CM-P00057598

Archives

Ref. TH. 767

SIMULTANEOUS s - t ANALYTICITY FOR INELASTIC TWO-BODY REACTIONS
FROM AXIOMATIC FIELD THEORY

G. Sommer
CERN - Geneva

A B S T R A C T

It is proved for inelastic two-body reactions $A+B \rightarrow C+D$, that the amplitude, if fulfilling an ordinary fixed negative dispersion relation, is simultaneously analytic in s and t in a topological product of the form

$$P_D = \left\{ s, t \mid \begin{array}{l} s \in \text{cut plane} \\ t \in D \text{ (independent of } s) \end{array} \right\}$$

As an example, we apply our method to the $\pi + \Lambda \rightarrow \pi + \Sigma$ amplitude which is shown to have a dispersion relation in a complex domain

$\overline{D}_{\pi \Lambda \rightarrow \pi \Sigma}$, drawn in Fig. 3

1. INTRODUCTION

For any elastic scattering amplitude with an ordinary fixed t dispersion relation it has been proved that the axiomatic analyticity domain contains the topological product of cut s plane and some complex region in the t plane surrounding $t=0$, the right extremity of this region being determined by the maximum of the right extremity of the small Lehmann ellipse ^{1),2)}. A necessary ingredient for finding these results is the positivity of the absorptive part of elastic scattering amplitudes

$$\left| \left(\frac{\partial}{\partial t} \right)^n A_s(s, t) \right| \leq \left[\left(\frac{\partial}{\partial t} \right)^n A_s(s, t) \right]_{t=0} \quad (1.1)$$

which follows from unitarity. For inelastic two-body reactions, $A+B \rightarrow C+D$, a corresponding positivity property does not hold as a consequence of unitarity. Therefore, the arguments which led to an extension of the analyticity domain for elastic scattering amplitudes are not applicable in this case. However, we shall show in this paper, by a very simple argument (making use of unitarity), that a corresponding result also holds for inelastic two-body reactions.

More precisely, if fixed t dispersion relations (with $-t_M \leq t \leq 0$) hold for

$$A + B \leftrightarrow C + D \quad (1.2a)$$

$$A + B \rightarrow A + B \text{ or } C + D \rightarrow C + D \quad (1.2b)$$

$$A + \bar{D} \rightarrow A + \bar{D} \text{ or } C + \bar{B} \rightarrow C + \bar{B} \quad (1.2c)$$

then we prove (in Section 2) that the domain of analyticity for the $A+B \rightarrow C+D$ amplitude contains the topological product

$$\left\{ s, t \mid s \notin \left[(s_0 \leq s \leq \infty) \cup (-\infty \leq s \leq \sum_K m_K^2 - t - u_0) \right] \right. \\ \left. t \in D = [D^{\text{right}} \cap D^{\text{left}}] \right\} \quad (1.3)$$

where s_0 , u_0 are the (unphysical) s and u channel thresholds, respectively, and D is a complex domain in the t plane containing the origin $t=0$. The exact shapes of D^{right} and D^{left} are given in (2.19) and (2.28).

In practice, if we have fixed real negative t dispersion relations for the inelastic two-body reaction (1.2a), then also, at least for one of the elastic scattering processes (1.2b) and also for one of the processes (1.2c). If dispersion relations are valid for both elastic reactions of (1.2b) and/or of (1.2c), then an improvement of the result (1.3) will be possible, as it will be seen in Section 3.

In Section 4 we shall discuss, as an application, the $\pi + \Lambda \rightarrow \pi + \Sigma$ amplitude. We first extend the analyticity domains for elastic $\pi \Lambda$ and elastic $\pi \Sigma$ scattering by means of unitarity. Then, applying the technique described in Sections 2, 3, we find that also the amplitude of the inelastic two-body reaction $\pi + \Lambda \rightarrow \pi + \Sigma$ fulfils dispersion relations inside a complex domain $\bar{D}_{\pi \Lambda \rightarrow \pi \Sigma}$, given by Eq. (4.16), shown in Fig. 3. The domain $\bar{D}_{\pi \Lambda \rightarrow \pi \Sigma}$ contains the real segment $-5.5 \leq t \leq +1.9$. Asymptotically, that is for $s \rightarrow \infty$, the analyticity domain of the absorptive part $A_{\pi \Lambda \rightarrow \pi \Sigma}$ extends to $t = +2.1$. (All numerical values are given in pion mass units.)

2. ANALYTICITY IN TWO VARIABLES FOR INELASTIC TWO-BODY REACTIONS

In this Section we intend to prove that under certain conditions there exists an analyticity domain of the type described by Eq. (1.3) for the amplitude of the reaction

For this it will be sufficient to assume:

- I. a fixed real negative t dispersion relation (with $-t_0 \leq t \leq 0$, $t_0 > 0$) can be deduced for the amplitude of the reaction (2.1) from classical methods of axiomatic field theory. (Of course, such a dispersion relation is simultaneously a dispersion relation for the crossed process

- II. at least for one each of the two pairs of elastic scattering processes

and

fixed t dispersion relations hold in $-t_M \leq t \leq 0$, $t_M > 0$. In particular, in the following we shall assume (2.3) and (2.5) to have dispersion relations.

[In practice, assumption II is automatically fulfilled, if assumption I is. This is partly due to the following: a breakdown of validity of large Lehmann ellipse, e.g., for process (2.3) always implies the breakdown for process (2.1), but not the reverse. Partly, this is also due to the fact that in the inelastic case none of the foci of the ellipses is fixed which leads, as compared with the elastic case, to an extra loss of information when performing the intersection of large Lehmann ellipses for $s_{\text{threshold}} \leq s \leq \infty$.]

In the following we denote the channels $A+B$, $C+D$, $A+\bar{D}$ and $C+\bar{B}$ by indices 1, 2, 3 and 4, respectively, and in particular, we use the notations $K_1(s)$, $K_2(s)$, $K_3(s)$, $K_4(s)$ for the c.m.s. momenta in the four channels.

From our assumption of validity of fixed real negative t dispersion relations for the elastic process (2.3), using positivity as expressed in Eq. (1.1), we know ^{1), 2)} that the amplitude $T_{A+B \rightarrow A+B}$ fulfils dispersion relations also in some circle

$$|t| < R_1 \quad (2.7)$$

where, in principle, we also have a lower limit for R_1 . This is also true ³⁾, if unphysical thresholds are present and therefore on a part of the s and/or u cut the absorptive part has no longer the positivity properties (1.1).

Next we may write the partial wave expansions for $T_{A+B \rightarrow A+B}(s, t)$ and $T_{A+B \rightarrow C+D}(s, t)$ as follows ^{*)}:

$$T_{A+B \rightarrow A+B}(s, t) = a_{11}(s) \sum_{\ell=0}^{\infty} (2\ell+1) (AB|AB)_{\ell}(s) P_{\ell}\left(1 + \frac{t}{2K_1^2(s)}\right) \quad (2.8)$$

$$T_{A+B \rightarrow C+D}(s, t) = a_{12}(s) \sum_{\ell=0}^{\infty} (2\ell+1) (CD|AB)_{\ell}(s) P_{\ell}\left(\frac{K_1^2(s) + K_2^2(s) + \frac{\delta}{2} + t}{2K_1(s)K_2(s)}\right) \quad (2.9)$$

where $a_{mn}(s)$ are unimportant kinematical factors and $\delta = -\frac{1}{4}(m_A^2 - m_B^2 - m_C^2 + m_D^2)^2$.

From (2.7) and the positivity property, the absorptive part $A_{A+B \rightarrow A+B}$ of $T_{A+B \rightarrow A+B}$ is analytic in the ellipse

*) We use spinless particles throughout this paper, but all arguments can be transcribed at least to the case of spin 0 spin $\frac{1}{2}$ reactions ³⁾.

$$E(R_1, s) = \left\{ |t| + |t + 4K_1^2(s)| < 4K_1^2(s) + 2R_1 \right\} \quad (2.10)$$

and as a consequence the imaginary parts of the partial amplitudes are bounded by

$$\lim_{l \rightarrow \infty} \left[\text{Im}(AB|AB)_l \right]^{1/l} \leq \frac{1}{\left(1 + \frac{R_1}{2K_1^2}\right) + \sqrt{\left(1 + \frac{R_1}{2K_1^2}\right)^2 - 1}} \quad (2.11)$$

But from unitarity

$$\text{Im}(AB|AB)_l \geq |(CD|AB)_l|^2 \quad (2.12)$$

and therefore

$$\lim_{l \rightarrow \infty} |(CD|AB)_l|^{1/l} \leq \left[\frac{1}{\left(1 + \frac{R_1}{2K_1^2}\right) + \sqrt{\left(1 + \frac{R_1}{2K_1^2}\right)^2 - 1}} \right]^{1/2} \quad (2.13)$$

which implies that $T_{A+B \rightarrow C+D}$ and in particular, $A_{A+B \rightarrow C+D}$ are analytic for fixed real s above physical threshold in the ellipse (unitarity ellipse):

$$E_{U12}(R_1, s) = \left\{ |t + (K_1 - K_2)^2 \frac{s}{4}| + |t + (K_1 + K_2)^2 \frac{s}{4}| < \frac{s}{4} 4K_1 K_2 \sqrt{1 + \frac{R_1}{4K_1^2}} \right\} \quad (2.14)$$

For $s \rightarrow \infty$ this ellipse goes over into

$$E_{U12}(R_1, \infty) = \left\{ |t| + |t + s - \sum_K m_K^2| < s - \sum_K m_K^2 + \frac{R_1}{2} \right\} \quad (2.15)$$

with the right extremity

$$r_{U12}(R_1, s = \infty) = \frac{R_1}{4} \quad (2.16)$$

The absorptive part of the amplitude for the reaction (2.1) is furthermore analytic in the large Lehmann ellipse

$$E_{L12}(s) = \left\{ |t + (K_1 - K_2) \frac{2}{s}| + |t + (K_1 + K_2) \frac{2}{s}| < 4Q(s) \right\} \quad (2.17a)$$

where

$$Q(s) = \sqrt{\left(K_1^2 + \frac{A_1}{s-B_1}\right)\left(K_2^2 + \frac{A_2}{s-B_2}\right)} + \sqrt{\frac{A_1}{s-B_1} \frac{A_2}{s-B_2}} + \frac{\delta}{2s} \quad (2.17b)$$

and A_i, B_i are some known constants ⁴⁾. From this it is clear that the function $I^{\text{right}}(s, t)$, defined to be the right cut dispersion integral of the dispersion relation for the $(A+B \rightarrow C+D)$ amplitude is

$$I^{\text{right}}(s, t) \equiv \frac{1}{\pi} \int_{s_{\text{unphys.}}}^{\infty} \frac{A_{A+B \rightarrow C+D}(s', t)}{s' - s} ds' \quad (2.18)$$

for $-t_0 \leq t \leq 0$ has an analytic continuation (in t) from this interval into a two-dimensional domain D^{right} and, in particular, also the representation (2.18) remains valid for all $t \in D^{\text{right}}$. Here D^{right} is given by

$$D^{\text{right}} = \left[\bigcap_{s_{\text{unphys.}} \leq s \leq s_{\text{phys}}} E_{L12}(s) \right] \cap \left[\bigcap_{s \geq s_{\text{phys}}} \left\{ E_{L12}(s) \cup E_{U12}(s) \right\} \right] \quad (2.19)$$

where s_{unphys} and $s_{\text{phys}} = (m_C + m_D)^2 > (m_A + m_B)^2$ denote the unphysical and physical threshold energies, respectively. It is easy to see that D^{right} is in fact a two-dimensional region in the t plane. At first, from the assumption I that classical methods yield fixed t dispersion relations for $A+B \rightarrow C+D$ in $-t_0 \leq t \leq 0$, it is obvious that

$$\bigcap_{s_{\text{unphys.}} \leq s \leq s_{\text{phys}}} E_{L12}(s)$$

contains the circle

$$\left| t + \frac{t_0}{2} \right| < \frac{t_0}{2} \quad (2.20)$$

For $s \geq s_{\text{phys}}$ the right extremity of the large Lehmann ellipse for process (2.1),

$$r_{L12}(s) = -K_1^2 - K_2^2 + 2Q(s) - \frac{\delta}{s} \quad (2.21)$$

with $Q(s)$ from Eq. (2.17b), has a positive value for $s = s_{\text{phys}}$ and drops down to $+0$ for $s \rightarrow \infty$, while the right extremity of the unitarity ellipse

$$r_{U12}(R_1, s) = -K_1^2 - K_2^2 + 2K_1K_2 \sqrt{1 + \frac{R_1}{4K_1^2}} \quad (2.22)$$

grows from $r_{U12} = -K_1^2$ ($s = s_{\text{phys}}$) at $s = s_{\text{phys}}$ to $r_{U12} = (1/4)R_1$ for $s \rightarrow \infty$. Therefore

$$t_{12} \equiv \min_{s \geq s_{\text{phys}}} \left\{ \max \left(\min \left[r_{L12}(s), r_{L12}(s) + \frac{(K_1 - K_2)^2 \delta}{s} \right], \min \left[r_{U12}(R_1, s), r_{U12}(R_1, s) + \frac{(K_1 - K_2)^2 \delta}{s} \right] \right) \right\} \quad (2.23)$$

and as a consequence (using the fact that the right focus of E_{L12} and E_{U12} lies at $t = -(K_1 - K_2)^2 - \frac{\delta}{s}$)

$$\{ |t| < t_{12} \} \subset \bigcap_{s \geq s_{\text{phys}}} \{ E_{L12}(s) \cup E_{U12}(R_1, s) \} \quad (2.24)$$

Thus, D^{right} contains at least the circle

$$\left| t + \frac{\hat{t}_{12}}{2} \right| < \frac{\hat{t}_{12}}{2}, \quad (2.25)$$

$$\hat{t}_{12} = \min(t_0, t_{12}). \quad (2.26)$$

Exactly the same reasoning applies to the left cut dispersion integral which is connected with processes (2.2), (2.5). If $T_{A+\bar{D} \rightarrow A+\bar{D}}$ turns out to be analytic in the topological product of cut s plane and the circle

$$|t| < R_3 \quad (2.27)$$

then the left cut dispersion integral in the dispersion relation for the reaction $A+B \rightarrow C+D$ is analytic in t in the two-dimensional region

$$D^{\text{left}} = \left[\bigcap_{\substack{u_{\text{unphys.}} \leq u \leq u_{\text{phys.}}}} E_{L34}(u) \right] \cap \left[\bigcap_{u \geq u_{\text{phys.}}} \left\{ E_{L34}(u) \cup E_{U34}(R_3, u) \right\} \right] \quad (2.28)$$

where all notations follow from substituting indices 1, 2 by 3, 4 in all formulae above. As a result the amplitude for the reaction $A+B \rightarrow C+D$ is analytic in the topological product

$$P_D = \left\{ s, t \mid \begin{array}{l} s \in \text{cut plane} \\ t \in D = [D^{\text{right}} \cap D^{\text{left}}] \end{array} \right\} \quad (2.29)$$

Before discussing possible improvements we want to point out the following. In principle, the method explained so far might under very favourable circumstances also lead to dispersion relations for the amplitude of the reaction (2.1), if classical methods only yield dispersion relations for the elastic scattering processes (2.3), (2.5), but not for (2.1). We have not checked whether such favourable cases exist. However, we have found that the reactions

$$\pi + \pi \leftrightarrow N + \bar{N}$$

$$\pi + \Lambda \leftrightarrow \bar{K} + N$$

are not of this nice feature, although we have dispersion relations for elastic $\pi\pi$ and $\pi\Lambda$ scattering at our disposal.

3. IMPROVEMENTS OF THE RESULTS OF SECTION 2

Now let us look how we can extend the result (2.29), if we have more information than assumptions I and II. For this we substitute II (Section 2) by one of the stronger assumptions III or IV.

III. fixed t dispersion relations hold in $-t_M \leq t \leq 0$ for the elastic scattering processes (2.3), (2.4), (2.5).

IV. fixed t dispersion relations hold in $-t_M \leq t \leq 0$ for the elastic scattering processes (2.3), (2.4), (2.5), (2.6).

From the validity of a dispersion relation for the elastic scattering (2.4) in $-t_M \leq t \leq 0$ (and positivity and small Lehmann ellipse), we find that it is also valid in

$$|t| < R_2 \quad (3.1)$$

As above we can use unitarity

$$\text{Im} (CD|CD)_e \geq |(CD|AB)_e|^2 \quad (3.2)$$

which yields that $A_{A+B \rightarrow C+D}$ is not only analytic in $E_{L12}(s)$ and $E_{U12}(R_1, s)$ but also in

$$E_{U21}(R_2, s) = \left\{ |t + (K_1 - K_2)^2 \frac{s}{4}| + |t + (K_1 + K_2)^2 \frac{s}{4}| \sqrt{1 + \frac{R_2}{4K_1^2}} \right\} \quad (3.3)$$

and D^{right} in (2.29) can be substituted by the bigger domain

$$\tilde{D}^{\text{right}} = \left[\bigcap_{s_{\text{unphys}} \leq s \leq s_{\text{phys}}} E_{L12}(s) \right] \cap \left[\bigcap_{s \geq s_{\text{phys}}} \left\{ E_{L12}(s) \bigcup_{U12} E(R_1, s) \bigcup_{U21} E(R_2, s) \right\} \right] \quad (3.4)$$

and, provided the situation is as assumed in IV, also D^{left} by the corresponding domain $\tilde{D}^{\text{left}}$.

So far we have applied unitarity only in such a way that we got an upper bound for $\text{Im}(CD/AB)_\ell$ in terms of the imaginary parts of one of the amplitudes $(AB/AB)_\ell$ and $(CD/CD)_\ell$, only. Thus, we have used the analyticity domains (2.7) and (3.1) for the processes (2.3), (2.4) only separately. It is clear, if we use both information simultaneously, we can find a bigger domain. What we shall do is to make use of an inequality due to Martin ^{5)*}

$$|\text{Im}(CD/AB)_\ell| \leq \sqrt{\text{Im}(CD/CD)_\ell \cdot \text{Im}(AB/AB)_\ell} \quad (3.5)$$

which is a direct consequence of unitarity and Schwarz inequality:

$$\begin{aligned} [\text{Im}(CD/AB)_\ell]^2 &= \left[\sum_n (CD|n)_\ell (n|AB)_\ell \right]^2 \leq \\ &\leq \sum_n (CD|n)_\ell (n|CD)_\ell \cdot \sum_{n'} (AB|n')_\ell (n'|AB)_\ell = \\ &= \text{Im}(CD/CD)_\ell \cdot \text{Im}(AB/AB)_\ell \end{aligned} \quad (3.6)$$

From (2.7) and (3.1) we have

$$\begin{aligned} \lim_{\ell \rightarrow \infty} [\text{Im}(AB/AB)_\ell]^{1/\ell} &\leq \left[\left(1 + \frac{R_1}{2K_1^2}\right) + \sqrt{\left(1 + \frac{R_1}{2K_1^2}\right)^2 - 1} \right]^{-1} \\ \lim_{\ell \rightarrow \infty} [\text{Im}(CD/CD)_\ell]^{1/\ell} &\leq \left[\left(1 + \frac{R_2}{2K_2^2}\right) + \sqrt{\left(1 + \frac{R_2}{2K_2^2}\right)^2 - 1} \right]^{-1} \end{aligned} \quad (3.7)$$

*) I am extremely thankful to Professor A. Martin who drew my attention to the possibility of applying this inequality in order to get a bigger domain.

Inserting these bounds into the inequality (3.5) we find that the absorptive part $A_{A+B \rightarrow C+D}$ of $T_{A+B \rightarrow C+D}$ for fixed $s \geq s_{\text{phys}}$ is analytic in t in the ellipse

$$\bar{E}_{U12}(R_1, R_2, s) = \left\{ |t + (K_1 - K_2)^2 \frac{\delta}{s}| + |t + (K_1 + K_2)^2 \frac{\delta}{s}| < \frac{2\delta}{s} + 4K_1 K_2 \frac{1+\zeta}{2\sqrt{s}} \right\} \quad (3.8)$$

with

$$\zeta = \left[1 + \frac{R_1}{2K_1^2} + \sqrt{\left(1 + \frac{R_1}{2K_1^2}\right)^2 - 1} \right] \left[1 + \frac{R_2}{2K_2^2} + \sqrt{\left(1 + \frac{R_2}{2K_2^2}\right)^2 - 1} \right] \quad (3.9)$$

The right extremity $\bar{r}_{U12}(R_1, R_2, s)$ of this ellipse tends for $s \rightarrow \infty$ to

$$\bar{r}_{U12} \rightarrow \frac{1}{4} \left(\sqrt{R_1} + \sqrt{R_2} \right)^2 \quad (3.10)$$

which is larger than the limiting values of the right extremities of both $E_{U12}(R_1, s)$ and $E_{U21}(R_2, s)$, as can be seen from (2.16). Then, if we substitute D^{right} in the arguments before by

$$\bar{D}^{\text{right}} = \left[\bigcap_{s_{\text{phys}} \leq s \leq s_{\text{phys}}} E_{L12}(s) \right] \cap \left[\bigcap_{s \geq s_{\text{phys}}} \left\{ E_{L12}(s) \cup \bar{E}_{U12}(R_1, R_2, s) \right\} \right] \quad (3.11)$$

and also, if we have the information corresponding to assumption IV, D^{left} by

$$\bar{D}^{\text{left}} \supseteq \bar{D}^{\text{right}} \quad (1, 2 \rightarrow 3, 4) \quad (3.12)$$

we get as the analyticity domain of the $(A+B \rightarrow C+D)$ amplitude

$$P_{\bar{D}} = \left\{ s, t \mid \begin{array}{l} s \in \text{cut plane} \\ t \in \bar{D} = [\bar{D}^{\text{right}} \cap \bar{D}^{\text{left}}] \end{array} \right\} \quad (3.13)$$

4. APPLICATION TO THE $\pi + \Lambda \rightarrow \bar{\pi} + \Sigma$ AMPLITUDE

This section is devoted to the application of the general prescription given in Sections 2 and 3 to the reaction

$$\Lambda + \pi \rightarrow \Sigma + \pi \quad (4.1)$$

In this case the corresponding crossed reaction (with respect to $s - u$ crossing) is the same as process (4.1). Furthermore we see that the assumptions I and IV of Sections 2 and 3 are fulfilled, that is classical methods yield fixed negative t dispersion relations for the inelastic reaction (4.1) [corresponding to (2.1) and (2.2)] and for the elastic processes

$$\Lambda + \pi \rightarrow \Lambda + \pi \quad [\text{corresponding to (2.3), (2.5)}] \quad (4.2)$$

$$\Sigma + \pi \rightarrow \Sigma + \pi \quad [\text{corresponding to (2.4), (2.6)}] \quad (4.3)$$

From the general rule 2), 3) that fixed t dispersion relations in s are valid in the circle

$$|t| < R \quad (4.4)$$

$$R = R_t \equiv \min \left[t_0, \max_{s \geq s_{thr.}} r_{SL}(s) \right] \quad (4.5)$$

[with $r_{SL}(s)$ = right extremity of the small Lehmann ellipse and $-t_0 \leq t \leq 0$ being the interval where dispersion relations are valid from classical methods] we have

$$R_t^{\pi\Lambda} \approx 1.83 \quad (4.6)$$

$$R_t^{\pi\Sigma} \approx 0.8 \quad (4.7)$$

But, for $R^{\pi\Lambda}$ there exists also a better value than follows from Lehmann ellipse, namely ⁶⁾

$$R^{\pi\Lambda} = 4 \quad (4.8)$$

Though not necessary for the following, we shall first give the results for the enlargements of the analyticity domains for elastic $\pi\Lambda$ and $\pi\Sigma$ scattering. Starting from the circles given by (4.4), (4.7), (4.8) and using the positivity properties of the absorptive parts (above physical threshold) and elastic unitarity ³⁾ [for $(m_\Lambda + 1)^2 \leq s \leq (m_\Sigma + 1)^2$], we immediately find the domains shown in Figs. 1 and 2 for $\pi + \Lambda \rightarrow \pi + \Lambda$ and $\pi + \Sigma \rightarrow \pi + \Sigma$, respectively. (We should mention that we did not try to find the best domains.)

Now we want to apply the method of Sections 2 and 3 in order to show that also for the $\pi + \Lambda \rightarrow \pi + \Sigma$ amplitude (which has no positivity properties from unitarity) dispersion relations are valid in a complex domain and to calculate this domain explicitly. In the notation of Sections 2 and 3 for the reaction (4.1) we have that

$$\begin{aligned} R_1 &= R_3 = 4 \\ R_2 &= R_4 = 0.8 \end{aligned} \quad (4.9)$$

and from Eqs. (3.14)-(3.16) that

$$\bar{D}^{\text{right}} = \bar{D}^{\text{left}} \quad (4.10)$$

i.e.,

$$\bar{D} \equiv \bar{D}^{\text{right}} \cap \bar{D}^{\text{left}} = \bar{D}^{\text{right}} \quad (4.11)$$

This is because for all three reactions (4.1)-(4.3) the left cut is connected with the same process as the right cut ^{*}).

The absorptive part of the amplitude for (4.1) is analytic in the large Lehmann ellipse

$$E_L^{\pi\Lambda \rightarrow \pi\Sigma}(s) = \left\{ \left| t + (K_{\pi\Lambda} - K_{\pi\Sigma})^2 - \frac{(m_\Lambda^2 - m_\Sigma^2)^2}{4s} \right| + \left| t + (K_{\pi\Lambda} + K_{\pi\Sigma})^2 - \frac{(m_\Lambda^2 - m_\Sigma^2)^2}{4s} \right| < \right. \quad (4.12)$$

$$\left. < -\frac{(m_\Lambda^2 - m_\Sigma^2)^2}{2s} + 4 \sqrt{\left(K_{\pi\Lambda}^2 + \frac{8(2m_\Lambda+1)}{s-(m_\Lambda-2)^2} \right) \left(K_{\pi\Sigma}^2 + \frac{8[(m_\Lambda+1)^2 - m_\Sigma^2]}{s-(m_\Lambda-2)^2} \right)} + 32 \frac{\sqrt{(2m_\Lambda+1)[(m_\Lambda+1)^2 - m_\Sigma^2]}}{s-(m_\Lambda-2)^2} \right\}$$

where $K_{\pi\Lambda}$, $K_{\pi\Sigma}$ are the c.m.s. momenta in the $\pi + \Lambda$ and $\pi + \Sigma$ channels, respectively, and m_Λ , m_Σ are the Λ and Σ masses (in pion mass units). From Eqs. (3.8), (3.9), (4.9) the absorptive part is also analytic in the ellipse (unitarity ellipse)

$$E_U^{\pi\Lambda \rightarrow \pi\Sigma}(4, 0.8, s) = \left\{ \left| t + (K_{\pi\Lambda} - K_{\pi\Sigma})^2 - \frac{(m_\Lambda^2 - m_\Sigma^2)^2}{4s} \right| + \left| t + (K_{\pi\Lambda} + K_{\pi\Sigma})^2 - \frac{(m_\Lambda^2 - m_\Sigma^2)^2}{4s} \right| < \right. \quad (4.13)$$

$$\left. < -\frac{(m_\Lambda^2 - m_\Sigma^2)^2}{2s} + [4K_{\pi\Lambda}^2 K_{\pi\Sigma}^2 + g] s^{-1/2} \right\}$$

with

$$g = \left[(2K_{\pi\Lambda}^2 + 4) + \sqrt{(2K_{\pi\Lambda}^2 + 4)^2 - (2K_{\pi\Lambda}^2)^2} \right] \left[(2K_{\pi\Sigma}^2 + 0.8) + \sqrt{(2K_{\pi\Sigma}^2 + 0.8)^2 - (2K_{\pi\Sigma}^2)^2} \right] \quad (4.14)$$

^{*}) We forget about the isospin and spin dependence of the pion, hyperon amplitudes, because they can be treated as in the πN case ³⁾.

From this and (3.11), (3.13), (4.11) we find that the $\pi + \Lambda \rightarrow \pi + \Sigma$ amplitude is analytic in the topological product

$$P_{\bar{D}_{\pi\Lambda \rightarrow \pi\Sigma}} = \left\{ s, t \mid s \notin \left\{ s = (m_\Lambda + 1)^2 + \eta, s = m_\Sigma^2 - 2m_\Lambda + 1 - t - \eta, \eta \geq 0 \right\} \right\} \quad (4.15)$$

$$\bar{D}_{\pi\Lambda \rightarrow \pi\Sigma} = \left[\bigcap_{(m_\Lambda + 1)^2 \leq s \leq (m_\Sigma + 1)^2} E_L^{\pi\Lambda \rightarrow \pi\Sigma}(s) \right] \cap \left[\bigcap_{s \geq (m_\Sigma + 1)^2} \left\{ E_L^{\pi\Lambda \rightarrow \pi\Sigma}(s) \cup E_U^{\pi\Lambda \rightarrow \pi\Sigma}(4.0, 8, s) \right\} \right] \quad (4.16)$$

We have drawn the domain $\bar{D}_{\pi\Lambda \rightarrow \pi\Sigma}$ in Fig. 3. On the real axis it extends from $t = -5.5$ to $t = +1.9$; the last value is just $t = t_{12}$, Eq. (2.23) specialized to $\pi + \Lambda \rightarrow \pi + \Sigma$. For $s \rightarrow \infty$ the analyticity domain of the absorptive part ranges further to the right than $\bar{D}_{\pi\Lambda \rightarrow \pi\Sigma}$, namely to $t = \gamma_U^{\pi\Lambda \rightarrow \pi\Sigma}(\infty) = +2.1$, as can be seen from (3.10), (4.9) or directly from (4.13).

Another application ($K + \bar{K} \rightarrow \eta + \pi$) of the method explained in Sections 2 and 3 will be discussed in a forthcoming paper ⁷⁾.

ACKNOWLEDGEMENTS

Enlightening discussions with Professor A. Martin are gratefully acknowledged. We are also thankful to Mr. W. Klein for numerical calculations.

R E F E R E N C E S

- 1) A. Martin, Nuovo Cimento 42, 930 (1966); 44, 1219 (1966).
- 2) G. Sommer, Nuovo Cimento 48A, 92 (1967).
- 3) G. Sommer, CERN Preprint TH. 731 (1966) (to appear in Nuovo Cimento).
- 4) H. Lehmann, Nuovo Cimento 10, 579 (1958).
- 5) A. Martin, private communication.
- 6) J.D. Bessis and V. Glaser, CERN Preprint TH. 728 (1966).
- 7) G. Sommer, CERN Preprint TH. 769 (1967).

FIGURE CAPTIONSFigure 1:

Domain of validity of fixed t dispersion relations for $\pi + \Lambda \rightarrow \pi + \Lambda$. (The interval following from classical methods is indicated by a dotted line.)

Figure 2:

Domain of validity of fixed t dispersion relations for $\pi + \Sigma \rightarrow \pi + \Sigma$. (The interval following from classical methods is indicated by a dotted line.)

Figure 3:

Domain $\bar{D}$ $\pi \Lambda \rightarrow \pi \Sigma$ of validity of fixed t dispersion relations for $\pi + \Lambda \rightarrow \pi + \Sigma$. (The interval following from classical methods is indicated by a dotted line.)

FIG.3