KINEMATICAL SINGULARITIES OF THE TRANSVERSITY AMPLITUDES

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ABSTRACT

All different mass cases for the two-body processes have been classified and kinematical singularities of the transversity amplitudes have been found, except the zero-mass cases. It is found that the singularities at the thresholds and pseudo-thresholds are factorizable while at the boundary of the physical region they are rather complicated, in contrast with the helicity amplitudes. Therefore there are no constraints for transversity at the thresholds and pseudo-thresholds, while there are constraints at the physical region boundary (i.e., at the forward and backward directions).
1. Introductio

Recently much attention has been paid to kinematical singularities of scattering amplitudes (cf. Williams, Hepp, Hara, Wang) and to the related problem of constraint equations (for a recent review cf. Bertocchi). The problem of finding all kinematical singularities and zeros of helicity amplitudes has been solved by Cohen-Tannoudji, Morad and Navoret [further quoted as CTN]. They also studied the transversity amplitudes in the general-mass case.

In this note we discuss the kinematical singularities of transversity amplitudes in the particular mass cases which are rather important in practical applications. To derive the behaviour of the amplitudes at singular points we find first the singularities of the crossing angles and then apply the crossing relations (cf. Kotani, 1, 2). We use throughout the conventions of CTN. In particular, their crossing relations differ from those derived by Trueman and Wick as their $y$-axis (perpendicular to the scattering plane) is reversed after the crossing and moreover the states of particles $2$ and $4$ in a two-body reaction are taken without the factors $(-1)^{S_2-S_4}$ and $(-1)^{S_4-S_2}$, respectively. Because of that, the transversities are now reversed after the crossing. This, however, does not influence either the positions or the nature of the kinematical singularities as the phase dependent on the Mandelstam variables $s$ and $t$ remains unchanged.

To be sure that all particular cases have been considered, we have classified them in Appendix A. We have found that for two-body processes there are altogether 14 cases which are characterized by different shapes of the border of physical regions on the Mandelstam plane, by the position of thresholds and pseudo-thresholds, and which can be recognized from the relations between the particle masses. Among them, we discern four groups with the following properties:
i) all particles with non-zero masses, no special equalities between the masses (as, for example, $m_1 + m_3 = m_2 - m_4$) — 5 cases,

ii) some particles with zero masses, no special equalities — 10 cases,

iii) massive particles, with special equalities — 5 cases,

iv) some particles with zero masses, special equalities to be satisfied — 2 cases.

Some cases can be realized in different ways and the sum of the above-listed individual cases does not equal 14.

We think that all these cases should be taken into account in any complete study of analytical properties of the amplitudes. We do not consider here groups (ii) and (iv) as we feel that the transversity amplitudes are not well suited to the study of massless particles.

In Appendix B we discuss the analytical properties of the crossing angles. Section 2 contains the discussion of the kinematical singularities derived from the crossing relations and in Section 3 some examples are given.

2. DERIVATION OF THE KINEMATICAL SINGULARITIES
FROM THE CROSSING RELATIONS

The crossing relations for the transversity amplitudes are

$$T_{\bar{s}_3 \bar{c}_4, c_1 \bar{c}_2} = (-1)^{s_1 + s_2 + s_3 + s_4} \epsilon \epsilon \bigl(\bar{s}_3 - \bar{c}_3\bigr) \epsilon^{-i \bigl(c_1 \bar{c}_1 - c_2 \bar{c}_2 + c_3 \bar{c}_3 - c_4 \bar{c}_4\bigr)} T_{\bar{s}_3 \bar{c}_4, c_1 \bar{c}_2} \text{crossed} \quad (2.1)$$

where $\sigma = 1$ if particles "1" and "4" are fermions and $\sigma = 0$ in all other cases [cf. CTMN\textsuperscript{6}, Kotrasoki\textsuperscript{7,8})].

Space reflection invariance requires

$$T_{\bar{s}_3 \bar{c}_4, c_1 \bar{c}_2} = 0 \quad \text{if} \quad \eta (-1)^{s_1 + s_2 - s_3 - s_4} = -1 \quad (2.2)$$

where $\eta$ is the product of intrinsic parities.
The transversity amplitudes can be expressed as linear combinations of helicity amplitudes with numerical coefficients

$$T_{\tau_3 \tau_4, \tau_1 \tau_2} = \sum_{\lambda} \mu(1)^{\lambda_1 \tau_1} \mu(2)^{\lambda_2 \tau_2} \mu^{*}(3)^{\lambda_3 \tau_3} \mu^{*}(4)^{\lambda_4 \tau_4} M_{\lambda_3 \lambda_4, \lambda_1 \lambda_2} \quad \ldots \quad (2.3)$$

with

$$\mu(\lambda, \tau) = \bigotimes \left( \frac{\tau}{2}, \frac{\tau}{2}, -\frac{\sqrt{\tau}}{2} \right) \lambda \tau \quad \ldots \quad (2.4)$$

The properties of matrices (2.4) have been discussed by Kotanski.$^7$.

We know now that the t-channel helicity amplitudes may be singular only at the border of the physical regions and on the t-channel thresholds and pseudo-thresholds. The same can be said about transversity amplitudes because of Eq. (2.3). This enables us to derive the behaviour of the transversity amplitudes at the s-channel thresholds and pseudo-thresholds (provided they do not coincide with the physical borders) by using the crossing relation (2.1) and the well-known formulae for the crossing angles $\chi$. In fact, all those singularities of $T^S$ have to be simultaneously singularities of the factor $e^{-i(\tau_1 X_1 - \tau_2 X_2 + \tau_3 X_3 - \tau_4 X_4)}$. We discuss now the singularities in particular cases, according to the classification given in Appendix B, but omitting those in which one of the masses is necessarily zero. To make the formulae compact, we discuss the singularities only on one Riemann sheet; namely, starting from the physical region where there are no singularities we proceed towards decreasing s passing above the singularities (i.e., turning around them in the positive direction). As the singularities are branch points or poles it is easy to find their behaviour when another continuation path is chosen. This is the case when one calculates the cross-section; the complex conjugate of the amplitude should be continued then along a different way and, as can be seen from the crossing relation, the threshold and pseudo-threshold factors (2.5) appear with different powers. Therefore in the cross-section there are no threshold or pseudo-threshold kinematical singularities [cf. Lin$^{14}$, Jackson and Hito$^{12}$].
Usually we give the explicit form of the singularity, but sometimes when the singularity is more complicated (for example non-factorizable) we show only how the amplitude changes after turning around the singularity in the positive direction. In particular, the behaviour at the boundary of the physical region was derived from the known behaviour of the helicity amplitudes. All our results have been checked for consistency with corresponding formulae for helicity (cf. CTMN).

1) Case 1 (III), general masses

This case was discussed in detail by CTMN. It is repeated here for completeness. From Eq. (2.4) and Appendix B we get

\[ T_{\epsilon_3 \epsilon_4, \epsilon_1 \epsilon_2} \sim \mathcal{F}_{12}^{\epsilon_1 + \epsilon_2} \mathcal{F}_{34}^{\epsilon_3 + \epsilon_4} \mathcal{Y}_{12}^{\epsilon_1 - \epsilon_2} \mathcal{Y}_{34}^{\epsilon_3 - \epsilon_4} \]  

(2.5)

where we adopt the notations

\[ \mathcal{F}_{ik} = \sqrt{s - (m_i + m_k)^2}, \]  

(2.6)

\[ \mathcal{Y}_{ik} = \sqrt{s - (m_i - m_k)^2}, \]  

(2.7)

\[ \epsilon = \text{sgn} \ N(s, t) \]  

(2.8)

and where \( N(s, t) \) is the numerator of the expression (B.1) for \( \cos \tilde{\theta}_5 \)

\[ N(s, t) = s(t - u) - (m_1^2 - m_2^2)(m_3^2 - m_4^2) \]  

(2.9)

and further

\[ \epsilon_{ik} = \text{sgn} \ (m_i - m_k). \]  

(2.10)

Equation (2.5) shows us what constraints should satisfy the scattering amplitudes at the thresholds and pseudo-thresholds.

The behaviour at the boundary of the physical region is a little more complicated. When turning around the line \( \tilde{\theta} = 0 \) (cf. Eq. (A.1)) in the positive directions the amplitude changes as

\[ T_{\epsilon_3 \epsilon_4, \epsilon_1 \epsilon_2} \rightarrow (-1)^{\epsilon_3} (-1)^{\epsilon_1-\epsilon_2} + \epsilon (\epsilon_3 - \epsilon_4) \frac{T_{-\epsilon_3 \epsilon_4, -\epsilon_1 \epsilon_2}}{T_{-\epsilon_3 \epsilon_4, -\epsilon_1 \epsilon_2}}. \]  

(2.11)
Here $\Sigma_0$ is the sum of the particle spins and $\epsilon$ is defined in Eq. (2.8). The exact behaviour at the boundary is rather complicated. It follows from the well-known behaviour of the helicity amplitudes

$$M_{\lambda_1 \lambda_2 \lambda_3 \lambda_4, \lambda_1 \lambda_2} \sim \left( \sin \frac{\beta_e}{2} \right)^{1+\lambda_2-\lambda_3+\lambda_4} \left( \cos \frac{\beta_e}{2} \right)^{1+\lambda_1-\lambda_2+\lambda_3-\lambda_4}$$

(2.12)

The transversity amplitudes are in this case not singular at $s = 0$.

ii) Case 2 (VII) or (IV), $m_1 - m_2 = m_3 - m_4$ or $m_1 + m_2 = m_3 + m_4$.

Now all the singularities are as above except at the coinciding pseudo-thresholds where

$$T_{\epsilon_2 \epsilon_4, \epsilon_1 \epsilon_2} \sim \gamma \epsilon \epsilon_{12} (\tau_1 - \tau_2) + \epsilon \epsilon_{34} (\tau_3 - \tau_4)$$

(2.13)

or at the coinciding thresholds where

$$T_{\epsilon_1 \epsilon_4, \epsilon_1 \epsilon_2} \sim \gamma \epsilon (\tau_1 + \tau_2 + \tau_3 + \tau_4)$$

(2.14)

iii) Case 4 (VI), $m_1 + m_2 = m_3 - m_4$.

Here a threshold coincides with a pseudo-threshold. There

$$T_{\epsilon_2 \epsilon_4, \epsilon_1 \epsilon_2} \sim \gamma \epsilon \epsilon_{12} (\tau_1 + \tau_2) + \epsilon \epsilon_{34} (\tau_3 - \tau_4)$$

(2.15)

The results (2.13) to (2.15) can also be obtained directly from Eq. (2.5) by putting $\psi_{12} = \psi_{34} = \psi$ or $\psi_{12} = \psi_{34} = \phi$ (case 2) or $\psi_{12} = \psi_{34} = \phi$ (case 4).

iv) Case 10 (VIII), $m_1 = m_2$ or 10' (IX), $m_1 = m_2 = m_3$.

Now $\psi_{12} = 0$ coincides with $s = 0$. At this point

$$T_{\epsilon_2 \epsilon_4, \epsilon_1 \epsilon_2} \sim (\sqrt{s}) \epsilon \epsilon_{34} (\tau_1 - \tau_2) = \gamma \psi_{12}$$

(2.16)

This cannot be obtained directly from Eq. (2.5) as now the numerators of both $\cos \chi_1$ and $\sin \chi_1$ vanish (as similarly do $\cos \chi_2$ and $\sin \chi_2$) and this changes the whole picture.
v) Case 11 (OIV), \(2m_4 = 2m_2 = m_3 + m_4\)

Here at \(s = 0\) the behaviour is as in Eq. (2.16) while at \(\rho = 0\) it is as in Eq. (2.14).

vi) Case 13 and 13' (OVI), \(2m_4 = 2m_2 = m_3 - m_4\)

Now the amplitude behaves at \(s = 0\) as in Eq. (2.16) and at \(\psi_{34} = 0\) as in Eq. (2.13).

vii) Case 1, channel s (VII), \(m_4 = m_4, m_2 = m_2\)

Both pseudo-thresholds coincide and also the two thresholds coincide.

\[
T_{e_3 e_4, e_1 e_2} \sim \psi^{\epsilon (e_1 + e_2 + e_3 + e_4)} \psi^{\epsilon (e_1 - e_2 + e_3 - e_4)}
\]  

(2.17)

viii) Case 1, channel t (OIII), \(m_4 = m_2, m_3 = m_4\)

The two pseudo-thresholds coincide at \(s = 0\). After turning round this point the amplitude changes as

\[
T_{e_3 e_4, e_1 e_2} \rightarrow (-1)^{\frac{s}{3}} T_{-e_3 - e_4, -e_1 - e_2}
\]  

(2.18)

As the helicity amplitudes can have in this case at most the singularity of the type \(\sqrt{s}\) (cf. CTMN), the same can be said about the transversity amplitudes. Therefore, defining

\[
T^\pm_{e_3 e_4, e_1 e_2} = T_{e_3 e_4, e_1 e_2} \pm (-1)^{\frac{s}{3}} T_{-e_3 - e_4, -e_1 - e_2}
\]  

(2.19)

[cf. Cohen-Tannoudji et al.] we get at \(s = 0\)

\[
T^+_{e_3 e_4, e_1 e_2} \quad \text{not singular,}
\]  

(2.20)

\[
T^-_{e_3 e_4, e_1 e_2} \sim \sqrt{s}
\]  

(2.21)

Other singularities are as in Case 1, but now \(N(s,t) = t - u\).
ix) Case III (00V), \( m_1 = m_2 = m_3 = m_4 \).

The pseudo-thresholds coincide at \( s = 0 \) (as in Ia) and besides
the thresholds coincide (as in Ib). Then at \( \varphi = 0 \)

\[
T_{\tau_1 \tau_2} \to (-1)^S e^{i \tau_1 \tau_2 \tau_3} - \varphi e^{i (\tau_1 + \tau_2 + \tau_3 + \tau_4)}
\]  

[cf. Eq. (2.18) to (2.20)] and at \( \varphi = 0 \)

\[
T_{\tau_1 \tau_2} \sim \varphi e^{i (\tau_1 + \tau_2 + \tau_3 + \tau_4)}
\]

(2.23)

In this case \( \varepsilon = \text{sgn } N(s,t) \) where \( N(s,t) = t - u \).

3. Examples

3.1 The pion-nucleon scattering

In this reaction we have only two scattering amplitudes different
from zero: \( T_{++} \) and \( T_{--} \). Subscripts denote the nucleon transversities:
+ means \( \frac{1}{2} \) and - means \( -\frac{1}{2} \). The pseudo-thresholds and the thresholds
coincide and this corresponds to the Case I (channel s). Denoting

\[
\varepsilon = \text{sgn } \left[ (t-u) + (\lambda^2 - m^2)^2 \right]
\]

(3.1)

we have at the thresholds and pseudo-thresholds [cf. Eq. (2.23)]

\[
T_{++} \sim \varphi e^\varepsilon, \quad T_{--} \sim \varphi e^{-\varepsilon}
\]

(3.2)

(3.3)

Now, at the physical region border, we get, from Eq. (2.11),

\[
T_{++} \to -T_{--}, \quad T_{--} \to -T_{++}
\]

(3.4)

but the exact behaviour of the amplitudes can be found from the well-known
behaviour of the helicity amplitudes and is

\[
T_{++} - T_{--} \sim \sqrt{\Phi}
\]

(3.5)
where $\phi$ has been defined in Eq. (A.1).

### 3.2 The nucleon-nucleon scattering

In this case four masses are equal and that corresponds to the case III. The thresholds coincide and from Eq. (2.23) we see that there is no branch point at the threshold. In fact,

$$T_{++} \sim \frac{1}{s-4m^2}$$  \hspace{1cm} (3.7)

while the other non-zero amplitudes, i.e. $T_{+-}$, $T_{-+}$, and $T_{--}$ are not singular at this point. The pseudo-thresholds coincide at $s = 0$, which is at the same time a part of the physical region border. Their behaviour is then more complicated. Three of the amplitudes are not singular there ($T_{++}, T_{+-}, T_{-+}$) while for the other two

$$T_{++} - T_{--} \sim \sqrt{s}$$  \hspace{1cm} (3.9)

and

$$T_{++} + T_{--} \text{ is not singular.} \hspace{1cm} (3.10)$$

The situation is similar for the boundary of the physical region. The only singular amplitudes are

$$T_{++} - T_{--} \sim \sqrt{s}$$  \hspace{1cm} (3.11)

while

$$T_{++} + T_{--} \text{ is not singular.} \hspace{1cm} (3.12)$$
Besides the singularities, there are still constraints at the forward and backward direction because of the angular momentum conservation. They are

\[ T_{++--} + T_{++-+} + T_{+-+-} - T_{+-+-} = 0 \]  \hspace{1cm} (3.13)

at the forward direction, and

\[ T_{++--} + T_{+-+-} + T_{+-+-} - T_{+-++} = 0 \]  \hspace{1cm} (3.14)

at the backward direction.

We see that in the nucleon-nucleon scattering three transversity amplitudes out of five have no kinematical singularities.

4. CONCLUSION

From our discussion we can conclude that the kinematical singularities of transversity amplitudes are rather simple at the thresholds and pseudo-thresholds; in fact, in most cases they are factorizable and are either simple branching points or poles. This has been also noticed by CTEQ in the general mass case. On the other hand, the singularities on the physical region border (which corresponds to the forward and backward scattering) are rather involved — typically, the amplitude is then a linear combination of several functions with different singularities.

Those properties are complimentary to those of helicity amplitudes which have factorizable singularities at the border of the physical region [cf. for example Gell-Mann et al.\(^1\)] and CTEQ\(^6\)] while on the thresholds and pseudo-thresholds they behave in a complicated way [cf. Harz\(^3\), Wang\(^4\), CTEQ\(^6\)]. In fact, they are not only singular, but also have to satisfy those constraint relations which are also relevant [for example they cancel the threshold and pseudo-threshold singularities in the cross-section — cf. Jackson and Hito\(^12\)]. This is because helicity amplitudes are combinations of transversity amplitudes of which some are singular and some vanish. The former cause the singularities in helicity while the latter diminish the number of independent amplitudes and cause constraints.
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Note

After this work was completed two papers on similar subjects came to our notice: a preprint of A. McLerran from Cambridge about kinematical singularities of helicity and transversity amplitudes and an article by I.A. Sakmar and J.H. Wojtaszek [Phys. Rev. 163, 1748 (1967)], about properties of the boundaries of the physical regions (especially in our Case 1).
CLASSIFICATION OF PARTICULAR MASS CASES

These cases are best classified by the shape of the physical region boundaries on the Mandelstam plane. The boundary line is given by the equation [cf. Kibble\textsuperscript{15}]:

\[ \Phi(s,t) = stu - \alpha s - \beta t - \gamma = 0 \]  \hspace{1cm} (A.1)

where

\[ \alpha = (m_b^2 - m_d^2)(m_a^2 - m_c^2), \]  \hspace{1cm} (A.2)
\[ \beta = (m_a^2 - m_b^2)(m_c^2 - m_d^2), \]  \hspace{1cm} (A.3)
\[ \gamma = (m_a^2 - m_b^2 - m_c^2 + m_d^2)(m_a^2 m_d^2 - m_b^2 m_c^2). \]  \hspace{1cm} (A.4)

Its shape was already discussed by several authors [Mandelstam\textsuperscript{10}, Kibble\textsuperscript{15}, Omnès and Froissart\textsuperscript{16}, Nyborg\textsuperscript{17}] who were mainly interested in the case when all the particles have different masses. Here we present a complete classification, recall some well-known but useful properties and finally display the shapes of the lines in Figs. 1 - 15. Our channels are defined as follows:

\[ s: \quad a + b \rightarrow c + d, \]
\[ t: \quad a + c \rightarrow b + d, \]
\[ u: \quad a + d \rightarrow c + b. \]  \hspace{1cm} (A.5)

Of course, \( s + t + u = m_a^2 + m_b^2 + m_c^2 + m_d^2 \).

If a particle has sufficiently large mass, a decay channel is possible. This condition may also be written in the form

\[ \text{either } m_a + m_b < |m_c - m_d| \text{ or } m_c + m_d < |m_a - m_b| \]  \hspace{1cm} (A.6)

(cf. Kibble, loc. cit.). In other words, a decay is possible if a pseudo-threshold lies higher than a threshold.
Equation (A.1) represents a cubic curve. The classification of such curves was proposed by Newton\(^1\) who divided them into 7 types, 14 classes and 72 cases. A compilation of the work on cubic curves was done by Smogorzhevsky and Stolova\(^2\). The degenerate cubics were classified by Stolova\(^3\) who divided them into 30 cases. Fortunately, Eq. (I.1) can represent only 14 cases, 10 of them belonging to Newton's type I (and called by him the "redundant hyperbolic hyperbolae") and 4 degenerate (cf. Table 1).

Newton classified the cubics according to the roots of their characteristic equations. It appears that these roots are equal to the values of a Mandelstam variable on the thresholds and pseudo-thresholds. For example in the a channel they are

\[
\lambda = (\mu a \pm \mu b)^2, \quad \lambda = (\mu c \pm \mu d)^2. \tag{A.7}
\]

The equalities between the roots or vanishing of some of them are decisive for the classification. To visualize these relations we use in the table a symbolic notation. Symbol (III) denotes four roots (or thresholds and pseudo-thresholds) different, (0IV) means that a pseudo-threshold is at zero and the two thresholds are equal, (VV) represents four roots equal in pairs, and (X) denotes four equal roots. Three sets of such symbols are given for the three channels in each case. The non-degenerate cases are labelled according to Newton, except that we distinguish between 10 and 10' and also between 13 and 13', to make the classification crossing-symmetric.

Further we recall that the lines \(s = 0, t = 0,\) and \(u = 0\) are asymptotes of the cubics which intersect them in at most three points lying on a straight line. For cases 10 through 13' this line is parallel to one of the asymptotes and the curve is symmetric.

The ovals in cases 1, 10, and 10' represent physical decay channels only if condition (A.6) is satisfied.

The threshold and pseudo-threshold lines (A.7) are tangent to the border curves. A set of such tangents (in one of the channels) is marked in Fig. 2. The tangency points lie simultaneously on the lines given by \(\cos \theta = 0\) and \(\cos \chi = 0\) where \(\theta\) is a scattering and \(\chi\) a crossing angle. The curves \(\cos \theta = 0\) and \(\cos \chi = 0\) are either hyperbolae or two intersecting straight lines.

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SINGULARITIES OF THE SCATTERING AND CROSSING ANGLES

For the scattering angle we have the formulae [cf. Kibble\textsuperscript{15}]

\[
\cos \vartheta_s = \frac{4(t-u) + (m_2^2-m_3^2)(m_3^2-m_4^2)}{S_{42} S_{34}}, \tag{B.1}
\]

\[
\sin \vartheta_s = \frac{2 \sqrt{\Phi(t)} \sqrt{5}}{S_{12} S_{34}}, \tag{B.2}
\]

where

\[
S_{ik}^2 = [s - (m_i - m_k)^2][s - (m_i + m_k)^2], \tag{B.3}
\]

\[
\Phi(t) = s(t + s)(m_1^2 - m_4^2)(m_1^2 - m_3^2) - t(m_2^2 - m_4^2)(m_3^2 - m_4^2) - \Delta (m_1^2 m_4^2 - m_2^2 m_3^2), \tag{B.4}
\]

\[
\Delta = m_2^2 + m_3^2 - m_1^2 - m_4^2, \tag{B.5}
\]

while the crossing angles are defined by [cf. Trueman and Wick\textsuperscript{9} and CTMN\textsuperscript{6}]

\[
\cos \chi_1 = -\frac{(s+m_1^2-m_2^2)(t+m_1^2-m_3^2) + 2 m_1^2 \Delta}{S_{14} T_{13}} \tag{B.6}
\]

\[
\sin \chi_1 = \frac{2 m_1 \sqrt{\Phi}}{S_{14} T_{13}}, \tag{B.7}
\]

\[
\cos \chi_2 = \frac{(s+m_2^2-m_1^2)(t+m_2^2-m_4^2) - 2 m_2^2 \Delta}{S_{12} T_{24}} \tag{B.8}
\]

\[
\sin \chi_2 = \frac{2 m_2 \sqrt{\Phi}}{S_{12} T_{24}}, \tag{B.9}
\]

\[
\cos \chi_3 = \frac{(s+m_3^2-m_4^2)(t+m_3^2-m_1^2) - 2 m_3^2 \Delta}{S_{34} T_{13}} \tag{B.10}
\]

\[
\sin \chi_3 = -\frac{2 m_3 \sqrt{\Phi}}{S_{34} T_{13}}, \tag{B.11}
\]
\[ \cos \chi_4 = \frac{(1 + m_4^2 - m_3^2)(1 + m_4^2 - m_2^2) + 2m_4^2 \Delta}{S_{34} T_{24}}, \quad (B.12) \]

\[ \sin \chi_4 = -\frac{2m_4 \sqrt{\Delta}}{S_{34} T_{24}}, \quad (B.13) \]

with

\[ T_{ik}^2 = \left[ t - (m_i - m_k)^2 \right] \left[ t - (m_i + m_k)^2 \right]. \quad (B.14) \]

To determine the singularities of the amplitudes it is not sufficient to know the behaviour of \( \cos \chi \) and \( \sin \chi \) as this gives information about the angle \( \chi \) only up to \( 2\pi \) and, therefore, about \( e^{i\chi} \) only up to a sign. We have studied thus the behaviour of \( \cos \frac{\chi}{2} \) and \( \sin \frac{\chi}{2} \). Below we show either the behaviour of \( e^{i\chi} \) at the singularity or the change of the angle when turning around the singularity in the positive (anticlockwise) direction.

In the following

\[ N(s,t) = s(t - u) + (m_1^2 - m_2^2)(m_3^2 - m_4^2) \quad (B.15) \]

is the numerator of the expression (B.1) for \( \cos \theta \). Equation \( N = 0 \) represents a hyperbola (which may degenerate to two straight lines) intersecting the threshold and pseudo-threshold lines at their common points with the border of the physical region. From Eqs. (B.6) to (B.13) we see that on the thresholds and pseudo-thresholds the sines and cosines have singularities in the denominators and behave like \( \varphi^{-1} \) or \( \varphi^{-1} \); therefore \( e^{i\chi} \) behaves also like \( \varphi^{-1} \) or \( \varphi^{-1} \) unless the singularity is not cancelled by a zero in the numerator. To determine exactly the singularities it is enough to compare the numerators of \( \sin \chi \) and \( \cos \chi \) on the threshold (pseudo-threshold). The factor \( \sqrt{\bar{\tau}} \) appearing in the sines is then purely imaginary. As everywhere in this paper we consider in detail the case of \( \sqrt{\bar{\tau}} \) positive imaginary. This corresponds to taking \( \bar{\tau} \) positive inside the physical region and to passing above the line \( \bar{\tau} = 0 \).

In other words, we give the singularities only on one Riemann sheet. It is easy to find the singularities on the other sheet – some of the powers would be then reversed.
We discuss now the singularities of the $s$-channel scattering angle and of the crossing angles in all the non-zero mass cases.

Case 1, (III), all masses different

i) The scattering angle

<table>
<thead>
<tr>
<th>SINGULARITY</th>
<th>BEHAVIOUR</th>
<th>CONDITION</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi = 0$</td>
<td>$\Theta \rightarrow -\Theta$</td>
<td>$\cos \Theta = -1, (N &lt; 0)$</td>
</tr>
<tr>
<td>thresholds and pseudo-thr. ($s$-channel)</td>
<td>$\Theta \rightarrow \Theta + \pi$</td>
<td>$N &gt; 0$</td>
</tr>
<tr>
<td>$s = 0$</td>
<td>$\Theta \rightarrow -\Theta$</td>
<td>$N &lt; 0$</td>
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<tr>
<td></td>
<td>$\Theta \rightarrow 2\pi - \Theta$</td>
<td>$N &gt; 0$</td>
</tr>
</tbody>
</table>

ii) The crossing angles

They are singular at the border of the physical region, and at the thresholds and pseudo-thresholds. We discuss here only the latter singularities as the former are not useful for our purposes.

\[
\begin{align*}
\epsilon^i \chi_1 & \sim \chi_{12}^i - \epsilon \chi_{12}^i \\
\epsilon^i \chi_2 & \sim \chi_{12}^i + \epsilon \chi_{12}^i \\
\epsilon^i \chi_3 & \sim \chi_{34}^i - \epsilon \chi_{34}^i \\
\epsilon^i \chi_4 & \sim \chi_{34}^i + \epsilon \chi_{34}^i
\end{align*}
\]

(Case 2, VII) or (IV), two thresholds or pseudo-thresholds coincide

i) The scattering angle

As in Case 1, except at the coinciding singularity, where

\[
\Theta \rightarrow \Theta + 2\pi.
\]

ii) The crossing angles

As in Case 1, but either $\phi_{12} = \phi_{34} = \phi$ or $\phi_{12} = \phi_{34} = \phi$. 

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SIS/66
Case IV, a threshold coincides with a pseudo-threshold

i) The scattering angle

At the coinciding singularity as in Eq. (B.20).

ii) The crossing angles

As in Case 1, but either $\phi_{12} = \phi_{34}$ or $\phi_{34} = \phi_{12}$.

Case 10 or 10' (0III), $m_3 = m_2$

i) The scattering angle

As in Case 1 except at $s = 0$ (and at $\phi_{12} = 0$) where

$$\Theta \rightarrow \pi - \Theta.$$  (B.21)

ii) The crossing angles

$$e^{ix_1} \sim \phi_{12} - \epsilon \gamma_{12} - \epsilon_{34},$$  (B.22)
$$e^{ix_2} \sim \phi_{12} + \epsilon \gamma_{12} - \epsilon_{34},$$  (B.23)

while $e^{ix_3}$ and $e^{ix_4}$ as in Case 1.

Case 11 (0IV), $2m_1 = 2m_2 = m_3 + m_2$

i) The scattering angle

At $s = 0$

$$\Theta \rightarrow -\Theta \text{ for } N > 0, \quad \Theta \rightarrow 2\pi - \Theta \text{ for } N < 0.$$  (B.24)

At the coinciding thresholds

$$\Theta \rightarrow \Theta + 2\pi.$$  (B.25)

ii) The crossing angles

$\phi_{12} = 0$ as for Case 10, at $\phi_{12} = \phi_{34}$ as for Case 2.
Case 13 or \(C_{13}'\) (OVI), \(2\theta_1 = 2\theta_2 = |\theta_3 - \theta_4|\) or \(3\theta_1 = 3\theta_2 = 3\theta_3 = \theta_4\)

i) The scattering angle

As in Case 11.

ii) The crossing angles

At \(\theta_{12} = 0\) as in Case 10, at \(\theta_{34} = \theta_{12} = 0\) as in Case 4.

**Case I (channel s) (VII):** \(\theta_1 = \theta_2, \theta_3 = \theta_4\)

i) The scattering angle

At coinciding thresholds or pseudo-thresholds

\[ \theta \to \theta + 2\pi \]  

(B.26)

ii) The crossing angles

\[ e^{i\chi_1} = |\psi - \psi - \epsilon \epsilon_{12}^2 \]

(B.27)

\[ e^{i\chi_2} = |\psi - \psi - \epsilon \epsilon_{12}^2 \]

(B.28)

\[ e^{i\chi_3} = |\psi - \psi - \epsilon \epsilon_{12}^2 \]

(B.29)

\[ e^{i\chi_4} = |\psi - \psi - \epsilon \epsilon_{12}^2 \]

(B.30)

**Case I (channel t) (OCII)**

i) The scattering angle

It is not singular at \(s = 0\) (where the two pseudo-thresholds coincide).

ii) The crossing angles

They cannot be used in this case (as in Case III) to determine the singularities of the amplitudes and therefore are not studied.

**Case III (OIV)**

The scattering angle

At coinciding thresholds

\[ \theta \to \theta + 2\pi \]  

(B.31)
and at the pseudo-thresholds \( s = 0 \) is not singular.

All the singularities which are not mentioned above are as in Case 1.
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20. E.S. Stolo\v{z}a, Trudy Kiev. gidromel. instit. \(\frac{4}{5}\), 173 (1954),
\(\frac{5}{5}\), 191 (1955).
<table>
<thead>
<tr>
<th>Type</th>
<th>Symbol</th>
<th>Conditions on masses</th>
<th>Remark</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(III, III, III)</td>
<td>( n_b = 0, n_c &lt;</td>
<td>n_c - n_d</td>
<td>) \ or ( n_d = 0, n_c &gt; n_b + n_o ) \ or (</td>
</tr>
<tr>
<td>2</td>
<td>(VII, VII, IV)</td>
<td>( n_b = 0, n_a =</td>
<td>n_c - n_d</td>
<td>) \ or ( n_b = 0, n_a = n_c + n_d )</td>
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<tr>
<td>3</td>
<td>(VII, VII, IV)</td>
<td>( n_b = 0, \frac{n_a}{2} = \frac{n_c - n_d}{2} )</td>
<td>OS</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>(IVI, IVI, IVI)</td>
<td>( n_b = 0,</td>
<td>n_c - n_d</td>
<td>&lt; n_a &lt; n_c + n_d ) \ or ( n_a + n_b =</td>
</tr>
<tr>
<td>10</td>
<td>(OIII, III, III)</td>
<td>( n_a = n_b )</td>
<td>N</td>
<td>( \pi N \to \rho N, \pi N \to \pi \Delta )</td>
</tr>
<tr>
<td>10'</td>
<td>(OIII, OIII, OIII)</td>
<td>( n_a = n_b = n_c )</td>
<td>N</td>
<td>( pp \to p \Delta, \omega \to 3\pi )</td>
</tr>
<tr>
<td>11</td>
<td>(OIV, VII, VII)</td>
<td>( n_d = 0, 2n_a = 2n_b &lt; n_c ) \ or ( 2n_a = 2n_b = n_c + n_d )</td>
<td>0</td>
<td>( \omega \to \pi^+ \pi^- )</td>
</tr>
<tr>
<td>13</td>
<td>(OVI, IVI, IVI)</td>
<td>( n_d = 0, n_c &lt; 2n_a = 2n_o ) \ or ( 2n_a = 2n_b =</td>
<td>n_c - n_d</td>
<td>)</td>
</tr>
<tr>
<td>13'</td>
<td>(OVI, OVI, OVI)</td>
<td>( n_a = n_b = n_c, n_d = 0 ) \ or ( 3n_a = 3n_b = 3n_c = n_d )</td>
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<td></td>
</tr>
<tr>
<td>I</td>
<td>(VII, 00II, VV)</td>
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<td>N</td>
<td>( \pi N \to \pi N )</td>
</tr>
<tr>
<td>II</td>
<td>(X, 000I, X)</td>
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<td>0</td>
<td>( \gamma p \to \gamma p )</td>
</tr>
<tr>
<td>III</td>
<td>(00V, 00V, 00V)</td>
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<td>N</td>
<td>( \pi^0 \to 3\gamma )</td>
</tr>
<tr>
<td>IV</td>
<td>(0000, 0000, 0000)</td>
<td>( n_a = n_b = n_c = n_d = 0 )</td>
<td>0</td>
<td>( \gamma \gamma \to \gamma \gamma )</td>
</tr>
</tbody>
</table>

N- normal cases, realized with massive particles, without special equalities
O- cases with a mass equal to zero
S- require special equalities between the masses