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EXTENSION OF THE AXIOMATIC ANALYTICITY DOMAIN :

PION-NUCLEON SCATTERING

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A B S T R A C T

Starting from analyticity in the topological product of the cut s plane and the circle $|t| < 1.83\mu^2$ and using unitarity several times, the domain of validity of fixed t dispersion relations for elastic πN scattering is extended to a larger domain $D^{\pi N}$ which is shown to contain the negative real axis from $t = 0$ to the "inelastic point" $t = -18.0\mu^2$. Thus, as in meson-meson scattering, the only reason which prevents one from continuing further to the left, is our lack of knowledge about possibly occurring singularities above inelastic threshold. Using also a recent result of Bessis and Glaser, the best domain is found to be the intersection of the large Lehmann ellipse for $s = (M+2\mu)^2$ with the large Lehmann ellipse for $s = (M+2\mu)^2 + 8\mu(M+\mu)$. In the elastic region the analyticity domain of the absorptive part extends to the border of the Mandelstam support. Partial wave amplitudes are found to be analytic in two disconnected regions which touch the circle $|s| = M^2 - \mu^2$ from inside and outside in $s = M^2 - \mu^2$.

In a previous paper ¹⁾, for any elastic scattering amplitude fulfilling a one-dimensional dispersion relation in s at fixed negative t , we have found a certain domain of simultaneous analyticity in s and t , namely the topological product of the cut s plane with cuts $s_{\text{threshold}} \leq s < \infty$, $u_{\text{threshold}} \leq u < \infty$ and the circle in the t plane

$$|t| < \max_{s_{\text{thr}} \leq s < \infty} r_{\text{SL}}(s) \equiv R_t \quad (1.1)$$

where $r_{\text{SL}}(s)$ is the right-hand extremity of the small Lehmann ellipse in the t plane ²⁾. In numerous cases ($\pi\pi$, πK , KK , $K\bar{K}$) ^{*)} one finds $R_t = 4$ (in pion mass units). For pion-nucleon scattering, which is anomalous in this sense, we get $R_t \cong 1.83$. A similar anomaly appears in $\pi\Lambda$ and $\pi\Sigma$ scattering ^{*)}, where $R_t^{\pi\Lambda} \cong R_t^{\pi N} \cong 1.83$, $R_t^{\pi\Sigma} \cong 0.8$.

It has already been mentioned that a domain of this type can be extended considerably by using unitarity and analyticity of the absorptive part in large Lehmann ellipses. In this paper we shall perform this extension for pion-nucleon scattering. Especially, we want to show that the domain $\mathcal{D}^{\pi N}$ of validity of fixed t dispersion relations extends to the maximum of the left extremities of all large Lehmann ellipses for s above inelastic threshold, i.e., to

*)

If unphysical regions occur, one has to be a little careful in applying the method of Ref. ¹⁾. At first, the absorptive part and its derivatives are positive only in the physical regions. So, if also a part of the left-hand cut is unphysical (in u), this part must be included in ϕ_β . Let us call this new function ϕ'_β . Then $T'_\beta = T - \phi'_\beta$ has again all desired properties. Secondly, in the unphysical region the left-hand extremity of the large Lehmann ellipse is closer to $t = 0$ than the right extremity. Thus the distance of the left extremity from $t = 0$ is to be considered to find the radius of the circle of analyticity around $t = 0$ for the function ϕ'_β . Therefore, if dispersion relations have been proved in the usual manner for $-t_M \leq t \leq 0$, it is found that $R_t = \min[t_M, \max r_{\text{SL}}(s)]$. But for all realistic cases $t_M > \max r_{\text{SL}}(s)$, so that the result of Ref. ¹⁾ is unchanged.

$$\max_{s \geq (M+2)^2} \ell_L(s) = - \min_{s \geq (M+2)^2} \left[4k^2 + \frac{32(2M+1)}{s - (M-2)^2} \right]_{(1.2)}$$

In the following we shall refer to this point as to the "inelastic point". The reason why our main interest concerns this property of $\mathcal{D}^{\pi\pi}$ is that the corresponding domains for $\pi\pi$ ³⁾ and KK ⁴⁾ scattering extend to the inelastic point. Thus, if the same is also true for πN scattering ^{*)}, we may conclude that this seems to be a general feature. This means that in all these cases the impossibility of an analytic continuation further to the left is only due to our lack of knowledge about the possibly occurring singularities above many particle threshold.

Very recently, when the proof that $\mathcal{D}^{\pi\pi}$ indeed extends to the inelastic point was finished, Bessis and Glaser ⁶⁾ succeeded to show that for any fixed s in $s_0 < s < (M+1)^2$ the amplitude for πN scattering is analytic in t in some complex neighbourhood of $0 < t < 4$. This new information simplifies considerably our proof. While without this result at least three successive steps of enlargement are necessary to reach the inelastic point, now the first two steps are sufficient. Nevertheless we shall present the proofs both without (Section 3) and with making use of the Bessis-Glaser result (Section 4) for the following reason. The method of enlargement we were forced to use in the first proof might help in less favourable cases than in πN scattering. Furthermore it gives an explicit confirmation of what one expects. Within a wide range the value of the radius R_t of the circle around $t = 0$ one starts with should not affect the possibility or impossibility of the analytic continuation to the inelastic point, as this is a problem of low energies, while the radius R_t is mainly connected with analytic properties at high energies.

^{*)} There exists already a paper ⁵⁾ in which a continuation of the range of validity of dispersion relations to the real interval $\ell_L[(M+2)^2] = -18.8 < t < \ell_L[(M+1)^2] = -12.4$ is given. But this does not cover a complex neighbourhood of this real segment and furthermore the proof turns out to be incomplete in some respects.

In Section 2 we shall treat some problems connected with the spin of the nucleon and in Section 5 a discussion of the results (especially the consequences for the analyticity domain for partial waves) is given.

4.

2. - POSITIVITY PROPERTIES FOR SPIN 0 SPIN $\frac{1}{2}$ SCATTERING AMPLITUDES

In both methods ^{1),3)} leading to a finite complex domain of validity of dispersion relations the complications connected with the spins of the particles were disregarded. Now, as we want to treat the case of πN scattering explicitly here, we have to make some remarks on this point.

In order to apply the Martin positivity argument one has to find a suitable linear combination F of amplitudes which has the positivity property (unitarity restriction) :

$$\left| \left(\frac{\partial}{\partial t} \right)^n \mathcal{F}_m \overline{F}(s,u,t) \right| \leq \left[\left(\frac{\partial}{\partial t} \right)^n \mathcal{F}_m \overline{F}(s,u,t) \right]_{t=0} \quad (2.1)$$

for all n in some fixed interval $-\tilde{t}(s) \leq t \leq 0$, $\tilde{t}(s) > 0$, not only on the right-hand cut, but also on the left-hand cut (u channel), that is with s and u interchanged in (2.1). Actually, this means that F must have

- a) - the positivity property (2.1) ;
- b) - good crossing properties ^{*)} of the type $F(s,u,t) = \pm F^*(u,s,t)$.

For elastic spin 0-particle-spin $\frac{1}{2}$ -particle scattering the amplitude has the general form

$$-A(s,u,t) + i \gamma \frac{q_1 + q_2}{2} B(s,u,t)$$

*) A common minus-sign of the absorptive part on the left-hand cut does not disturb the arguments of ^{1),3)}.

where A and B have opposite crossing behaviour. Especially, for πN scattering

$$\begin{aligned} A^{(\pm)}(u, s, t) &= \mp A^{(\pm)*}(s, u, t) \\ B^{(\pm)}(u, s, t) &= \pm B^{(\pm)*}(s, u, t) \end{aligned} \quad (2.2)$$

where $(\pm)$ denotes the isospin combinations $\frac{1}{2}(T_{\pi^- p} + T_{\pi^+ p})$. Therefore, to fulfil condition b), F must have the form

$$F(s, u, t) = A(s, u, t) + h(s, u, t) B(s, u, t) \quad (2.3)$$

with $h(x, t)$ being odd in the first variable

$$h(-x, t) = -h(x, t) \quad (2.4)$$

A possible choice ⁷⁾ for h is

$$h(s, u, t) = \frac{s-u}{4M} \quad (2.5)$$

this choice is such that for $t = 0$, F reduces to the no-spin flip amplitude. Then $F(s, u, t)$ can be written

$$F(s, u, t) = \frac{\pi}{M} \left\{ (\sqrt{k^2 + M^2} + M)(f_1 + z f_2) + (\sqrt{k^2 + M^2} - M)(z f_1 + f_2) + 2\sqrt{k^2 + 1}(f_1 + f_2) \right\} \quad (2.6)$$

Here $z = \cos \theta_{cm}$ and the three terms have the following partial wave decompositions*:

$$f_1 + z f_2 = \sum_{\ell=0}^{\infty} \left[(\ell+1) f_{\ell+}^{(+)} + \ell f_{\ell-}^{(+)} \right] P_{\ell}(z) \quad (2.7a)$$

$$zf_1 + f_2 = \sum_{\ell=0}^{\infty} (\ell+1) f_{\ell+}^{(s)} P_{\ell+1}(z) + \sum_{\ell=1}^{\infty} \ell f_{\ell-}^{(s)} P_{\ell-1}(z) \quad (2.7b)$$

$$f_1 + f_2 = \sum_{\ell=0}^{\infty} [f_{\ell+}^{(s)} + f_{(\ell+1)-}^{(s)}] [P'_{\ell+1}(z) - P'_{\ell}(z)] \quad (2.7c)$$

Especially, $f_1 + zf_2$ is the no-spin-flip, and f_2 is the spin-flip amplitude. The first two terms in (2.6) have the desired property (2.1) in $-4K^2 \leq t \leq 0$, as $a_{\ell\pm}(s) = \text{Im} f_{\ell\pm}(s)$ are positive. The third term has also the property (2.1), if it can be shown that in $0 \leq z \leq +1$ (i.e., $\tilde{t} = 2K^2$) :

$$\left| \left(\frac{\partial}{\partial z} \right)^n [P'_{\ell+1}(z) - P'_{\ell}(z)] \right| \leq \left(\frac{\partial}{\partial z} \right)^n [P'_{\ell+1}(z) - P'_{\ell}(z)]_{z=+1} \quad (2.8)$$

Now there is a relation between the $(n+1)^{\text{th}}$ and the n^{th} derivatives of the Legendre functions :

$$\begin{aligned} P_{\ell+1}^{(n+1)} - P_{\ell}^{(n+1)} &= (\ell+1) [P_{\ell}^{(n)} - P_{\ell-1}^{(n)}] + (\ell-3) [P_{\ell-2}^{(n)} - P_{\ell-3}^{(n)}] + \dots + \\ &+ 2 P_{\ell-1}^{(n)} + 2 P_{\ell-3}^{(n)} + \dots \end{aligned} \quad (2.9)$$

Therefore, (2.8) is fulfilled if

$$|P'_{\ell+1}(z) - P'_{\ell}(z)| \leq P'_{\ell+1}(1) - P'_{\ell}(1) \quad (2.10)$$

in $0 \leq z \leq +1$. But (2.10) is in fact true ⁷⁾. Thus (2.3) with (2.5) satisfies all conditions sufficient to conclude that F obeys a dispersion relation in $|t| < 1.83$.

In order to enlarge this domain we have to use the information that $A_F(s,t)$, the absorptive part of $F(s,u,t)$, is analytic in the large Lehmann ellipse. Therefore, from (2.6), (2.7) the function defined by

$$\tilde{A}_F(s,t) = (1+z) A_F(s,t) \quad , \quad z = 1 + \frac{t}{2K^2} \quad (2.11)$$

has the following properties :

- i) - $\tilde{A}_F(s,t)$ is analytic in $|t| < 1.83$ for any fixed real $s > (M+1)^2$;
- ii) - $\tilde{A}_F(s,t)$ is analytic in the large Lehmann ellipse $E_L(s)$ for any fixed real $s > (M+1)^2$;
- iii) - $\tilde{A}_F(s,t)$ has a Legendre expansion $\sum_{\ell=0}^{\infty} (2\ell+1) c_{\ell}(s) P_{\ell}(z)$ with positive coefficients : $c_{\ell}(s) \geq 0$.

From (ii), (iii) the Legendre expansion of $\tilde{A}_F(s,t)$ must have a singularity at the right extremity of its ellipse of convergence and from (i) this ellipse must extend to $t = +1.83$. This means that $\tilde{A}_F(s,t)$ is analytic for fixed $s > (M+1)^2$ in

$$E(1.83, s) \equiv \{0, -4K^2 / 1.83\} \quad (2.12)$$

From this we conclude that $A_F(s,t) = (1+z)^{-1} \tilde{A}_F(s,t)$ is also analytic in $E(1.83, s)$, except for a pole which might occur at $z = -1$. But, from Lehmann ellipse we know that there is no pole at $z = -1$.

The coefficients $c_{\ell}^{(I)}(s)$ of the Legendre expansion of $\tilde{A}_F^{(I)}(s,t)$ (I denoting definite isospin $\frac{1}{2}$ or $\frac{3}{2}$) are sums of the positive quantities $a_{\ell+}^{(I)}(s)$ and $a_{\ell-}^{(I)}(s)$. Therefore, from the fact that $\tilde{A}_F^{(I)}(s,t)$ is analytic in (2.12) it follows that $a_{\ell+}^{(I)}$ and $a_{\ell-}^{(I)}$ are separately bounded by

$$\lim_{\ell \rightarrow \infty} [a_{\ell \pm}^{(I)}(s)]^{1/\ell} \leq \frac{1}{\left(1 + \frac{1.83}{2K^2}\right) + \sqrt{\left(1 + \frac{1.83}{2K^2}\right)^2 - 1}} \quad (2.13)$$

8.

This yields that the absorptive parts of the no-spin-flip amplitude

$$f_1^{(I)} + z f_2^{(I)} = \sum_{l=0}^{\infty} \left[(l+1) f_{l+}^{(I)}(s) + l f_{l-}^{(I)}(s) \right] P_l(z) \quad (2.7a')$$

and the spin-flip amplitude

$$f_2^{(I)} = \sum_{l=0}^{\infty} \left[f_{l-}^{(I)}(s) - f_{l+}^{(I)}(s) \right] P'_l(z) \quad (2.7d)$$

are separately analytic in (2.12) and the amplitudes themselves obey dispersion relations in $|t| < R_t = 1.83$.

3. - EXTENSION OF THE ANALYTICITY DOMAIN ISTEP 1

We know from the discussion of the last Section that the absorptive parts of any independent πN scattering amplitudes are analytic in the large Lehmann ellipse $E_L(s)$ as well as in the ellipse $E(1.83, s)$. We take the best information. Then the analyticity domain of the absorptive parts A_s looks like :

$$E_L(s) = \left\{ 0, -4K^2 \mid \frac{32(2M+1)}{s-(M-2)^2} \right\} \quad \text{for } (M+1)^2 \leq s \leq s_{\text{crit}}$$

$$E(1.83, s) = \left\{ 0, -4K^2 \mid 1.83 \right\} \quad \text{for } s_{\text{crit}} \leq s \leq \infty \quad (3.1)$$

Here s_{crit} is given by the condition that the right-hand extremity of $E_L(s_{\text{crit}})$ passes through $t = 1.83$, that is *)

$$s_{\text{crit}} = (M-2)^2 + \frac{32(2M+1)}{1.83} \quad (3.2)$$

The analyticity domain (3.1) for A_s immediately implies that dispersion relations in s hold for fixed $t \in \mathcal{D}_1^{\pi N}$

$$\mathcal{D}_1^{\pi N} = \left[\bigcap_{(M+1)^2 \leq s \leq s_{\text{crit}}} E_L(s) \right] \cap \left[\bigcap_{s \geq s_{\text{crit}}} E(1.83, s) \right] \quad (3.3a)$$

*) It should be mentioned that as $s_{\text{crit}} > (M+1)^2$ to get the enlarged analyticity domain (3.1) unitarity (positivity) has effectively been used only above inelastic threshold.

The intersection of all $E(1.83, s)$ for $s \gg s_{\text{crit}}$ is just the "critical" ellipse $E_L(s_{\text{crit}})$. So we are left with

$$\mathcal{D}_1^{\pi N} \equiv \bigcap_{(M+1)^2 \leq s \leq s_{\text{crit}}} E_L(s) \quad (3.3b)$$

This turns out to be a very simple domain (shown in Fig. 1a) :

$$\mathcal{D}_1^{\pi N} = E_L[(M+1)^2] \cap E_L(s_{\text{crit}}) \quad (3.4)$$

This can be verified by checking that the point of intersection of $E_L[(M+1)^2]$ and $E_L(s_{\text{crit}})$ in the upper-half plane (denoted as point A in Fig. 1) is inside all ellipses $E_L(s)$ for $(M+1)^2 < s < s_{\text{crit}}$. (This is sufficient, because all ellipses under consideration have the right focus $t = 0$ in common, and therefore any two of them cannot have more than one point of intersection in the upper-half plane.)

The domain $\mathcal{D}_1^{\pi N}$ of validity of dispersion relations contains the real segment

$$-\frac{32}{3} \cdot \frac{2M+1}{2M-1} = -12.4 < t < 1.83 \quad (3.5)$$

while the range where dispersion relations can be proved in conventional manner ²⁾ is

$$-\frac{32}{3} \cdot \frac{2M+1}{2M-1} = -12.4 < t \leq 0 \quad (3.6)$$

Thus we did not get any improvement for real, negative t ; our improvement only includes the non-real and the real, positive points of $\mathcal{D}_1^{\pi N}$. But while for $\pi\pi$ scattering, as has been shown by Martin ³⁾ using the same technique of enlargement of $|t| < R_t$, one cannot get further to the left, for πN scattering a further enlargement of $\mathcal{D}_1^{\pi N}$ is possible. This is easily understood. The domain $\mathcal{D}_1^{\pi N}$ is bordered to the left by the Lehmann ellipse at threshold energy, $E_L[(M+1)^2]$. Now if one can improve the Lehmann ellipses for the absorptive part at and near threshold, then it is clear that also the left-hand border of $\mathcal{D}_1^{\pi N}$ can be improved. But such an improvement of $E_L(s)$ is indeed possible.

STEP 2

For real s in the elastic region, the independent πN scattering amplitudes (no-spin-flip and spin-flip amplitude) are analytic in t inside the ellipses $E(1.83, s)$, as these ellipses are inside the domain $\mathcal{D}_1^{\pi N}$ for all s in $(M+1)^2 \leq s \leq (M+2)^2$ (see Fig. 2). From elastic unitarity ^{*)}

$$\operatorname{Im} f_{e\pm}(s) \propto \frac{K}{\sqrt{s}} |f_{e\pm}(s)|^2 \quad (3.7)$$

^{*)} Because of the distribution character of the amplitude, it is not quite clear (though generally accepted) that it is allowed to use elastic unitarity to enlarge the analyticity domain for the absorptive part in the region between elastic and inelastic threshold. Some further assumptions on the boundedness of cross-section in this region are necessary to exclude some pathologic situations ⁸⁾. But we hope that also from general principles this pathology can be excluded.

the absorptive parts are analytic in

$$E_{\nu}(1.83, s) = \left\{ 0, -4K^2 / 1.83 \left(4 + \frac{1.83}{K^2} \right) \right\} \\ \text{for } (M+1)^2 \leq s < (M+2)^2 \quad (3.8)$$

This implies that dispersion relations are valid in

$$\mathcal{D}_2^{\pi N} \equiv \left[\bigcap_{(M+1)^2 \leq s < (M+2)^2} E_{\nu}(1.83, s) \right] \cap \left[\bigcap_{(M+2)^2 \leq s \leq s_{\text{crit}}} E_L(s) \right] \cap \left[\bigcap_{s \geq s_{\text{crit}}} E(1.83, s) \right] \quad (3.9a)$$

$$= \left[\bigcap_{(M+1)^2 \leq s < (M+2)^2} E_{\nu}(1.83, s) \right] \cap \left[\bigcap_{(M+2)^2 \leq s \leq s_{\text{crit}}} E_L(s) \right] \quad (3.9b)$$

The left extremity of $E_{\nu}(R_t = 1.83, s)$

$$\ell_{\nu}(s) = - \left[4K^2 + 4R_t \left(1 + \frac{R_t}{4K^2} \right) \right] \quad (3.10)$$

cannot move further to the right than

$$\ell_{\nu}(s_m) = -8R_t = -14.6 \quad (3.11)$$

where s_m corresponds to

$$K^2(s_m) \equiv \frac{[s_m - (M+1)^2][s_m - (M-1)^2]}{4s_m} = \frac{R_t}{2} \quad (3.12)$$

As the left extremity $\ell_L(s)$ of $E_L(s)$ for $s \gg (M+2)^2$ always lies to the left of $\ell_L[(M+2)^2] = -18.0$, the domain $\mathcal{D}_2^{\pi N}$ contains the real interval

$$-14.6 \leq t \leq +1.83 \quad (3.13)$$

The explicit shape of the domain $\mathcal{D}_2^{\pi N}$ is shown in Fig. 1b ; it is bordered

- a) - for $-10.5 \leq \text{Re } t \leq +1.83$ by the "critical ellipse" $E_L(s_{\text{crit}})$;
- b) - for $-14.6 \leq \text{Re } t \leq -10.5$ by the envelope of the ellipses $E_U(1.83, s)$ with s between s_m and $(M+2)^2$.

STEP 3

We want to use again elastic unitarity, but more extensively than in Step 2. From the unitarity relation for $s < (M+2)^2$

$$A_s(s, -2K^2(1-\cos\theta)) = \frac{K}{\sqrt{s}} \int_{-1}^{+1} dz_1 \int_{-1}^{+1} dz_2 \frac{\Theta(1 + 2z_1 z_2 \cos\theta - z_1^2 - z_2^2 - \cos^2\theta)}{\sqrt{1 + 2z_1 z_2 \cos\theta - z_1^2 - z_2^2 - \cos^2\theta}} \cdot T_{\pi N}(s, -2K^2(1-z_1)) \cdot T_{\pi N}^*(s, -2K^2(1-z_2)) \quad (3.14)$$

the singularities of $A_s[s, -2K^2(1-\cos\theta)]$ are given by ⁹⁾

$$\cos\theta = \bar{z}_1 \bar{z}_2 + \sqrt{\bar{z}_1^2 - 1} \sqrt{\bar{z}_2^2 - 1} \quad (3.15)$$

if $\bar{z}_1, \bar{z}_2$ is any pair of singularities of $T_{\pi N}$ in the $\cos\theta$ plane. The transformation into the t plane yields :

$$\bar{t} = 2K^2 \left\{ \left(1 + \frac{\bar{t}_1}{2K^2}\right) \left(1 + \frac{\bar{t}_2}{2K^2}\right) + \sqrt{\left(1 + \frac{\bar{t}_1}{2K^2}\right)^2 - 1} \sqrt{\left(1 + \frac{\bar{t}_2}{2K^2}\right)^2 - 1} - 1 \right\} \quad (3.16)$$

If one wanted to get the best result, one had to put the co-ordinates of any two points of the border of $\mathcal{D}_2^{\pi N}$ into the relation (3.16). However, in this Section we do not want to find the exact shape of the best domain $\mathcal{D}^{\pi N}$, but only to check, how far it ranges to the left on the real axis. Therefore we are seeking for singular points of A_s at negative values of t , i.e., we have to consider only those pairs of points $\bar{t}_1, \bar{t}_2$ of the border of the analyticity domain of $T_{\pi N}$, which lead to a phase $-\pi$ for t from (3.16). Such points $\bar{t}_1, \bar{t}_2$ are lying on the different branches of any hyperbola with foci 0, $-4K^2$ in the t plane. The simplest method we can use then, is to put two half-ellipses, both with foci 0 and $-4K^2$ into $\mathcal{D}_2^{\pi N}$, as is shown in Fig. 3. That means, instead of $\mathcal{D}_2^{\pi N}$ we work with the more convenient domain bordered by arcs of the ellipses

$$\begin{aligned} E(1.83, s) &\equiv \{0, -4K^2/1.83\} \quad \text{for } -2K^2 \leq \text{Re } t < 1.83 \\ E_D(s) &\equiv \{0, -4K^2/\sqrt{b^2 + (2K^2)^2} - 2K^2\} \quad (3.17) \\ &\quad \text{for } -2K^2 + \sqrt{b^2 + (2K^2)^2} \leq \text{Re } t \leq -2K^2 \end{aligned}$$

where $b(s)$ is some s dependent parameter (the semi-minor axis of E_D). We can use a simple formula for $b(s)$ throughout the elastic region of s :

$$b(s) = \sqrt{(2K_{\text{crit}}^2 + 1.83)^2 - (2K_{\text{crit}}^2)^2} \cdot \sqrt{1 - \left(\frac{2K_{\text{crit}}^2 - 2K(s)^2}{2K_{\text{crit}}^2 + 1.83} \right)^2} \quad (3.18)$$

with $K_{\text{crit}}^2 = K^2(s_{\text{crit}})$. Then (3.17) is inside $\mathcal{D}_2^{\pi N}$ for all real s in $(M+1)^2 \leq s \leq (M+2)^2$. As $T_{\pi N}$ is analytic in (3.17), the relation (3.16) yields that for s in the elastic region the absorptive part $A_s(s, t)$ can have singularities on the negative real t axis only for

$$t < - \min \left\{ \frac{\sqrt{(2K^2 + 1.83)^2 - (2K^2)^2}}{2K^2} b(s) + \frac{2K^2 + 1.83}{2K^2} \sqrt{b(s) + (2K^2)^2} + 2K^2 \right\} \\ \cong - 20.8 \quad (3.19)$$

We had already seen above that for $s \geq (M+2)^2$ there are no singularities of $A_s(s, t)$ in the interval $-18.0 < t < 1.83$. We have thus found the expected result : dispersion relations are valid in a domain (roughly sketched in Fig. 4) which extends on the negative real axis down to the "inelastic point"

$$\max_{s \geq (M+2)^2} \ell_L(s) = -(2M+1) \left[\frac{3(2M+3)}{(M+2)^2} + \frac{4}{M} \right] \\ \cong - 18.0 \quad (3.20)$$

4. -

EXTENSION OF THE ANALYTICITY DOMAIN II

Using the Bessis-Glaser result that for fixed s in $s_0 < s < (M+1)^2$ the πN scattering amplitude (and therefore also the absorptive part) has no singularity in $0 < t < 4$ we see from the discussion in Section 2 that the ellipse of convergence of the partial wave expansion of $\tilde{A}_P(s, t)$ effectively extends to $t = +4$, that is the absorptive parts of any one of the independent πN amplitudes are analytic in

$$E(4, s) = \{0, -4K^2 | 4\} \quad (4.1)$$

STEP 1

Taking again the best information we know that the absorptive parts are analytic in

$$\begin{aligned} E_L(s) &= \left\{0, -4K^2 \mid \frac{32(2M+1)}{s-(M-2)^2}\right\} & \text{for } (M+1)^2 \leq s \leq s'_{\text{crit}} \\ E(4, s) &= \{0, -4K^2 | 4\} & \text{for } s'_{\text{crit}} \leq s \leq \infty \end{aligned} \quad (4.2)$$

with

$$s'_{\text{crit}} = (M+2)^2 + 8(M+1) \quad (4.3)$$

From this, dispersion relations are valid in

$$\mathcal{D}'_{\pi N} = \left[\bigcap_{(M+1)^2 \leq s \leq s'_{\text{crit}}} E_L(s) \right] \cap \left[\bigcap_{s \geq s'_{\text{crit}}} E(4, s) \right] \quad (4.4)$$

which (analogously to Section 3) reduces to

$$\mathcal{D}'_{\pi N} = \bigcap_{(M+1)^2 \leq s \leq s'_{\text{crit}}} E_L(s) \quad (4.5)$$

Like $\mathcal{D}'_{\pi N}$ the new domain $\mathcal{D}'_{\pi N}$ extends to the left to $-\frac{32}{3} \cdot \frac{2M+1}{2M-1}$, but $\mathcal{D}'_{\pi N}$ is no longer simply the intersection of only two ellipses. We have drawn $\mathcal{D}'_{\pi N}$ in Fig. 1c.

STEP 2

Analogously to Step 2 of Section 3 we put the ellipses

$$E(r, s) = \left\{ 0, -4K^2/r \right\} \quad (4.6)$$

into the domain $\mathcal{D}'_{\pi N}$. For all real s in $(M+1)^2 \leq s < (M+2)^2$ these ellipses are inside $\mathcal{D}'_{\pi N}$, if we choose $0 < r \leq 3$. Then again from elastic unitarity the absorptive parts are analytic in

$$E_U(r, s) = \left\{ 0, -4K^2 / 4r \left(1 + \frac{r}{4K^2} \right) \right\} \quad (4.7)$$

for $(M+1)^2 \leq s < (M+2)^2$. This implies that we have dispersion relations in

$$\mathcal{D}'^{\pi N}_2(r) \equiv \left[\bigwedge_{(M+1)^2 \leq s < (M+2)^2} E_v(r, s) \right] \bigwedge \left[\bigwedge_{(M+2)^2 \leq s \leq s'_{\text{int}}} E_L(s) \right] \bigwedge \left[\bigwedge_{s \geq s'_{\text{int}}} E(4, s) \right] \quad (4.8a)$$

$$= \left[\bigwedge_{(M+1)^2 \leq s < (M+2)^2} E_v(r, s) \right] \bigwedge \left[\bigwedge_{(M+2)^2 \leq s \leq s'_{\text{int}}} E_L(s) \right] \quad (4.8b)$$

Now we could choose r between 2.5 and 3. Then from (3.10) and (3.11), it follows that already this second step of our proof yields that $\mathcal{D}^{\pi N}$ extends to the inelastic point -18.0 , as $-8r < -20$.

However, as r can be chosen only smaller than 3, we have to iterate the procedure once more in order to get the best domain. For simplicity, we take again $r = 1.83$. Then we already know from Section 3 that $\mathcal{D}'^{\pi N}_2(r = 1.83)$ extends to -14.6 and therefore contains the ellipse

$$E(4, s) = \left\{ 0, -4K^2/4 \right\} \quad (4.1)$$

for all s in the elastic region which implies that from elastic unitarity the absorptive parts in $(M+1)^2 \leq s < (M+2)^2$ are analytic in the ellipses

$$E_v(4, s) = \left\{ 0, -4K^2/16(1 + \frac{1}{K^2}) \right\} \quad (4.9)$$

What then follows is : dispersion relations for elastic πN scattering are valid in

$$\mathcal{D}^{\pi N} = \left[\bigwedge_{(M+1)^2 \leq s < (M+2)^2} E_v(4, s) \right] \bigwedge \left[\bigwedge_{(M+2)^2 \leq s \leq s'_{\text{int}}} E_L(s) \right] \quad (4.8c)$$

In the same manner as has been done in Section 3 (Step 1) it can be shown that $\mathcal{D}^{\pi N}$ is an intersection of two ellipses only, namely (see Fig. 5)

$$\mathcal{D}^{\pi N} = E_L[(M+2)^2] \bigwedge E_L[(M+2)^2 + 8(M+1)] \quad (4.10)$$

5. - DISCUSSION

Let us summarize what we have found. Starting from analyticity in the topological product of cut s plane and $|t| < 1.83$ we have proved that the analyticity domain can be extended to the inelastic point -18.0 , that is, only the fact that not enough is known about analyticity of production amplitudes prevents us from a further enlargement. The proof consists of three steps, in the first of which unitarity (positivity) is used only in the inelastic region. In the following steps, elastic unitarity was applied in order to get a larger domain for the absorptive part once the analyticity domain of the amplitude $T_{\pi N}$ was known. In Step 2, ellipses were taken as analyticity domain of $T_{\pi N}$, while in Step 3 we used domains of the form shown in Fig. 3. This relatively complicated third step can be avoided if we use the very new information ⁶⁾ that for unphysical s $T_{\pi N}$ has no singularities in $0 < t < 4$.

The best result one can find is that dispersion relations for $T_{\pi N}$ are valid in

$$\mathcal{D}^{\pi N} = E_L[(M+2)^2] \cap E_L[(M+2)^2 + 8(M+1)] \quad (5.1)$$

The absorptive part of $T_{\pi N}$ in the elastic region $(M+1)^2 \leq s < (M+2)^2$ is analytic in the ellipse

$$E_L(4, s) = \left\{ 0, -4K^2 / 16 \left(1 + \frac{1}{K^2} \right) \right\} \quad (5.2)$$

This exactly extends to the border of the support of the Mandelstam spectral functions, which corresponds to the diagram

The domain (5.1) implies some analytic properties of the partial wave amplitudes. Of course, except for the cuts coming from the poles and s and u cuts in the fixed t dispersion relation, the partial wave amplitudes are analytic for all values of s for which the line $t = -4K^2\alpha$ with $0 \leq \alpha \leq 1$ is in $\mathcal{D}^{\pi N}$. The result is shown in Fig. 6.

The analyticity domain for the partial waves consists of two disconnected regions, one outside and one inside the well-known circle $|s| = M^2 - 1$ of partial wave singularities. Both regions touch this circle (from outside or inside) in $s = M^2 - 1$. The region inside the circle extends from $\text{Re } s = 22.3$ to $\text{Re } s = M^2 - 1 = 45.5$, the region outside from $\text{Re } s = 45.5$ to $\text{Re } s = 88.7$. The elastic cut ($60.0 \leq s \leq 76.5$) is therefore well inside the analyticity domain and an analytic continuation into the second (elastic) sheet is possible.

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FIGURE CAPTIONS

- Figure 1 Some domains of validity of fixed t πN dispersion relations, which occur as intermediate step in enlarging the analyticity domain.
- a - domain $\mathcal{D}_1^{\pi N}$ (full line, extending from -12.4 to $+1.83$)
 - b - domain $\mathcal{D}_2^{\pi N}$ (marked with dashed lines, extending from -14.6 to $+1.83$)
 - c - domain $\mathcal{D}_1^{\prime \pi N}$ (marked with dot-dashed lines, extending from -12.4 to $+4$).
- Figure 2 For fixed s in $(M+1)^2 \leq s < (M+2)^2$ ellipses with foci $0, -4K^2$ (shadowed) are put into the analyticity domain $(\mathcal{D}_1^{\pi N}$ or $\mathcal{D}_1^{\prime \pi N})$ of the amplitude. Then elastic unitarity yields certain larger ellipses for the absorptive part.
- Figure 3 In this figure two half-ellipses (both with foci $0, -4K^2$) are put into the analyticity domain $(\mathcal{D}_2^{\pi N})$ of the amplitude, in order to get an enlargement of the analyticity domain of the absorptive part in the elastic region.
- Figure 4 The domain of validity of dispersion relations is roughly sketched, as it is found in Section 3. To the left it extends to the "inelastic point" -18.0 .

Figure_5 The final domain $\mathcal{D}^{\pi N}$ (in t plane) of validity of fixed t πN dispersion relations. It is bordered by two large Lehmann ellipses $E_L(s)$, to the left by that at $s = (M+2)^2$, to the right by that at $s = (M+2)^2 + 8(M+1)$. It extends from $\text{Re } s = -18.0$ to $\text{Re } s = +4$ (in pion mass units, with nucleon mass $M = 6.75$).

Figure_6 Analyticity domain for πN partial waves (shadowed region). It consists of two disconnected parts, one inside and one outside the circle $|s| = M^2 - 1$. The figure also shows the real cuts and the circle of singularities, as expected from unitarity.

FIG.1

FIG. 2

FIG. 3

FIG. 4

FIG. 5

FIG. 6