec{p}^2}$

Instead of dealing with the functions $f(y)$ we can now use the Fourier transforms $\hat{f}$ and the correspondingly simplified Lie algebra

$$\begin{aligned} P_0 &= -ip_0 = -i\sqrt{m^2 + \vec{p}^2} \\ P_i &= -ip_i \\ M_{ij} &= p_i \frac{\partial}{\partial p_j} - p_j \frac{\partial}{\partial p_i} + S_{ij} \\ M_{0i} &= p_0 \frac{\partial}{\partial p_i} + S_{0i} \end{aligned} \quad (22)$$

where S_{ij} and S_{0i} are reduced by putting $\partial/\partial\psi = ik$ and $\partial/\partial s = (i\nu - 1)$. The fourth of Eqs. (20) is most easily satisfied by choosing a basis in which $\vec{p}$ is diagonal, solve the eigenvalue equation for $\vec{p} = 0$ and then by means of a "boost", i.e., a pure Lorentz transformation, define wave functions with given momentum and spin. Calling the basis $|\vec{p}, j_3\rangle$ one finds for $\vec{p} = 0$ the following eigenvectors to $W_\mu W^\mu$ (and M_{12})

$$|\vec{0}, j_3\rangle = \text{Const} e^{im\tau} e^{(i\nu-1)s} D_{j_3 k}^j(\varphi, \theta, \psi) \quad (23)$$

where the Wigner function $D_{j_3 k}^j$ is defined in the Appendix. To get to a state with definite non-zero momentum along the third axis, we just have to make the substitutions of Eq. (15)

$$\begin{aligned} |p^1=p^2=0, p^3=m \operatorname{sh} \epsilon, j_3\rangle &= \\ &= \text{Const} e^{i(p_0 t - p^3 x^3)} e^{(i\nu-1)s} \frac{D_{j_3 k}^j(\varphi, \theta', \psi)}{[p_0 - p^3 \cosh \epsilon]^{1-i\nu}} \end{aligned} \quad (24)$$

where

$$\cosh \epsilon' = \frac{\cosh \epsilon \cosh \epsilon - \operatorname{sh} \epsilon}{\cosh \epsilon - \cosh \epsilon \operatorname{sh} \epsilon}$$

Then, by applying a rotation, we get to an arbitrary momentum $\vec{p}$ and the angular variables have to be changed correspondingly. Our definition of one-particle states is thus the same as in the helicity formalism. Notice, however, that the emphasis is here on the explicit realization of the wave functions on the carrier space and also that this realization is dependent on two parameters k and ν , external to P . Notice also the typical behaviour in spin space for ν real. When the momentum is very large $p^3 \gg m$, then the wave function is sharply peaked in the direction of $\theta = 0$. This characteristic feature will be used as a possible explanation of the high-energy behaviour of the differential cross-section in high-energy scattering.

4. - UNITARITY

Unitarity refers to a scalar product and a Hilbert space. Whether the representations defined in the preceding Section correspond to unitary representations is settled by asking for a scalar product which makes the transformations (16) unitary and the generators (17) anti-hermitian. We shall in this Section introduce the most general scalar product. It is found that each irreducible representation of P_0 can be given in terms of several scalar products which, however, depend on k and ν .

For practical reasons, we shall use the momentum space wave-functions. As an introduction to the problem, we first guess a scalar product

$$(f_1, f_2) \equiv \iint \frac{d^3p}{p_0} d\Omega \hat{f}_1^*(\vec{p} \varphi \theta) \hat{f}_2(\vec{p} \varphi \theta)$$

(25)

where $d\Omega = \sin\theta d\theta d\varphi$. In fact, as is easily shown, this scalar product makes the generators anti-hermitian as long as ν is real. But not only this. The orbital part and the spin part $S_{\mu\nu}$ of $M_{\mu\nu}$ are separately anti-hermitian.

In the general case we have to assume a scalar product of the form

$$(f_1, f_2)_K = \iiint \frac{d^3 p_1}{p_{10}} \frac{d^3 p_2}{p_{20}} d\Omega_1 d\Omega_2 \times \hat{f}_1^*(\vec{p}_1, \varphi_1, \theta_1) K(\vec{p}_1, \varphi_1, \theta_1 | \vec{p}_2, \varphi_2, \theta_2) \hat{f}_2(\vec{p}_2, \varphi_2, \theta_2) \quad (26)$$

where K is a positive definite Hermitian integral kernel. Now K has to be a function of the two-point invariants of the carrier space $\mathcal{C}$. This is so because unitarity means that the scalar product remains invariant under the co-ordinate transformation (16) on both vectors f_1 and f_2 . By a substitution of variables under the integral, this transformation can be thrown over on K so that K has to satisfy ⁸⁾

$$K(q \cdot y_1 | q \cdot y_2) d\Omega'_1 d\Omega'_2 = K(y_1 | y_2) d\Omega_1 d\Omega_2 \quad (27)$$

where $d\Omega'$ is the transformed volume element ($d^3 p/p_0$ is always invariant). Thus $K d\Omega_1 d\Omega_2$ has to be constructed out of invariants depending on two points y_1 and y_2 . These can be formed either by looking at the physical definition of $\mathcal{C}$ or by using its group theoretical significance. Physically the carrier space is characterized by a four-vector p^μ and the two three-vectors $\vec{e}$ and $\vec{h}$. Since they are the electric and magnetic field vectors of a photon it is clear that they also define a four-vector namely the momentum of the photon

$$k^\mu = e^\mu (1, \sin\theta \cos\varphi, \sin\theta \sin\varphi, \cos\theta) \quad (28)$$

Notice that $\vec{k}$ is parallel to $\vec{e} \times \vec{h}$. From the four four-vectors p_1 , p_2 , k_1 and k_2 we can now form the four two-point invariants

$$(p_1 p_2), (k_1 k_2), (p_2 k_1), (p_1 k_2) \quad (29)$$

out of scalar products. But these do not exhaust the possibilities. The combination $\vec{e}_1 \cdot \vec{h}_2 + \vec{e}_2 \cdot \vec{h}_1$ is also a two-point invariant. Besides $(k_1 k_2)$ and $\vec{e}_1 \cdot \vec{h}_2 + \vec{e}_2 \cdot \vec{h}_1$ we do not expect any further invariants in the internal space since in the topological product of this space with itself the homogeneous Lorentz group "uses up" six dimensions so that only two invariants can be constructed.

Using group theoretical arguments one finds in the internal space the complex invariant

$$c = e^{\frac{1}{2}(i_1 + i_2)} e^{\frac{i}{2}(\varphi_1 + \varphi_2)} \left[c \frac{\theta_1}{2} s \frac{\theta_2}{2} e^{\frac{i}{2}(\varphi_1 - \varphi_2)} - s \frac{\theta_1}{2} c \frac{\theta_2}{2} e^{-\frac{i}{2}(\varphi_1 - \varphi_2)} \right] \quad (30)$$

The connection between c and $(k_1 k_2)$ and $\vec{e}_1 \cdot \vec{h}_2 + \vec{e}_2 \cdot \vec{h}_1$ is given by

$$(k_1 k_2) = 2|c|^2$$

$$\frac{\vec{e}_1 \cdot \vec{h}_2 + \vec{e}_2 \cdot \vec{h}_1}{(k_1 k_2)} = \frac{1}{2} \Im \frac{c}{c^*} \quad (31)$$

Going back to formula (26), for the scalar product, we now have to construct K out of the five invariants in Eqs. (29) and (31).

A first reduction occurs by requiring that P_μ is anti-hermitian. Since P_μ acts multiplicatively, K must be local in the momentum

variables so that $p_1 = p_2 = p$ and we are left with the invariants (pk_1) , (pk_2) , $(k_1 k_2)$ and $\text{Im} \frac{c}{c^*}$. Now in an irreducible representation $\partial/\partial s_1 = \partial/\partial s_2 = (-1+i\nu)$ and $\partial/\partial \psi_1 = \partial/\partial \psi_2 = ik$ so that the remaining four invariants can only enter into K in certain powers. A closer examination shows that $\text{Im} \frac{c}{c^*}$ must come in with the exponent zero because k is real. Thus we are left with a scalar product of the form

$$(f_1, f_2)_{\gamma\delta\varepsilon} \equiv \iiint \frac{d^3 p}{p_0} d\Omega_1 d\Omega_2$$

$$(k_1 k_2)^\gamma (pk_1)^\delta (pk_2)^\varepsilon f_1(\vec{p}, \varphi_1, \theta_1) \hat{f}_2(\vec{p}, \varphi_2, \theta_2).$$
(32)

Now anti-hermiticity of the generators $M_{\mu\nu}$ requires

$$\begin{aligned}\delta + \gamma &= -1 + i\nu \\ \varepsilon + \gamma &= -1 - i\nu\end{aligned}$$
(33)

The exponents $\gamma, \delta, \varepsilon$ are not completely fixed by this requirement. Hermiticity of the scalar product adds the conditions

$$\begin{aligned}\delta &= \varepsilon^* \\ \gamma &= \gamma^*\end{aligned}$$
(34)

A more difficult problem is to decide for which values $\gamma, \delta, \varepsilon$ K is positive definite, i.e., defines a scalar product. We shall not take up this problem in full generality here but only discuss two cases

- a) - $\gamma = -1$ and local measure ;
- b) - $\delta = \varepsilon = 0$ and non-local measure.

In the first case we thus have the scalar product

$$(f_1, f_2)_\nu = \iint \frac{d^3 p}{p_0} d\Omega \frac{\hat{f}_1^*(\vec{p}, \varphi, \theta) \hat{f}_2(\vec{p}, \varphi, \theta)}{(pk)^{-2\gamma m \nu}} \quad (35)$$

It is clear that $(f, f) \geq 0$ since $(pk) > 0$. In case b), we have

$$(f_1, f_2)_\gamma = \iiint \frac{d^3 p}{p_0} d\Omega_1 d\Omega_2 \frac{\hat{f}_1^*(\vec{p}, \varphi_1, \theta_1) \hat{f}_2(\vec{p}, \varphi_2, \theta_2)}{(k_1, k_2)^{-\gamma}} \quad (36)$$

From the analysis in Ref. 5), we know that this scalar product is positive definite for $0 < -\gamma < 1$, i.e., for $0 < -i\nu < 1$.

Returning to the scalar product (35), we first remark that it reduces to that given in Eq. (25) for ν real. Secondly we can now characterize precisely the Hilbert space in which the representation $U(m, j)$ of P_0 is given. Define first a big Hilbert space $\mathcal{H}_B$ as follows. For given k and ν $\mathcal{H}_B$ contains all functions $\hat{f}(\vec{p}, \varphi, \theta)$ which are square integrable in the norm defined by Eq. (35). Consider then the subspace $\mathcal{H}_j$ of $\mathcal{H}_B$ whose elements are eigenfunctions of $W_\mu W^\mu$ with the eigenvalue $m^2 j(j+1)$ ($k \leq j$). Then the operators of Eq. (22) define an infinitesimal irreducible anti-hermitian representation of P_0 and the transformation

$$T_g \hat{f}(\vec{p}, \varphi, \theta) = \frac{e^{(i\nu-1)s'} e^{ik\varphi'}}{e^{(i\nu-1)s} e^{ik\varphi}} \hat{f}(\vec{p}', \varphi', \theta'), \quad (37)$$

where $\vec{p}', \varphi', \theta', \varphi'$ and s' are obtained as in Eq. (14), defines a unitary irreducible representation of P_0 .

5. - THE DISCRETE TRANSFORMATIONS

For completeness we should also discuss the full Poincaré group including space reflection Σ and time reversal τ . We use the same technique of construction as for the continuous transformations. Thus we first define Σ on the carrier space $\mathcal{C}$ and then put

$$T_{\Sigma} f(y) = \eta f(\Sigma \cdot y) \quad (38)$$

where $\eta^2 = \pm 1$ since $T_{\Sigma}^2 = T_{\Sigma}^2 = \pm 1$. Using the physical definition of the carrier space, one has

$$\Sigma \cdot (t, \vec{x}, \vec{e}, \vec{h}) = (t, -\vec{x}, -\vec{e}, \vec{h}) \quad (39)$$

since the electric field behaves as a vector and the magnetic as an axial vector. Equation (39) is equivalent to

$$\Sigma \cdot (t, \vec{x}, \varphi, \theta, \psi, s) = (t, -\vec{x}, \varphi + \pi, \pi - \theta, \pi - \psi, s) \quad (40)$$

Similarly for time-reversal

$$\tau \cdot (t, \vec{x}, \vec{e}, \vec{h}) = (-t, \vec{x}, \vec{e}, -\vec{h}) \quad (41)$$

or

$$\tau \cdot (t, \vec{x}, \varphi, \theta, \psi, s) = (-t, \vec{x}, \varphi + \pi, \pi - \theta, -\psi, s) \quad (42)$$

For a state at rest one now obtains using the formulae of the Appendix

$$\begin{aligned} T_Z \{ e^{imt} e^{(i\nu-1)s} \mathcal{D}_{j_3 k}^j(\varphi, \theta, \psi) \} = \\ = \eta (-1)^{j-k} e^{imt} e^{(i\nu-1)s} \mathcal{D}_{j_3 -k}^j(\varphi, \theta, \psi) \end{aligned} \quad (43)$$

If $k \neq 0$ one must double the function space by including also the vectors where $k \rightarrow -k$. If the mesons are assumed to belong to representations with $k = 0$, one finds that they have definite parity and furthermore that this parity alternates with the spin j . The definition of time reversal is chosen to be

$$T_Z f(y) = f^*(T \cdot y) \quad (44)$$

in order to avoid the introduction of negative energies. A state at rest is consequently transformed according to

$$\begin{aligned} T_Z \{ e^{imt} e^{(i\nu-1)s} \mathcal{D}_{j_3 k}^j(\varphi, \theta, \psi) \} = \\ = \pm (-1)^{j+k-j_3} e^{imt} e^{(-i\nu^*-1)s} \mathcal{D}_{-j_3 k}^j(\varphi, \theta, \psi) \end{aligned} \quad (45)$$

where the $(-)$ sign refers to half-integer spin. Thus unless ν is imaginary (as it is in the Dirac representation), we must double the representation space once more. Notice that it might well happen that the vector in Eq. (45) belongs to a representation equivalent to the one we start from. But since our emphasis is on explicit representations we do not want to invoke unitary equivalence.

DISCUSSION

We have given the positive mass representations of the Poincaré group P in an explicit form which depends on two parameters k and ν . Yet, of course, two representations characterized by the same mass m and spin j but by different k and ν values are unitarily equivalent. Since we hope to give a physical significance to these explicit representations, we have to interpret the parameters k and ν . We shall make an attempt at this in a forthcoming paper.

The connection with the customary spin 0 and Dirac representations is obtained as follows. For $j = 0$, $k = 0$ follows, and furthermore from Eq. (23) the wave-function at rest is

$$|\vec{0}, 0\rangle = \text{Const} e^{imt} e^{(i\nu - 1)s} \quad (46)$$

If we therefore choose $i\nu = 1$ the boost to an arbitrary momentum just gives

$$|\vec{p}, 0\rangle = \text{Const} e^{ip_\mu x^\mu} \quad (47)$$

Thus for $k = 0$, $i\nu = 1$ the transformation in the spin space is invisible. Also the scalar product (35) reduces to

$$(f_1, f_2) = \frac{4\pi}{m^2} \int \frac{d^3p}{p_0} \hat{f}_1^* \hat{f}_2 \quad (48)$$

and we are left with the usual treatment of the spin 0 case. In the spin $\frac{1}{2}$ case, $j = \frac{1}{2}$ and thus $k = \pm\frac{1}{2}$. The wave-function at rest is

$$|\vec{0}, \pm\frac{1}{2}\rangle = \text{Const} e^{i\omega t} e^{i(\vec{0}-\vec{v})\cdot\vec{x}} \mathcal{D}_{\pm\frac{1}{2}}^{\frac{1}{2}}(\varphi, \theta, \psi) \quad (49)$$

Now

$$\mathcal{D}_{\pm\frac{1}{2}}^{\frac{1}{2}}(\varphi, \theta, \psi) = e^{\pm\frac{i}{2}\varphi} P_{\pm\frac{1}{2}}^{\frac{1}{2}}(\cos\theta) e^{ik\psi}$$

and

$$P_{\pm\frac{1}{2}}^{\frac{1}{2}}(\cos\theta) = \begin{cases} \cos\frac{\theta}{2} & \text{or} \\ \pm \sin\frac{\theta}{2} \end{cases}$$

If we now choose $i\omega = \frac{3}{2}$ the transformation (15) gives

$$|(0,0, mSh\epsilon), \pm\frac{1}{2}\rangle =$$

$$e^{im(tSh\epsilon - x^j Sh\epsilon)} \frac{1}{2} e^{\mp\frac{\epsilon}{2}} \mathcal{D}_{\pm\frac{1}{2}}^{\frac{1}{2}}(\varphi, \theta, \psi)$$

(50)

if $k = \frac{1}{2}$. The transformation is again absent in spin space. The scalar product is as before given by Eq. (35). By working out the effect of the infinitesimal operators in the chosen basis one finds

$$M_{ij} = x_i \frac{\partial}{\partial x^j} - x_j \frac{\partial}{\partial x^i} - i \epsilon_{ijk} \frac{1}{2} \sigma_k$$

$$M_{0j} = x_0 \frac{\partial}{\partial x^j} - x_j \frac{\partial}{\partial x^0} - k \sigma_j$$

(51)

where σ_i are the Pauli spin $\frac{1}{2}$ matrices. From here to the spinor calculus the step is not long.

It has been customary for some time to regard the appearance of spinors in the representations of the Poincaré group as an effect of the homogeneous Lorentz group $\mathcal{L}$ in some auxiliary capacity⁹⁾. While it is possible to take this point of view, it is apparent from our analysis that this is not necessary at all. Starting from the realization (17) of the Lie algebra (which has nothing to do with $\mathcal{L}$), it just turns out that for certain values of the free parameters k and ν it is sufficient to have a finite basis $\mathcal{D}_{j,k}^j$ (with j and k fixed) in the spin space. We have found this situation above for $j = 0$ with $i\nu = 1$, $k = 0$ and, for $j = \frac{1}{2}$ with $i\nu = \frac{3}{2}$, $k = \pm\frac{1}{2}$. In general, we would have found a similar behaviour for arbitrary spin j with $k = \pm j$ and $i\nu = j + 1$.

Our results also show that one can very well choose the parameters k and ν arbitrary (however, $|k| \leq j$) without having to introduce infinite spin multiplets. The fact that the wave function (24) in general has an expansion of the form

$$e^{i(p_0 t - p^3 x^3)} \sum_{l=0}^{\infty} a_l^j \mathcal{D}_{j,k}^l (\varphi \oplus \psi) \quad (52)$$

of course does not mean that all spins are there, since it is still an eigenfunction of $W_\mu W^\mu$ with the eigenvalue $m^2 j(j+1)$. On the other hand, the big Hilbert space $\mathcal{H}_B$ is large enough to carry the reducible representation

$$\sum_{2j=0}^{\infty} \oplus \mathcal{U}(m, j)$$

of P .

Therefore, we get very easily an explicit realization of the group $P \times \text{SL}(2, \mathbb{C})$ considered by Fronsdal, Rühl and others¹⁰⁾. This group is obtained by requiring not only $M_{\mu\nu}$ to be generators but

also $S_{\mu\nu}$ separately. Of course, we now no longer impose the condition $W_{\mu} W^{\mu} = m^2 j(j+1)$. If we insist on unitary representation $M_{\mu\nu}$ and $S_{\mu\nu}$ must be separately anti-hermitian which is possible only if ν is real or $0 < i\nu < 1$ as is discussed in connection with Eqs. (25) and (36).

APPENDIX

The functions $\mathcal{D}_{mn}^j(\varphi, \theta, \psi)$ are defined for

$$\begin{aligned} 2l &= 0, 1, 2, \dots & 0 \leq \varphi + \psi \leq 4\pi \\ -l \leq m, n \leq l & & -2\pi \leq \varphi - \psi \leq 2\pi \end{aligned}$$

and have the following properties

$$\mathcal{D}_{mn}^l(\varphi, \theta, \psi) = e^{im\varphi} P_{mn}^l(\cos\theta) e^{in\psi}$$

$$P_{mn}^l(z) = \frac{(-1)^{l-n}}{2^l(l-n)!} \sqrt{\frac{(l-n)!(l+m)!}{(l+n)!(l-m)!}} \times$$

$$(1-z)^{-\frac{m-n}{2}} (1+z)^{-\frac{m+n}{2}} \frac{d^{l-m}}{dz^{l-m}} [(1-z)^{l-n} (1+z)^{l+n}]$$

$$P_{mn}^l(z) = (-1)^{m-n} P_{nm}^l(z)$$

$$P_{-m-n}^l(z) = (-1)^{m-n} P_{mn}^l(z)$$

$$P_{00}^0(z) = 1 \quad P_{mn}^{\frac{1}{2}} = \begin{array}{c|cc} m & n = -\frac{1}{2} & \frac{1}{2} \\ \hline -\frac{1}{2} & C \frac{\theta}{2} & S \frac{\theta}{2} \\ \frac{1}{2} & -S \frac{\theta}{2} & C \frac{\theta}{2} \end{array}$$

$$\int_0^\pi \int_0^{4\pi} \int_{-2\pi}^{2\pi} S \theta d\theta d(\varphi+\psi) d(\varphi-\psi) \mathcal{D}_{m_1 n_1}^{*l_1} \mathcal{D}_{m_2 n_2}^{l_2} =$$

$$= \frac{32\pi^2}{2l_1+1} \delta_{l_1 l_2} \delta_{m_1 m_2} \delta_{n_1 n_2}$$

- REFERENCES AND FOOTNOTES -

- 1) - E.P. Wigner - Ann.Math. 40, 149 (1939).
- 2) - See for instance : J.C. Guillot and J.L. Petit - Helv.Phys.Acta 39, 281-324 (1966) ;
J. Nilsson and A. Beskow - Preprint 66-14, Institute of Theoretical Physics Göteborg, where a program similar to ours in spirit but different on essential points is outlined.
- 3) - By the Dirac representation we mean the explicit spin $\frac{1}{2}$ representation of P defined through the Dirac equation. Quantum electrodynamics is the only successful elementary particle theory and makes explicit use of the Dirac equation.
- 4) - By carrier space we mean a topological space on which the wave functions are defined.
- 5) - A. Kihlberg - Arkiv.Fysik 28, 121 (1964).
- 6) - S.S. Schweber - "An Introduction to Relativistic Quantum Field Theory", Evanston, Ill., Row, Peterson (1961).
- 7) - K. Iwasawa - Ann.Math. 50, 507 (1949). See also : A. Kihlberg, Arkiv.Fysik 30, 121 (1965).
- 8) - Strictly speaking, Eq. (27) follows only if $f_1(y)$ and $f_2(y)$ are arbitrary functions on the carrier space. This is not so since the dependence on φ and θ is restricted as soon as we consider a given spin. We shall not try to analyze the possibilities for K beyond the restriction (27).
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