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AN EXPANSION OF THE SCATTERING AMPLITUDE
AT VANISHING FOUR - MOMENTUM TRANSFER
USING THE REPRESENTATIONS OF THE LORENTZ GROUP

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A B S T R A C T

The scattering amplitude at vanishing four-momentum transfer is expanded by means of the matrix elements of the irreducible representations of the Lorentz group including space reflection and the PCT transformation. These representations are labelled by a complex parameter λ and other "quantum numbers". The high-energy asymptotic behaviour of the amplitude is assumed to be determined by the contributions of some poles in the complex λ plane (Lorentz poles).

The contribution of a Lorentz pole is decomposed into a family of Regge pole contributions with given parities and signatures. The connection with other formalisms which give rise to families of Regge trajectories is briefly discussed. Particles with arbitrary spin and parity are considered. The selection rules for the quantum numbers of the Lorentz poles are treated in detail. The role of the internal symmetries and of the symmetry with respect to the permutation of identical particles is discussed. The general formalism is applied to the meson-nucleon and nucleon-nucleon scattering.

In the last few years, some attention has been devoted by various authors ¹⁾⁻⁷⁾ to a group-theoretical description of the scattering amplitude, which is, in a sense, complementary to the usual partial wave expansion. The two descriptions are based on different "coupling schemes" for the irreducible representations of the Poincaré group ⁸⁾ which describe the one-particle states (see Fig. 1). In the usual partial wave analysis, the representations which describe the incoming particles (and in a similar way those which describe the outgoing particles) are coupled between themselves. In the alternative treatment mentioned above, which we call "crossed partial wave analysis", each of the two incoming particles is associated with one of the two outgoing particles. The exact meaning of these coupling schemes is given in Refs. ^{1),5),7)}.

The formalism explicitly based on the representations of the Poincaré group is equivalent to a more elementary procedure ⁴⁾ which is a direct generalization of the method used by Jacob and Wick ⁹⁾ to perform the partial wave expansion of the helicity amplitudes. This procedure consists in two steps :

- (i) - to write the amplitude for a fixed value of the four-momentum transfer Q as a function defined over the little group corresponding to Q , i.e., the subgroup of the Lorentz group which contains all the elements which leave the four vector Q unchanged ;
- (ii) - to expand this function in terms of the matrix elements of the irreducible representations of the little group.

We shall consider the process

$$(3) + (4) \rightarrow (1c) + (2c) \quad (1.1)$$

(see Fig. 1), where $(1c)$ means the antiparticle corresponding to a particle of the kind (1) . It is convenient to treat separately the equal mass case in which the conditions

2.

$$M_3 = M_1, \quad M_2 = M_4, \quad (1.2)$$

between the masses are satisfied and the unequal mass case in which at least one of the conditions (1.2) is not satisfied.

In the equal mass case, the four-momentum transfer Q is spacelike or vanishes. In the first case the little group is the three-dimensional Lorentz group $O(2,1)$ and the corresponding expansion has been extensively treated in the papers ¹⁾⁻⁷⁾. The results are strongly connected ⁶⁾ with the results which can be obtained from the Regge theory of complex angular momentum ¹⁰⁾. If $Q = 0$, the little group is the Lorentz group $O(3,1)$ and the corresponding expansion is the main object of the present paper. The $O(3,1)$ expansion which we shall describe has been suggested in Ref. ⁴⁾ and has been developed in two unpublished reports ^{11),12)}. The present paper contains the most important part of these reports completely rewritten making use of a more powerful formalism and of some mathematical results recently obtained ^{13),14)}.

An expansion of the relativistic scattering amplitude by means of the matrix elements of the representations of the Lorentz group has been proposed some years ago by Dolginov and Topygin ¹⁵⁾. However, these authors do not start from the symmetry with respect to the Poincaré group and, as a consequence, they use the $O(3,1)$ expansion in a more general way, but with less physical justification. A similar remark can be made for the expansions proposed in Ref. ³⁾.

In the unequal mass case, the four-momentum transfer Q can never vanish and therefore the $O(3,1)$ expansion cannot simply be defined. In this case, Q may be spacelike and if

$$(M_3 - M_1)(M_2 - M_4) > 0, \quad (1.3)$$

it may also be timelike or of the light type. In the last case, the little group is the two-dimensional Euclidean group $E(2)$. This is the case treated recently by Freedman and Wang¹⁶⁾, using the analyticity properties of the amplitude. A group theoretical treatment of this problem seems promising, but has not yet been given. In this paper we treat the amplitude for $Q^2 = 0$ only in the equal mass case.

The importance of the Lorentz symmetry in the treatment of the scattering amplitude at $Q = 0$ has been pointed out by Wick¹⁷⁾ and by Cutkosky¹⁸⁾ in their investigation of the Bethe-Salpeter equation¹⁹⁾. In Wick's work, the symmetry of the Bethe-Salpeter equation with respect to the Lorentz group $O(3,1)$ is transformed (by means of the analytic continuation to imaginary values of the energy variable) into a symmetry with respect to the four-dimensional rotation group $O(4)$. The introduction of the group $O(4)$ has to be considered as a mathematical trick which permits to avoid the difficulties connected with the non-compactness of $O(3,1)$ which is the physically significant group. Note, however, that this simplification has its counterpart in the more subtle difficulties connected with the analytic continuation.

The $O(4)$ symmetry has been used by Ceolin et al.²⁰⁾ and, more explicitly, by Bjorken²¹⁾, in order to derive from a multiperipheral^{22),23)} (or Bethe-Salpeter) model the high-energy behaviour of the amplitude at $Q = 0$. A very interesting consequence of the $O(4)$ symmetry of the Bethe-Salpeter equation has been pointed out by Domokos and Suranyi²⁴⁾⁻²⁶⁾. Continuing analytically the angular momentum and a parameter n which labels the irreducible representations of $O(4)$, these authors have found that, in some models, the Regge trajectories¹⁰⁾ can be classified into families, whose members intersect the $Q^2 = 0$ axis at points displaced by integral numbers. It is not correct to say that a family of this kind is described by a representation of $O(4)$, because the parameter n in general is not integral and only integral values of n correspond to representations of $O(4)$. However, going back to the original amplitude, which is symmetric with respect to

$O(3,1)$, it is possible to assign to every family of trajectories an irreducible representation of this group. Therefore it is clear that the $O(3,1)$ expansion is a very appropriate instrument for the description of the behaviour of the Regge singularities in the limit $Q \rightarrow 0$.

A second argument in favour of the existence of families of Regge singularities has been given, independently of any field theoretical model, by Gribov and Volkov ²⁷⁾, studying the scattering of particles with equal masses and non-vanishing spins. Also in presence of spin, it is possible, as we shall show, to construct families of Regge trajectories described by irreducible representations of $O(3,1)$. It is possible to show that these families satisfy the conditions imposed by Gribov and Volkov.

A third argument in favour of the families of trajectories has been given recently by Freedman and Wang ¹⁶⁾ in their investigation of the Regge pole expansion in the unequal mass case. It is clear that, as the Regge trajectories do not depend on the external particles, the argument by Freedman and Wang requires the existence of the families also in the equal mass scattering. As we shall see, the trajectories belonging to the families obtained from the irreducible representations of $O(3,1)$ have the quantum numbers required by Freedman and Wang.

In this paper, we concentrate our attention on the group theoretical aspects of the problem. However, it is clear that more complete results can be obtained taking into account also the analyticity properties of the amplitude. An investigation of this kind has been done in the spinless case by Rubin ²⁸⁾. Note that a minimum of analyticity (or at least of continuity) properties is necessary in order to give a physical meaning to the amplitude in the set defined by $Q = 0$, which has a vanishing measure. In the pure Hilbert space approach of Ref. ⁷⁾ it was not possible to introduce the $O(3,1)$ expansion.

Another limitation of the present paper is that we do not consider asymptotic contributions which correspond to integral or half-integral values of the complex angular momentum ℓ or of the parameter λ which labels the representations of $O(3,1)$. For such values of the parameters, some of our formulae become singular and a special discussion (which can give rather interesting results) is needed. This will be the argument of a forthcoming paper.

In Section 2, we introduce a powerful formalism for the description of the scattering amplitude for particles with arbitrary spins. This formalism is a slight generalization of a formalism introduced by Moussa and Stora²⁹⁾. We take into account also the symmetry of the scattering amplitude with respect to space reflection, PCT and permutation of identical particles. At the end of the Section, we write the cross-sections and the density matrices which describe polarizations, in terms of the amplitude previously defined.

In Section 3, we describe the little groups for $W = Q^2 \neq 0$ and their relevant irreducible representations. In Section 4, following the program sketched above, we write the scattering amplitude for fixed momentum transfer as a function defined on a little group and find some very important conditions which this function must satisfy.

In Section 5, we perform the crossed partial wave analysis for $W = Q^2 < 0$. This argument was treated in Refs. 1)-6) with some important restrictions on the asymptotic behaviour of the amplitude. In Ref. 7), the analysis is performed without any physically unjustified restriction, but the formalism used is too general for our purposes. Here we specify the formalism of Ref. 7) by means of some assumptions suggested by Ref. 2). These assumptions are roughly equivalent to the meromorphy assumption of the usual Regge theory and permit the definition of "Regge pole contributions" which determine the high-energy behaviour of the amplitude.

In Section 6, we describe the irreducible representations of the Lorentz group including space reflection and PCT. The possibility of associating a unitary operator to the PCT transformation has been discussed in Ref. ⁷⁾. In Section 7, we perform the crossed partial wave expansion for $Q = 0$, using the representations described in Section 6. These representations are labelled by some parameters, which we call the "quantum numbers" of the "object" exchanged in the scattering process. These quantum numbers are

- (i) - a complex parameter λ which is analogous to the complex angular momentum ;
- (ii) - a parameter τ , which can assume two values and has the properties of the signature ;
- (iii) - a parameter $M = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$;
- (iv) - if $M = 0$, there is a further parameter $\sigma = \pm 1$ connected with the parity.

The new quantum number M appears in consequence of the four-dimensional symmetry and can be different from zero only if the particles involved in the process have non-vanishing spin. Its physical meaning is not yet clear, but we shall see that it is connected with the helicity exchange at very high energy. In writing the $O(3,1)$ expansion, we use some assumptions, which are a straightforward generalization of the assumptions made in Section 5 for the case $W < 0$. In consequence of these assumptions it is possible to describe the high energy behaviour of the amplitude by means of the contributions of some poles in the complex λ plane. We call them "Lorentz poles". From the symmetry properties of the amplitude we derive some selection rules, which prevent contributions with certain quantum numbers from appearing in certain processes.

In Section 8, we derive a formula which connects the matrix elements of the representations of the groups $O(3,1)$ and $O(2,1)$ including space and time reflections. This formula is a direct generalization of a similar formula obtained in Ref. ¹³⁾ without taking

into account the reflections. In Section 9, we use this formula to decompose a Lorentz pole contribution into Regge pole contributions. The result is that a Lorentz pole generates a family of Regge poles which are superposed or displaced by integral numbers. Probably this connection between different Regge poles is valid only for $Q^2 = 0$ (see, however, ³⁰⁾), but it is sufficient in order to classify the Regge trajectories into families.

In Section 10, we apply the results obtained in the preceding Sections to the meson-nucleon and nucleon-nucleon scattering and we discuss some possible consequences of the Lorentz pole hypothesis.

2. - THE SCATTERING AMPLITUDE

In this Section, we introduce a formalism for the description of the scattering amplitude for particles with arbitrary spins, which is very suitable for the following investigations. It is an extension of the formalism introduced by Moussa and Stora²⁹⁾ for the wave functions which describe particles with arbitrary spins. It can also be considered as a generalization of the helicity formalism⁹⁾.

We call one-particle states of the kind (i) the states which belong to the Hilbert space of an irreducible unitary representation $U^{(i)}$ of the symmetry group G . The group G contains the Poincaré group (or, more exactly, its covering group), but it does not contain symmetries involving time reversal. We call G' the complete symmetry group which contains G and the PCT operation, which we indicate by t . The group G' is composed of the subgroup G and the residue class tG . It is the semi-direct product of the (four dimensional) translation group T and of an homogeneous group A' , which contains the group $SL(2C)$ (homomorphic to the connected part of the Lorentz group), the internal symmetry group, the PCT element t and the space reflection s . We call A the subgroup of A' which contains all the elements which do not involve time reversal. Of course, A' is composed of the subgroup A and the residue class tA .

We use for the elements of G' the notation (x, a) , where $x \in T$ and $a \in A'$. The multiplication rule is given by the usual formula

$$(x, a) (x', a') = (x + L(a)x', aa') \quad , \quad (2.1)$$

where $L(a)$ is the 4×4 Lorentz matrix which corresponds to the element a of A' . We shall often write a instead of $(0, a)$.

For every element $g \in G'$, there is an operator $U_g^{(i)}$ which acts on the space of the particles of the kind (i). This operator is unitary if $g \in G$ and anti-unitary if $g \in tG$. Note that if G does not

contain elements which transform particles into antiparticles (e.g., the charge conjugation), then particles and antiparticles are not "particles of the same kind", and the operator $U_g^{(i)}$ for $g \in tG$ transforms a one-particle state of the kind (i) into a state of the different kind (ic) .

Now we introduce the non-normalizable states $\varphi_m^{(i)}$ which describe particles of the kind (i) at rest. The subscript m stands for the projection of the spin along the z axis and for the internal quantum numbers which distinguish the members of a multiplet (e.g., the third component of the isotopic spin). The four momentum of these states is given by

$$P^{o(i)} = (M_i, 0, 0, 0) \quad (2.2)$$

We call H_+ the little group corresponding to this four momentum. It contains all the elements h of A' which satisfy the condition

$$L(h) P^{o(i)} = P^{o(i)} \quad (2.3)$$

H_+ is a subgroup of A and contains the group $SU(2)$ (homomorphic to the proper rotation group), the internal symmetry group and the space reflection s . The states $\varphi_m^{(i)}$ satisfy the following transformation property

$$U_h^{(i)} \varphi_m^{(i)} = \sum_{m'} R_{m'm}^{(i)}(h) \varphi_{m'}^{(i)}, \quad h \in H_+, \quad (2.4)$$

where $R^{(i)}$ is an irreducible unitary representation of the little group H_+ .

If we consider the six processes

$$\begin{aligned} (3)+(4) &\rightarrow (1c)+(2c), & (1)+(2) &\rightarrow (3c)+(4c), \\ (2)+(4) &\rightarrow (1c)+(3c), & (1)+(3) &\rightarrow (2c)+(4c), \\ (2)+(3) &\rightarrow (1c)+(4c), & (1)+(4) &\rightarrow (2c)+(3c), \end{aligned} \quad (2.5)$$

we may describe the corresponding scattering amplitude by means of a function $M_{m_1 m_2 m_3 m_4}(a_1, a_2, a_3, a_4)$ defined for $a_i \in A'$ in the following way. It does not vanish only if two arguments (e.g., a_r and a_s) belong to tA and the other two (we call them a_j, a_k) belong to A . In this case, it is given by

$$\begin{aligned} i(2\pi)^{-2} \delta^4\left(\sum_i L(a_i) P^{o(i)}\right) M_{m_1 m_2 m_3 m_4}(a_1, a_2, a_3, a_4) = \\ = \pm \left(\left[U_{a_r}^{(r)} \varphi_{m_r}^{(r)} \otimes U_{a_s}^{(s)} \varphi_{m_s}^{(s)} \right], (S-1) \left[U_{a_j}^{(j)} \varphi_{m_j}^{(j)} \otimes U_{a_k}^{(k)} \varphi_{m_k}^{(k)} \right] \right), \end{aligned} \quad (2.6)$$

where S is the scattering operator and the minus sign has to be used if the permutation $(1,2,3,4) \rightarrow (r,s,j,k)$ involves an odd number of fermion exchanges. The function M is defined only for

$$\sum_i L(a_i) P^{o(i)} = 0. \quad (2.7)$$

From Eq. (2.4) we obtain four covariance conditions of the kind

$$\begin{aligned} M_{m_1 m_2 m_3 m_4}(a_1, h, a_2, a_3, a_4) &= \sum_{m'_1} R_{m'_1 m_1}^{(1)}(h) \\ M_{m'_1 m_2 m_3 m_4}(a_1, a_2, a_3, a_4), & \quad h \in H_+ \end{aligned} \quad (2.8)$$

If, for instance, the particles (1) and (2) are of the same kind, from the definition (2.6) it follows easily that the amplitude has the following symmetry property

$$M_{m_1 m_2 m_3 m_4}(a_1, a_2, a_3, a_4) = (-1)^{2j_1} M_{m_2 m_1 m_3 m_4}(a_2, a_1, a_3, a_4), \quad (2.9)$$

where j_i is the spin of the particle (i). The invariance condition of the amplitude with respect to the group A can easily be written in the form

$$M_{m_1 m_2 m_3 m_4}(aa_1, aa_2, aa_3, aa_4) = M_{m_1 m_2 m_3 m_4}(a_1, a_2, a_3, a_4). \quad (2.10)$$

The invariance with respect to an element of tA requires more attention, because it involves the exchange of the initial and the final states. If for instance $a \in tA$ and $a_r, a_s \in tA$, a factor $(-1)^{2(j_r+j_s)}$ appears in the right-hand side of Eq. (2.10). In order to avoid this factor, it is convenient to introduce a new group $\tilde{G}$, which contains the same element as G' , but has a different definition of the product of two elements containing time reversal. This product differs from the product calculated according to the rules of the group G' for a factor e' , where e' is the element of $SL(2C)$ which corresponds to a "rotation of an angle 2π ". If we take into account the covariance relation (2.8) and the identity

$$R_{m m'}^{(i)}(e') = (-1)^{2j_i} \delta_{m m'}, \quad (2.11)$$

we see that, if we adopt the new multiplication rule, Eq. (2.10) holds also in the case $a \in tA$. In the following, we shall always use the multiplication rules of the group $\tilde{G}$. Note that the property (2.8) is not affected by the new convention. A more detailed discussion of the necessity of introducing the group $\tilde{G}$ is given in Ref. 7). We call $\tilde{A}$ the subgroup of $\tilde{G}$ which contains the same element as A' .

If we introduce the states $\varphi_m^{(ic)}$, which describe an antiparticle of the type (1c) at rest and have properties similar to the properties of the states $\varphi_m^{(i)}$, we see that the PCT operator $U_t^{(i)}$ has the property

$$U_t^{(i)} \varphi_m^{(i)} = \sum_{m'} C_{m'm}^{(i)} \varphi_{m'}^{(ic)}, \quad (2.12)$$

where $C_{mm'}^{(i)}$ is a unitary matrix. Taking into account Eq. (2.4) and the anti-unitary character of the operator $U_t^{(i)}$, we see that this matrix must satisfy the following condition, which determines it within a phase factor

$$\sum_{m'} C_{mm'}^{(i)} \overline{R_{m'm''}^{(i)}(t^{-1}ht)} = \sum_{m'} R_{mm'}^{(ic)}(h) C_{m'm''}^{(i)}, \quad (2.13)$$

$$h \in H_+.$$

If the group G contains elements which transform particles into antiparticles, we may take simply

$$\begin{cases} \varphi_m^{(i)} = \varphi_m^{(ic)} \\ R_{mm'}^{(i)}(h) = R_{mm'}^{(ic)}(h) \end{cases}. \quad (2.14)$$

For each four-momentum P which can be assumed by the particle (i), we choose an element a_P of $SL(2C)$ with the property

$$L(a_P) P^{o(i)} = P. \quad (2.15)$$

Then we define the states

$$\varphi_m^{(i)}(P) = U_{a_P}^{(i)} \varphi_m^{(i)}, \quad (2.16)$$

which describe particles with four-momentum P . We write their normalization property in the form

$$(\varphi_m^{(i)}(P), \varphi_{m'}^{(i)}(P')) = 2 (\vec{P}^2 + M_i^2)^{\frac{1}{2}} \delta^3(\vec{P} - \vec{P}') \delta_{mm'} \quad (2.17)$$

In a similar way, we define the states $\varphi_m^{(ic)}(P)$.

If we are interested in the process $(1,1)$, we may write the corresponding matrix element in the form

$$\begin{aligned} & \left([\varphi_{m_1}^{(1c)}(P^{(1)}) \otimes \varphi_{m_2}^{(2c)}(P^{(2)})], (S-1) [\varphi_{m_3}^{(3)}(P^{(3)}) \otimes \varphi_{m_4}^{(4)}(P^{(4)})] \right) = \\ & = i (2\pi)^{-2} \delta^4(P^{(1)} + P^{(2)} - P^{(3)} - P^{(4)}) \sum_{m'_1 m'_2} C_{m_1 m'_1}^{(1)} C_{m_2 m'_2}^{(2)} \end{aligned} \quad (2.18)$$

$$M_{m'_1 m'_2 m_3 m_4}(a_{P^{(1)}} t, a_{P^{(2)}} t, a_{P^{(3)}}, a_{P^{(4)}}) \quad .$$

In this case, the subscript m_i denotes (besides the internal quantum numbers) the projection of the spin on the z axis measured in the rest frame of the particle (i) [or (ic)], which is obtained from an arbitrary initial frame by means of the Lorentz transformation $a_i = a_{P(i)}$. If the Lorentz transformations are properly chosen, Eq. (2.18) gives directly the helicity amplitudes. One of the advantages of this formalism is that we can choose in every case the elements a_i in the most convenient way.

If we denote by $\rho_{m_3 m'_3}^{(3)}$ and $\rho_{m_4 m'_4}^{(4)}$ the density matrices which describe the polarizations (and the internal quantum numbers) of the incoming particles in the above described rest frames, the differential cross-section for the process $(1,1)$ is given by

$$2\pi \frac{d\sigma}{dt d\varphi} = (64\pi)^{-1} [(P^{(3)}, P^{(4)})^2 - M_3^2 M_4^2]^{-1}.$$

$$\sum_{m_1 m_2} \sum_{m_3 m_4} \sum_{m'_3 m'_4} M_{m_1 m_2 m_3 m_4} (a_1 t, a_2 t, a_3, a_4).$$

$$M_{m_1 m_2 m'_3 m'_4} (a_1 t, a_2 t, a_3, a_4) \rho_{m_3 m'_3}^{(3)} \rho_{m_4 m'_4}^{(4)}. \quad (2.19)$$

The density matrix $\rho_{m_1 m_2 m'_1 m'_2}^{(F)}$, which describes the polarizations (and the internal quantum numbers) of the outgoing particles, measured in the rest frames described above, is given by

$$\rho_{m_1 m_2 m'_1 m'_2}^{(F)} = K \sum_{m''_1 m''_2} \sum_{m'''_1 m'''_2} C_{m_1 m'_1}^{(1)} \overline{C_{m'_1 m_1}^{(1)}} C_{m_2 m'_2}^{(2)} \overline{C_{m'_2 m_2}^{(2)}}.$$

$$\sum_{m_3 m_4} \sum_{m'_3 m'_4} M_{m''_1 m''_2 m_3 m_4} (a_1 t, a_2 t, a_3, a_4)$$

$$M_{m''_1 m''_2 m'_3 m'_4} (a_1 t, a_2 t, a_3, a_4) \rho_{m_3 m'_3}^{(3)} \rho_{m_4 m'_4}^{(4)}, \quad (2.20)$$

where K is a normalization factor.

If the particles (3), (1c) and the particles (4), (2c) are of the same kind and if we take

$$\begin{cases} \varphi_m^{(3)} = \varphi_m^{(1c)} \\ \varphi_m^{(4)} = \varphi_m^{(2c)} \end{cases}, \quad (2.21)$$

the optical theorem gives the following formula for the total cross-section

$$\sigma^T = \frac{1}{2} [(P^{(3)} P^{(4)})^2 - M_3^2 M_4^2]^{-\frac{1}{2}},$$

$$\cdot \sum_{m_1 m_2} \sum_{m_3 m_4} \sum_{m'_1 m'_2} C_{m_1 m'_1}^{(1)} C_{m_2 m'_2}^{(2)}$$

$$\cdot M_{m'_1 m'_2 m_3 m_4} (a_3 t, a_4 t, a_3, a_4) \rho_{m_3 m_1}^{(3)} \rho_{m_4 m_2}^{(4)} \Big].$$

(2.22)

Similar formulae can be obtained for the other five reactions of Eq. (2.5).

3. - THE REPRESENTATIONS OF THE LITTLE GROUPS

In this Section, we summarize some of the results of Ref. 7) and describe the relevant irreducible representations of some of the little groups. From an analysis performed by Lee and Wick³¹⁾, it follows that, if we consider only electromagnetic and strong interactions, at the present stage of our knowledge, it is reasonable to assume that the elements of the internal symmetry group commute with s , t , and with the elements of the Poincaré group. It follows that the internal symmetry group can be treated separately. Therefore in the following, we shall identify $\tilde{A}$ with the group generated by $SL(2C)$, s and t , and we shall use the subscripts m_i only to indicate the spin variables.

We use, for some elements of $SL(2C)$, the following notation^{7),13)}

$$a_z(\zeta) = \exp\left(\frac{1}{2}\zeta\sigma^z\right), \quad u_z(\mu) = \exp\left(-\frac{i}{2}\mu\sigma^z\right),$$

$$e = u_z(0), \quad e' = u_z(2\pi). \quad (3.1)$$

In a similar way, we define the elements $a_x(\zeta)$, $a_y(\zeta)$, $u_x(\mu)$, $u_y(\mu)$. We adopt, for the Pauli matrices, the following form

$$\sigma_{mm'}^x = \delta_{m,-m'}, \quad \sigma_{mm'}^y = -2im\delta_{m,-m'},$$

$$\sigma_{mm'}^z = 2m\delta_{mm'}, \quad m, m' = \pm \frac{1}{2} \quad (3.2)$$

The multiplication rules of the group $\tilde{A}$ can be obtained from the multiplication rules of $SL(2C)$ and from the following equations⁷⁾

$$\begin{cases} ta = at, & a \in SL(2C); \\ \lambda u = u\lambda, & u \in SU(2); \\ \lambda a_z(\zeta)\lambda^{-1} = [a_z(\zeta)]^{-1}. \end{cases} \quad (3.3)$$

$$\begin{cases} t^2 = e, \\ \lambda^2 = e', \\ \lambda t = e' t \lambda. \end{cases} \quad (3.4)$$

The matrices $L(a)$ for $a \in SL(2C)$ are defined by

$$\begin{cases} L(a)(t, x, y, z) = (t', x', y', z') \text{ if} \\ t'e + x'\sigma^x + y'\sigma^y + z'\sigma^z = a(te + x\sigma^x + y\sigma^y + z\sigma^z)a^+, \end{cases} \quad (3.5)$$

and it is obviously

$$\begin{aligned} L(t)(t, x, y, z) &= (-t, -x, -y, -z), \\ L(\sigma)(t, x, y, z) &= (t, -x, -y, -z). \end{aligned} \quad (3.6)$$

It follows that $L(a_z(\xi))$ is a pure Lorentz transformation along the z axis with velocity $\tanh \xi$ and $L(u_z(\mu))$ is a rotation of an angle μ around the z axis.

Now we consider the four vectors Q_W^0 defined by

$$\begin{cases} Q_W^0 = (\sqrt{W}, 0, 0, 0), & W > 0; \\ Q_0^0 = 0; \\ Q_W^0 = (0, 0, 0, \sqrt{-W}), & W < 0; \end{cases} \quad (3.7)$$

and the corresponding little groups H_W defined by

$$\begin{cases} h \in H_W \text{ if} \\ h \in \tilde{A}, \quad L(h)Q_W^0 = Q_W^0. \end{cases} \quad (3.8)$$

Only the sign of W is relevant in the definition of H_W and therefore we shall use the symbols H_+ and H_- . Clearly, it is $H_0 = \tilde{A}$. The groups H_+ and H_- and their irreducible unitary representations have been described in Ref. 7). Here we summarize the results.

The group H_+ contains the elements of the form u and su where u belongs to the subgroup $SU(2)$ of $SL(2C)$. Its unitary irreducible representations depend on the parameters j (spin) and π (parity) and are given by

$$\begin{cases} R_{mm'}^{\pi j}(u) = R_{mm'}^j(u), & u \in SU(2); \\ R_{mm'}^{\pi j}(\Delta) = \pi \delta_{mm'}; \\ j = 0, \frac{1}{2}, 1, \dots; & \pi^2 = (-1)^{2j}; \end{cases} \quad (3.9)$$

where R^j are the usual representations of $SU(2)$ ³²⁾. Note that, according to our conventions, the parities of fermions are imaginary.

In order to find the connection with the more general notation used in the preceding Section, we write explicitly the internal quantum numbers α , α' and decompose an element of the group H_+ of Section 2 into the product of an element h of the purely kinematical group H_+ of the present Section and an element k of the internal symmetry group. Then we have

$$R_{m\alpha m'\alpha'}^{(i)}(hk) = R_{mm'}^{\pi_i j_i}(h) \hat{R}_{\alpha\alpha'}^{(i)}(k), \quad (3.10)$$

where $\hat{R}^{(i)}$ is a representation of the internal symmetry group. In a similar way, we have

$$C_{m\alpha m'\alpha'}^{(i)} = C_{mm'}^{j_i} \hat{C}_{\alpha\alpha'}^{(i)}, \quad (3.11)$$

where

$$C_{mm'}^j = (-1)^{j-m} \delta_{m, -m'} \quad (3.12)$$

and

$$\sum_{\alpha'} \hat{C}_{\alpha\alpha'}^{(i)} \overline{\hat{R}_{\alpha'\alpha''}^{(i)}(k)} = \sum_{\alpha'} \hat{R}_{\alpha\alpha'}^{(ic)}(k) \hat{C}_{\alpha'\alpha''}^{(i)} . \quad (3.13)$$

In the following, we shall no more consider the indices α , α' , and the corresponding matrices.

The group H_- contains the elements of the form v , $s'v$, $t'v$, $s't'v$, where v belongs to the subgroup $SU(1,1)$ of $SL(2C)$ and the elements s' , t' are given by

$$\begin{cases} s' = \mathcal{U}_y(\pi) s , \\ t' = \mathcal{U}_x(\pi) t . \end{cases} \quad (3.14)$$

The group $SU(1,1)$ [homomorphic to the connected part of the three-dimensional Lorentz group $O(2,1)$] contains all the elements of the form

$$v = \mathcal{U}_z(\mu) \mathcal{A}_x(\xi) \mathcal{U}_z(\nu) . \quad (3.15)$$

A set of unitary irreducible representations of H_- which is sufficient to write a Plancherel formula, has been described in Ref. 7) in terms of the representations of $SU(1,1)$ 33). We use for the representations of $SU(1,1)$ the notations and the phase conventions introduced in Ref. 13). There are two classes of irreducible unitary representations of H_- :

(i) - the continuous class defined by

$$\left\{ \begin{array}{l} \tilde{D}_{mm'}^{\sigma\tau\epsilon l}(v) = D_{mm'}^{\epsilon l}(v) \quad , \quad v \in SU(1,1) \\ \tilde{D}_{mm'}^{\sigma\tau\epsilon l}(\sigma') = \sigma \delta_{m,-m'} \quad ; \\ \tilde{D}_{mm'}^{\sigma\tau\epsilon l}(t') = \tau \delta_{m,-m'} \quad ; \\ \operatorname{Re} l = -\frac{1}{2} \quad , \quad \operatorname{Im} l > 0 \quad ; \quad \epsilon = 0, \frac{1}{2} \quad ; \\ m, m' = \epsilon, \epsilon \pm 1, \epsilon \pm 2, \dots, \\ \sigma = \pm 1 \quad ; \quad \tau^2 = (-1)^{2\epsilon} \end{array} \right. \quad (3.16)$$

(ii) - the discrete class defined by

$$\left\{ \begin{array}{l} \tilde{D}_{mm'}^{\pi k}(v) = \begin{cases} D_{mm'}^{k+}(v) & \text{for } m, m' = k, k+1, \dots \\ D_{mm'}^{k-}(v) & \text{for } m, m' = -k, -k-1, \dots \\ 0 & \text{for other values of } m, m' \end{cases} \\ \tilde{D}_{mm'}^{\pi k}(\sigma') = \pi (-1)^{m+k} \delta_{m,-m'} \quad ; \\ \tilde{D}_{mm'}^{\pi k}(t') = (-1)^{m-k} \delta_{m,-m'} \quad ; \\ k = \frac{1}{2}, 1, \frac{3}{2}, \dots \quad ; \quad \pi^2 = (-1)^{2k} \quad ; \\ m, m' = \pm k, \pm(k+1), \dots \end{array} \right. \quad (3.17)$$

As shown in Ref. 4), the matrix elements $D_{mm'}^{\epsilon l}(v)$ and therefore also the matrix elements $\tilde{D}_{mm'}^{\sigma\tau\epsilon l}(h)$ can be continued analytically in the whole complex l plane. They satisfy the following condition 7)

$$(U_m^l)^{-1} \tilde{D}_{mm'}^{\sigma\tau\epsilon l}(h) U_{m'}^l = \tilde{D}_{mm'}^{\sigma'\tau'\epsilon l'}(h), \quad h \in H_-, \quad (3.18)$$

where

$$\sigma' = (-1)^{2\epsilon} \sigma, \quad \tau' = (-1)^{2\epsilon} \tau = \overline{\tau}, \quad l' = -l-1, \quad (3.19)$$

and

$$U_m^l = \frac{\Gamma(l+m+1)}{\Gamma(m-l)}. \quad (3.20)$$

In order to study the asymptotic properties of the functions $D_{mm'}^{\epsilon l}(v)$ it is useful to consider the decomposition 4), 13)

$$D_{mm'}^{\epsilon l}(v) = A_{mm'}^l(v) + U_m^l A_{mm'}^{-l-1}(v) (U_{m'}^l)^{-1}, \quad (3.21)$$

where

$$\begin{aligned} A_{mm'}^l(\mu_z(\mu) a_x(\zeta) \mu_z(\nu)) &= \\ &= \exp(-i m \mu - i m' \nu) a_{mm'}^l(\zeta), \quad \zeta > 0. \end{aligned} \quad (3.22)$$

The functions $a_{mm'}^l(\zeta)$ are described in Refs. 4), 13). Their asymptotic behaviour is given by

$$a_{mm'}^l(\zeta) = 4^{l+1} (-1)^{m-m'} \Gamma(-2l-1) [\Gamma(-l-m) \Gamma(-l+m)]^{-1} \cdot \exp[\zeta(-l-1)] \left[1 - 2 \frac{mm'}{l+1} \exp(-\zeta) + O[\exp(-2\zeta)] \right] \quad (3.23)$$

From Eqs. (3.16) and (3.21), we easily obtain the decomposition

$$\tilde{D}_{mm'}^{\sigma\tau\ell}(h) = \tilde{A}_{mm'}^{\sigma\tau\ell}(h) + U_m^l \tilde{A}_{mm'}^{\sigma'\tau'\ell'}(h) (U_{m'}^l)^{-1}, \quad (3.24)$$

where σ' , τ' and ℓ' are given by Eq. (3.19) and we have defined the new functions

$$\left\{ \begin{array}{l} \tilde{A}_{mm'}^{\sigma\tau\ell}(v) = A_{mm'}^l(v), \\ \tilde{A}_{mm'}^{\sigma\tau\ell}(\sigma'v) = \sigma A_{-m,m'}^l(v), \\ \tilde{A}_{mm'}^{\sigma\tau\ell}(\tau'v) = \tau A_{-m,m'}^l(v), \\ \tilde{A}_{mm'}^{\sigma\tau\ell}(\sigma'\tau'v) = \sigma\tau A_{mm'}^l(v), \end{array} \right. \quad v \in SU(1,1) \quad (3.25)$$

Note that the functions $a_{mm'}^l(\zeta)$ in general are singular for $\zeta = 0$. It follows that the functions (3.25) cannot be defined if the argument has one of the forms $u_z(\mu)$, $s'u_z(\mu)$, $t'u_z(\mu)$, $s't'u_z(\mu)$. In these cases the decomposition (3.24) cannot be written.

4. - THE SCATTERING AMPLITUDE AS A FUNCTION DEFINED ON A
LITTLE GROUP

As we have said in the Introduction, the first step to perform the crossed partial wave analysis is to write the scattering amplitude at fixed momentum transfer $Q = Q_W^0$ [see Eq. (3.7)] as a function defined on the little group H_W . This problem was treated in Ref. 4). Here we use an equivalent procedure, which, making use of the formalism described in Section 2, permits to obtain more easily more complete results.

Following Ref. 7), for every possible value of W , we define the four elements b_{iW} of $\tilde{A}$ with the following property

$$L(b_{1W})P^{o(1)} + L(b_{3W})P^{o(3)} = L(b_{4W})P^{o(4)} + L(b_{2W})P^{o(2)} = Q_W^0. \quad (4.1)$$

The most convenient choice is, for $W \leq 0$,

$$\begin{cases} b_{1W} = a_z(\zeta_{1W}) t \mu_z(\pi), \\ b_{2W} = a_z(\zeta_{2W}), \\ b_{3W} = a_z(\zeta_{3W}), \\ b_{4W} = a_z(\zeta_{4W}) t \mu_z(\pi), \end{cases} \quad (4.2)$$

where the parameters ζ_{iW} can be obtained from Eq. (4.1) and are given explicitly in Ref. 7). In the equal mass case, i.e., if the conditions (1.2) are satisfied, for $W = 0$ we have

$$\begin{cases} b_{10} = b_{40} = t \mu_z(\pi), \\ b_{30} = b_{20} = e. \end{cases} \quad (4.3)$$

In the unequal mass case, the parameters $\sum iW$ become singular at $W = 0$. This is due to the fact that, if the conditions (1.2) are not satisfied, the four-momentum transfer can never vanish. Then, we should take

$$Q_0^0 = (q, 0, 0, q) \quad , \quad q \neq 0 \quad , \quad (4.4)$$

where q is arbitrary, and also the group H_0 would have a different structure. We do not treat here this case.

It is possible to prove that, given four arbitrary elements of $\tilde{A}$ which satisfy Eq. (2.7), they can be written in the following way

$$\begin{cases} a_1 = a \, h \, b_{1W} \, u_1 \, , \\ a_2 = a \, b_{2W} \, t \, u_2 \, , \\ a_3 = a \, h \, b_{3W} \, u_3 \, , \\ a_4 = a \, b_{4W} \, t \, u_4 \, , \end{cases} \quad \begin{cases} a \in \tilde{A} \, , \\ h \in H_W \, , \\ u_i \in H_+ \, . \end{cases} \quad (4.5)$$

It follows that, if we define the function

$$T_{m_1 m_2 m_3 m_4}^W(h) = \sum_{m'_2 m'_4} C_{m_2 m'_2}^{(2)} C_{m_4 m'_4}^{(4)} \cdot M_{m_1 m'_2 m_3 m'_4}(h b_{1W}, b_{2W} t, h b_{3W}, b_{4W} t) \quad , \quad h \in H_W. \quad (4.6)$$

the function M can be completely reconstructed starting from the function T and using the invariance property (2.10) and the covariance properties (2.8). In other words, the scattering amplitude is completely described by the function T , which is defined on the little group H_W .

The covariance conditions (2.8) for the function M give rise to some covariance conditions for the function T . In fact, if we define the groups

$$\begin{cases} K_W^A = l_{1W} H_+ l_{1W}^{-1} \cap l_{3W} H_+ l_{3W}^{-1} , \\ K_W^B = l_{4W} H_+ l_{4W}^{-1} \cap l_{2W} H_+ l_{2W}^{-1} , \end{cases} \quad (4.7)$$

from Eqs. (4.6), (2.8) and (2.13), we obtain

$$\begin{aligned} T_{m_1 m_2 m_3 m_4}^W(l, k) &= \sum_{m'_1 m'_3} T_{m'_1 m_2 m'_3 m_4}^W(l) R_{m'_1 m_1}^{(1)}(l_{1W}^{-1} k l_{1W}) \cdot \\ &\cdot R_{m'_3 m_3}^{(3)}(l_{3W}^{-1} k l_{3W}) , \quad l \in H_W , k \in K_W^A , \end{aligned} \quad (4.8)$$

$$\begin{aligned} T_{m_1 m_2 m_3 m_4}^W(k, l) &= \sum_{m'_2 m'_4} R_{m_2 m'_2}^{(2c)}(l_{2W}^{-1} k l_{2W}) \cdot \\ &\cdot R_{m_4 m'_4}^{(4c)}(l_{4W}^{-1} k l_{4W}) T_{m_1 m'_2 m_3 m'_4}^W(l) , \quad l \in H_W , k \in K_W^B . \end{aligned} \quad (4.9)$$

These are the only properties of the function T which can be derived from the invariance and covariance properties of the function M . If we use the choice (4.2), for $W < 0$ the groups K_W^A and K_W^B coincide and are composed by the elements of the forms $u_z(\mu)$ and $s'u_z(\mu)$. Then the conditions (4.8) and (4.9) can be written in the more explicit form

$$\begin{cases} T_{m_1 m_2 m_3 m_4}^W(l, u_z(\mu)) = \exp[-i\mu(m_1 + m_3)] T_{m_1 m_2 m_3 m_4}^W(l) , \\ T_{m_1 m_2 m_3 m_4}^W(u_z(\mu) l) = \exp[-i\mu(m_2 + m_4)] T_{m_1 m_2 m_3 m_4}^W(l) , \end{cases} \quad (4.10)$$

$$\begin{cases} T_{m_1 m_2 m_3 m_4}^W(l, \gamma) = \pi_1 \pi_3 (-1)^{j_1 - m_1 + j_3 - m_3} T_{-m_1, m_2, -m_3, m_4}^W(l) , \\ T_{m_1 m_2 m_3 m_4}^W(\gamma', l) = \pi_2 \pi_4 (-1)^{j_2 + m_2 + j_4 + m_4} T_{m_1, -m_2, m_3, -m_4}^W(l) . \end{cases} \quad (4.11)$$

We have used the identity

$$R_{m m'}^{\pi j} [u_y(\pi)] = (-1)^{j+m} \delta_{m, -m'} . \quad (4.12)$$

If the particles (1) and (3) are of the same kind, using the identity ⁷⁾

$$\begin{cases} b_{1W} = t' b_{3W} u_y(\pi) , \\ b_{3W} = t' b_{1W} u_y(\pi) , \end{cases} \quad (4.13)$$

we obtain, from Eq. (2.9), for $W \leq 0$

$$T_{m_1 m_2 m_3 m_4}^W(l) = \xi^A (-1)^{m_1 + m_3} T_{-m_3, m_2, -m_1, m_4}^W(l t') . \quad (4.14)$$

In a similar way, if the particles (2) and (4) are of the same kind we have

$$T_{m_1 m_2 m_3 m_4}^W(l) = \xi^B (-1)^{m_2 + m_4} T_{m_1, -m_4, m_3, -m_2}^W(t' l) . \quad (4.15)$$

The factors ξ^A and ξ^B have been inserted in order to take into account the fact that also the internal quantum numbers have to be permuted. We have assumed that the amplitude has been decomposed into

two parts, one symmetric and the other antisymmetric with respect to the permutation of the internal quantum numbers of the particles (1) and (3) [or of the particles (2) and (4)]. The factor ζ^A (or ζ^B) has to be taken equal to +1 for the symmetric part and equal to -1 for the antisymmetric part.

If $W = 0$, from Eq. (4.3), we easily see that it is $K_0^A = K_0^B = H_+$. Then, if we take into account the identity

$$R_{mm'}^{\pi j}(t^{-1}ht) = R_{mm'}^{\pi j}(h) \quad , \quad h \in H_+ \quad , \quad (4.16)$$

Eqs. (4.8) and (4.9) take the form

$$T_{m_1 m_2 m_3 m_4}^0(a) = \sum_{m'_1 m'_3} T_{m'_1 m_2 m'_3 m_4}^0(a) (-1)^{m_1 - m'_1} R_{m'_1 m_1}^{\pi_1 j_1}(h) R_{m'_3 m_3}^{\pi_3 j_3}(h) \quad , \quad a \in \tilde{A} \quad , \quad h \in H_+ \quad , \quad (4.17)$$

$$T_{m_1 m_2 m_3 m_4}^0(ha) = \sum_{m'_2 m'_4} (-1)^{m_2 - m'_2} R_{m'_2 m_2}^{\pi_2 j_2}(h) T_{m_1 m'_2 m_3 m'_4}^0(a) \quad , \quad (4.18)$$

We can use these conditions to write Eqs. (4.14) and (4.15) (for $W = 0$) in the simpler form

$$T_{m_1 m_2 m_3 m_4}^0(a) = \zeta^A (-1)^{m_1 + m_3} T_{m_3 m_2 m_1 m_4}^0(at) \quad , \quad (4.19)$$

$$T_{m_1 m_2 m_3 m_4}^0(a) = \zeta^B (-1)^{m_2 + m_4} T_{m_1 m_4 m_3 m_2}^0(ta) \quad , \quad (4.20)$$

The properties of the amplitude at $W = 0$ can more easily be described by means of the new function

$$\begin{aligned} \tau_{j_B m_B j_A m_A}^W(a) &= \sum_{m_1 m_3} \sum_{m_2 m_4} C(j_1, j_3, j_A; m_1, m_3, m_A) \cdot \\ &\cdot C(j_4, j_2, j_B; m_4, m_2, m_B) (-1)^{j_1 - m_1 + j_4 - m_4} \cdot \\ &\cdot T_{m_1 m_2 m_3 m_4}^W(a) \end{aligned} \quad (4.21)$$

Using well-known properties of the Clebsch-Gordan coefficients ³²⁾, from Eqs. (4.17) and (4.18), we have

$$\tau_{j_B m_B j_A m_A}^0(a) = \sum_{m'_A} \tau_{j_B m_B j_A m'_A}^0(a) R_{m'_A m_A}^{\pi_1 \pi_3, j_A}(h), \quad (4.22)$$

$$\tau_{j_B m_B j_A m_A}^0(ha) = \sum_{m'_B} R_{m'_B m_B}^{\pi_4 \pi_2, j_B}(h) \tau_{j_B m'_B j_A m_A}^0(a). \quad (4.23)$$

If we use the following symmetry property of the Clebsch-Gordan coefficients ³²⁾

$$C(j, j, j_A; m_1, m_3, m_A) = (-1)^{j_A - 2j} C(j, j, j_A; m_3, m_1, m_A), \quad (4.24)$$

from Eqs. (4.19) and (4.20) we obtain

$$\tau_{j_B m_B j_A m_A}^0(a) = \xi^0 (-1)^{j_A} \tau_{j_B m_B j_A m_A}^0(at), \quad (4.25)$$

$$\tau_{j_B m_B j_A m_A}^0(a) = \xi^B (-1)^{j_B} \tau_{j_B m_B j_A m_A}^0(ta). \quad (4.26)$$

The functions defined above are the most suitable for the group theoretical investigations. However, if we want to calculate the measurable quantities regarding the process (1,1), from Eq. (2.18) we see that it is more convenient to use the function

$$\tilde{T}_{m_1 m_2 m_3 m_4}^w(l_1) = \sum_{m'_1 m'_2} C_{m_1 m'_1}^{(1)} C_{m_2 m'_2}^{(2)} M_{m'_1 m'_2 m_3 m_4}(l_1 a_z(\xi_{1w})t, a_z(\xi_{2w})t, l_1 a_z(\xi_{3w}), a_z(\xi_{4w})) . \quad (4.27)$$

In this new function the indices m_i have directly the meaning of spin components of the particles involved in the process (1,1) measured in the rest systems which are obtained from the initial system by means of the Lorentz transformations which are the arguments of the M function in Eq. (4.27). Comparing this equation with Eq. (4.6) and using Eqs. (4.2) and (3.12), we have

$$\tilde{T}_{m_1 m_2 m_3 m_4}^w(l_1) = (i)^{-2(j_1 + j_4)} T_{-m_1, m_2, m_3, -m_4}^w(l_1) . \quad (4.28)$$

From Eq. (4.21), we have also

$$\begin{aligned} \tilde{T}_{m_1 m_2 m_3 m_4}^w(l_1) &= (i)^{2(m_1 + m_4)} \sum_{j_A m_A} \sum_{j_B m_B} \\ &C(j_1, j_3, j_A; -m_1, m_3, m_A) C(j_4, j_2, j_B; -m_4, m_2, m_B) \\ &\cdot \tilde{\sigma}_{j_B m_B j_A m_A}^w(l_1) . \end{aligned} \quad (4.29)$$

If the internal quantum numbers are taken into account, we should introduce into the right-hand side of Eqs. (4.28) and (4.29) the matrices $\hat{C}^{(1)} [\hat{C}^{(4)}]^{-1}$ [see Eq. (3.11)].

From the optical theorem (2.22), we have that, if Eq. (2.21) is satisfied, the total cross-section for the process (3) + (4) with unpolarized beam and target is given by

$$\sigma^T = [2 M_3 M_4 \sinh \zeta (2j_3+1)(2j_4+1)]^{-1} \cdot \int_{-1}^1 \sum_{m_3 m_4} \tilde{T}_{m_3 m_4 m_3 m_4}^0(a_z(\zeta)) d\zeta \quad (4.30)$$

where $\tanh \zeta$ is the velocity of the incident beam in the laboratory system. If we take into account the identity

$$\sum_m (-1)^{j+m} C(j, j, j_A; -m, m, m_A) = (2j+1)^{\frac{1}{2}} \delta_{0j_A} \delta_{0m_A}, \quad (4.31)$$

which follows from the formula ³²⁾

$$C(j, j, 0; m, m', 0) = (-1)^{j-m} (2j+1)^{-\frac{1}{2}} \delta_{m, -m'} = (2j+1)^{-\frac{1}{2}} C_{mm'}^j, \quad (4.32)$$

using Eq. (4.29) we obtain

$$\sigma^T = [2 M_3 M_4 \sinh \zeta]^{-1} [(2j_3+1)(2j_4+1)]^{-\frac{1}{2}} \cdot \int_{-1}^1 (-i)^{2(j_3+j_4)} \tau_{0000}^0(a_z(\zeta)) d\zeta \quad (4.33)$$

Note that in Eqs. (4.30) and (4.33) the internal quantum numbers which can appear in the right-hand side have to be taken fixed with $\alpha_1 = \alpha_3$, $\alpha_2 = \alpha_4$. In Eq. (4.33) we should also introduce the matrices $\hat{C}(1) [\hat{C}(4)]^{-1}$.

The square of the centre-of-mass energy (the Mandelstam variable s) can be written in terms of W (i.e., the Mandelstam variable t) and the variable ζ , which corresponds to the group element $h \in H_W$ [see Eqs. (3.15) and (6.1)]. In the equal mass case, we have

$$\left\{ \begin{array}{l} (p^{(3)} + p^{(4)})^2 = \omega_3^2 + \omega_4^2 + 2\omega_3 \omega_4 \cosh \zeta, \\ \omega_i = \left[M_i^2 - \frac{W}{4} \right]^{\frac{1}{2}}. \end{array} \right. \quad (4.34)$$

5. - THE CROSSED PARTIAL WAVE ANALYSIS FOR SPACE-LIKE FOUR-MOMENTUM TRANSFER

In this Section we perform, in the case $W < 0$, the second step of the crossed partial wave analysis, i.e., we expand the functions defined in the preceding Section in terms of the matrix elements of the representations of the little group H_- . It is well known that it is not sufficient to consider integrals containing the irreducible unitary representations of H_- . The origin of this difficulty has been examined in Ref. 7) and it was shown that the crossed partial wave amplitude cannot in general be considered as a function, but it is necessary to define it as a linear functional in a certain space of functions of the parameters which label the representations of H_- . Some possible functionals were studied in detail and their contributions to the function $T_{m_1 m_2 m_3 m_4}^W(h)$ were given explicitly. These contributions could be written as integrals containing matrix elements of some non-unitary irreducible representations of H_- . However, the initial assumptions were very general and a very wide class of functionals was suggested. In order to obtain some interesting result, it is necessary to make further assumptions based on some dynamical model.

We shall use the multiperipheral model 20), 22), 23), which can be treated in a very natural way by means of the crossed partial wave analysis. An explicit group theoretical treatment of the multiperipheral model has been given only in a very special case 2), in which all the particles were spinless. However, we think that this analysis is sufficient to suggest the form of the result which should be obtained in the more general case. These considerations lead to the following expansion formula

$$\begin{aligned}
 T_{m_1 m_2 m_3 m_4}^W(h) &= (4i)^{-1} \sum_{\sigma, \tau} \int_{C_W} F_{m_1 m_2 m_3 m_4}(W, \sigma, \tau, \ell) \cdot \\
 &\cdot \tilde{D}_{m_4+m_2, m_1+m_3}^{\sigma \tau \ell}(h) \eta(\epsilon, \ell) d\ell + \sum_{\pi, k} (2k-1) \cdot \\
 &\cdot F_{m_1 m_2 m_3 m_4}(W, \pi, k) \tilde{D}_{m_4+m_2, m_1+m_3}^{\pi k}(h),
 \end{aligned} \tag{5.1}$$

where

$$\eta(\epsilon, l) = (2l+1) \cotg \pi(l-\epsilon) \quad , \quad (5.2)$$

and

$$(-1)^{2\epsilon} = (-1)^{2(\gamma_1 + \gamma_3)} = (-1)^{2(\gamma_2 + \gamma_4)} \quad ; \quad (5.3)$$

the sum over k has to be extended to all the values larger than $\frac{1}{2}$ such that $|m_4+m_2| - k$ and $|m_1+m_3| - k$ are non-negative integers. The function F satisfies the condition

$$\begin{aligned} U_{m_4+m_2}^l F_{m_1 m_2 m_3 m_4}(W, \sigma, \tau, l) (U_{m_1+m_3}^l)^{-1} = \\ = F_{m_1 m_2 m_3 m_4}(W, (-1)^{2\epsilon} \sigma, (-1)^{2\epsilon} \tau, -l-1) \quad , \end{aligned} \quad (5.4)$$

and is meromorphic in the strip

$$-2 \leq \operatorname{Re} l \leq 1 \quad . \quad (5.5)$$

The integration path C_W goes from $-\frac{1}{2}-i\infty$ to $-\frac{1}{2}+i\infty$ and is symmetric for $l \rightarrow -l-1$. When W varies, the path has to be deformed in such a way that the poles of F do not cross it. Of course the path must not cross the poles of $\eta(\epsilon, l)$. A more detailed justification of these assumptions will be given elsewhere. Here we remark that the expansion (5.1) can be derived from one of the functionals proposed in Ref. ⁷⁾ and in the spinless case it coincides with the expansion obtained in Ref. ²⁾.

The conditions (4.10) are automatically satisfied by the expansion (5.1). The conditions (4.11) are satisfied if

$$\begin{cases}
F_{m_1 m_2 m_3 m_4}(W, \sigma, \tau, \ell) = \sigma \pi_1 \pi_3 (-1)^{j_1 - m_1 + j_3 - m_3} \\
\quad \cdot F_{-m_1, m_2, -m_3, m_4}(W, \sigma, \tau, \ell), \\
F_{m_1 m_2 m_3 m_4}(W, \pi, k) = \pi \pi_1 \pi_3 (-1)^{k + j_1 + j_3} F_{-m_1, m_2, -m_3, m_4}(W, \pi, k),
\end{cases} \quad (5.6)$$

$$\begin{cases}
F_{m_1 m_2 m_3 m_4}(W, \sigma, \tau, \ell) = \sigma \pi_2 \pi_4 (-1)^{j_2 + m_2 + j_4 + m_4} \\
\quad \cdot F_{m_1, -m_2, m_3, -m_4}(W, \sigma, \tau, \ell), \\
F_{m_1 m_2 m_3 m_4}(W, \pi, k) = \pi \pi_4 \pi_2 (-1)^{k + j_2 + j_4} F_{m_1, -m_2, m_3, -m_4}(W, \pi, k).
\end{cases} \quad (5.7)$$

The conditions (4.14) and (4.15) (which hold only if two particles are of the same kind) are satisfied if it is respectively

$$\begin{cases}
F_{m_1 m_2 m_3 m_4}(W, \sigma, \tau, \ell) = \\
\quad = \sum^A \tau (-1)^{m_1 + m_3} F_{-m_3, m_2, -m_1, m_4}(W, \sigma, \tau, \ell), \\
F_{m_1 m_2 m_3 m_4}(W, \pi, k) = \sum^A (-1)^k F_{-m_3, m_2, -m_1, m_4}(W, \pi, k),
\end{cases} \quad (5.8)$$

$$\begin{cases}
F_{m_1 m_2 m_3 m_4}(W, \sigma, \tau, \ell) = \\
\quad = \sum^B \tau (-1)^{m_4 + m_2} F_{m_1, -m_4, m_3, -m_2}(W, \sigma, \tau, \ell), \\
F_{m_1 m_2 m_3 m_4}(W, \pi, k) = \sum^B (-1)^k F_{m_1, -m_4, m_3, -m_2}(W, \pi, k).
\end{cases} \quad (5.9)$$

From Eqs. (5.6)-(5.9), it follows that some of the functions F must vanish. If we look for the values of the arguments for which these functions do not vanish identically, we find again the results summarized in the tables I and II of Ref. 7).

It is interesting to investigate the asymptotic behaviour of the expression (5.1) when the velocity corresponding to the Lorentz transformation h approaches the velocity of light. We neglect the contribution of the discrete class ³⁴⁾ and use Eqs. (3.24) and (5.4) to write the contribution of the continuous class in the form

$$(2i)^{-1} \sum_{\sigma, \tau} \int_{C_W} F_{m_1 m_2 m_3 m_4}(W, \sigma, \tau, -l-1) \cdot$$

$$\tilde{A}_{m_2+m_4, m_1+m_3}^{\sigma, \tau, -l-1}(h) \eta(\epsilon, l) dl.$$

(5.10)

Taking into account Eq. (3.23), we see that, if we shift the integration path on the left, the contributions of the poles crossed by the path give an asymptotic approximation of the scattering amplitude. We assume that for different values of σ and τ the poles of the function $F_{m_1 m_2 m_3 m_4}(W, (-1)^{2\epsilon} \sigma, (-1)^{2\epsilon} \tau, -l-1)$ appear (prescinding from accidental coincidences) in different points of the complex l plane, so that we can assign to every pole definite values of the quantum numbers σ , τ and ϵ . We assume that the residues of these poles can be written in a factorized form and we define the residue of a pole at $l = l_0(W)$ with quantum numbers σ and τ in the following way

$$\lim_{l \rightarrow l_0(W)} [\pi F_{m_1 m_2 m_3 m_4}(W, (-1)^{2\epsilon} \sigma, (-1)^{2\epsilon} \tau, -l-1) \cdot$$

$$(l - l_0(W))] = \beta_{m_4 m_2}^{(B)W} \beta_{m_1 m_3}^{(A)W}.$$

(5.11)

From Eqs. (5.6)-(5.9) we can derive immediately some important properties of the quantities $\beta_{m_4 m_2}^{(B)W}$ and $\beta_{m_1 m_3}^{(A)W}$. From Eqs. (5.10) and (5.11) we have that the contribution of the pole we are considering is given by

$$T_{m_1 m_2 m_3 m_4}^W(h) \sim \beta_{m_4 m_2}^{(B)W} \beta_{m_1 m_3}^{(A)W} \eta(\epsilon, l(w)).$$

$$\tilde{A}_{m_2+m_4, m_1+m_3}^{\sigma', \tau', -l(w)-1}(h),$$

(5.12)

where σ' and τ' are defined by Eq. (3.19). Of course, only the poles which lie on the left of the integration path have to be considered. If we examine in detail the expression (5.12), we see that it has the form of a Regge pole contribution with signature τ and parity $\sigma\tau$. Note that, according to our conventions, the signature and the parity of a fermion trajectory are imaginary. In the following we shall call directly "Regge poles" the poles of the function F which lie on the left of the integration path.

From Eq. (5.4), we see that if there is a Regge pole at $l = l_0$, there is also a pole at $l = -l_0 - 1$ which lies on the right of the integration path. We call poles of this kind "mirror poles". They do not contribute to the asymptotic behaviour of the amplitude, but can give rise to interesting effects when they cross a Regge pole. From Eq. (5.4), we see that if $\epsilon = 0$ the mirror pole has the same quantum numbers as the corresponding Regge pole and if $\epsilon = \frac{1}{2}$ it has opposite σ and τ .

We do not consider here the contributions of the poles of the function $\eta(\epsilon, l)$. They are very similar to the contributions of the representations of the discrete class. As briefly suggested in Ref. 4), if some conditions (which we shall examine elsewhere) are satisfied, the contribution of the poles of $\eta(\epsilon, l)$ cancel the contributions of the representations of the discrete class.

It may be useful to remark that the functions $\tilde{A}_{mm'}^{\sigma\tau l}(h)$ have poles for half integral values of $l - m$, but these poles are cancelled by the zeros of $\eta(\epsilon, l)$.

6. -

THE REPRESENTATIONS OF THE GROUP $\tilde{A}$

The unitary irreducible representations of the group $\tilde{A}$ can be built by means of the unitary irreducible representations of its subgroup $SL(2C)$, which have been studied and classified by Gelfand and Naimark ³⁵⁾. The explicit form of the matrix elements of the irreducible representations of $SL(2C)$ has been investigated by Ström ^{36), 37)}. We use here the notations and the phase conventions introduced in Ref. ¹³⁾. Every element of $SL(2C)$ can be written in the form

$$\begin{aligned} a &= u_z(\mu_1) u_y(\theta_1) u_z(\gamma_1) a_z(\zeta) u_y(\theta_2) u_z(\gamma_2) = \\ &= u_1 a_z(\zeta) u_2 \quad ; \quad u_1, u_2 \in SU(2). \end{aligned} \quad (6.1)$$

The matrix elements of the unitary irreducible representations, which depend on the two parameters M and λ , can be written in the following way

$$\begin{aligned} \mathcal{D}_{j m j' m'}^{M \lambda}(u_1 a_z(\zeta) u_2) &= \sum_{m''} R_{m m''}^j(u_1) d_{m'' j j'}^{M \lambda}(\zeta) R_{m'' m'}^{j'}(u_2); \\ \operatorname{Re} \lambda &= 0; \operatorname{Im} \lambda > 0; M = 0, \pm \frac{1}{2}, \pm 1, \dots; \quad j, j' = |M|, |M|+1, \dots \end{aligned} \quad (6.2)$$

where $R_{mm'}^j(u)$ are the irreducible representations of $SU(2)$ ³²⁾ and the functions $d_{mj j'}^{M \lambda}(\zeta)$ have been defined in Ref. ¹³⁾. We shall not consider here the representations of the complementary series and the corresponding representations of $\tilde{A}$.

The matrix elements (6.2) can be continued analytically in the whole complex λ plane and they satisfy the condition ¹³⁾

$$\mathcal{D}_{j m j' m'}^{-M, -\lambda}(a) = \left(\mathcal{U}_{j'}^{M \lambda} \right)^{-1} \mathcal{D}_{j m j' m'}^{M \lambda}(a) \mathcal{U}_{j'}^{M \lambda}, \quad (6.3)$$

where

$$U_{\dot{j}}^{M\lambda} = U_{\dot{j}}^{-M,\lambda} = \prod_{\Delta=|M|}^{\dot{j}} \frac{\Delta-\lambda}{\Delta+\lambda} \quad (6.4)$$

The representations of the group which contains $SL(2\mathbb{C})$ and the space reflection are described in Ref. ³⁵⁾; it is very easy to modify slightly this treatment in order to introduce also the element t and to take into account the fact that $s^2 = e' \neq e$. We obtain in this way the representations $\tilde{\mathcal{D}}^{\sigma\tau 0\lambda}$ and $\tilde{\mathcal{D}}^{\tau M\lambda}$ of $\tilde{A}$ defined by the following formulae

$$\left\{ \begin{aligned} \tilde{\mathcal{D}}_{\dot{j}m\dot{j}'m'}^{\sigma\tau 0\lambda}(a) &= \mathcal{D}_{\dot{j}m\dot{j}'m'}^{0\lambda}(a), \quad a \in SL(2\mathbb{C}), \\ \tilde{\mathcal{D}}_{\dot{j}m\dot{j}'m'}^{\sigma\tau 0\lambda}(\Delta) &= \sigma (-1)^{\dot{j}} \delta_{\dot{j}\dot{j}'} \delta_{mm'}, \\ \tilde{\mathcal{D}}_{\dot{j}m\dot{j}'m'}^{\sigma\tau 0\lambda}(t) &= \tau \delta_{\dot{j}\dot{j}'} \delta_{mm'}, \\ \sigma &= \pm 1, \quad \tau = \pm 1, \\ \operatorname{Re} \lambda &= 0, \quad \operatorname{Im} \lambda > 0, \\ \dot{j}, \dot{j}' &= 0, 1, 2, \dots \end{aligned} \right. \quad (6.5)$$

$$\left\{ \begin{aligned} \tilde{\mathcal{D}}_{\dot{j}mq\dot{j}'m'q'}^{\tau M\lambda}(a) &= \delta_{qq'} \mathcal{D}_{\dot{j}mq\dot{j}'m'}^{qM,\lambda}(a), \quad a \in SL(2\mathbb{C}), \\ \tilde{\mathcal{D}}_{\dot{j}mq\dot{j}'m'q'}^{\tau M\lambda}(\Delta) &= q^{2M} \delta_{q,-q'} (-1)^{\dot{j}+M} \delta_{\dot{j}\dot{j}'} \delta_{mm'}, \\ \tilde{\mathcal{D}}_{\dot{j}mq\dot{j}'m'q'}^{\tau M\lambda}(t) &= \tau (-iq)^{2M} \delta_{qq'} \delta_{\dot{j}\dot{j}'} \delta_{mm'}, \\ M &= \frac{1}{2}, 1, \frac{3}{2}, \dots; \quad \tau^2 = (-1)^{2M} \\ \operatorname{Re} \lambda &= 0, \quad \operatorname{Im} \lambda > 0, \\ \dot{j}, \dot{j}' &= M, M+1, \dots; \quad q, q' = \pm 1. \end{aligned} \right. \quad (6.6)$$

Starting from the identity

$$\mathcal{D}_{j m j' m'}^{M \lambda}([a_z(z)]^{-1}) = (-1)^{j-j'} \mathcal{D}_{j m j' m'}^{-M, \lambda}(a_z(z)) , \quad (6.7)$$

it is easy to show that these equations define unitary representations of $\tilde{A}$. It is also easy to show that these representations are irreducible and two by two inequivalent. They are not all the unitary irreducible representations of $\tilde{A}$, because we have not used the complementary series of representation of $SL(2\mathbb{C})$. However, we shall show in the Appendix A that they are sufficient to write a Plancherel formula.

The matrix elements defined by Eqs. (6.5) and (6.6) can be continued analytically in the whole complex λ plane and from Eq. (6.3) we can derive the conditions

$$\tilde{\mathcal{D}}_{j m j' m'}^{\sigma, \tau, 0, -\lambda}(a) = (\mathcal{U}_j^{0\lambda})^{-1} \tilde{\mathcal{D}}_{j m j' m'}^{\sigma \tau 0 \lambda}(a) \mathcal{U}_{j'}^{0\lambda} \quad (6.8)$$

and

$$\begin{aligned} & \tilde{\mathcal{D}}_{j m q j' m' q'}^{\bar{\tau}, M, -\lambda}(a) = \\ & = \sum_{q'' q'''} [\tilde{\mathcal{U}}_j^{M\lambda}]_{q q''}^{-1} \tilde{\mathcal{D}}_{j m q'' j' m' q'''}^{\tau M \lambda}(a) \tilde{\mathcal{U}}_{j' q'' q'}^{M \lambda} , \end{aligned} \quad (6.9)$$

where we have used the identity

$$\bar{\tau} = (-1)^{2M} \tau \quad (6.10)$$

and

$$\begin{cases} \tilde{\mathcal{U}}_{j q q'}^{M \lambda} = q^{2M} \delta_{q, -q'} \mathcal{U}_j^{M \lambda} , \\ \left[\tilde{\mathcal{U}}_j^{M \lambda} \right]_{q q'}^{-1} = \delta_{q, -q'} (q')^{2M} (\mathcal{U}_j^{M \lambda})^{-1} . \end{cases} \quad (6.11)$$

In order to study the asymptotic behaviour of the matrix elements $\mathcal{D}_{jmj'm'}^{M\lambda}(a)$ it is useful to consider the following decomposition ¹³⁾

$$\mathcal{D}_{jmj'm'}^{M\lambda}(a) = a_{jmj'm'}^{M\lambda}(a) + \mathcal{U}_j^{M\lambda} a_{jmj'm'}^{-M,-\lambda}(a) (\mathcal{U}_{j'}^{M\lambda})^{-1}, \quad (6.12)$$

where

$$a_{jmj'm'}^{M\lambda}(u_1, a_2(\zeta), u_2) = \sum_{m''} R_{mm''}^j(u_1) a_{m''j'j'}^{M\lambda}(\zeta) R_{m''m'}^{j'}(u_2), \quad \zeta > 0. \quad (6.13)$$

The functions $a_{mj j'}^{M\lambda}(\zeta)$ have been defined in Ref. ¹³⁾ and their asymptotic behaviour for $\zeta \rightarrow \infty$ is given by

$$\begin{aligned} a_{mj j'}^{M\lambda}(\zeta) &= a_{Mj j'}^{m\lambda}(\zeta) = (2j+1)^{\frac{1}{2}} (2j'+1)^{\frac{1}{2}} \cdot \\ &\cdot \left[(j-m)!(j+M)!(j'-m)!(j'+M)! \right]^{\frac{1}{2}} \left[(j+m)!(j-M)!(j'+m)!(j'-M)! \right]^{-\frac{1}{2}} \cdot \\ &\cdot (-1)^{j'+m} \Gamma(\lambda+j'+1) \Gamma(-\lambda-M) \left[(M-m)! \Gamma(\lambda-m+1) \Gamma(-\lambda+j'+1) \right]^{-1} \cdot \\ &\cdot \exp[\zeta(-\lambda-1-M+m)] \left[1 + O(\exp(-2\zeta)) \right], \\ &M \geq m. \end{aligned} \quad (6.14)$$

A similar decomposition for the matrix elements of the representations of $\tilde{A}$ is

$$\tilde{\mathcal{D}}_{jmj'm'}^{\sigma\tau\phi\lambda}(a) = \tilde{a}_{jmj'm'}^{\sigma\tau\phi\lambda}(a) + \mathcal{U}_j^{\phi\lambda} \tilde{a}_{jmj'm'}^{\sigma,\tau,\phi,-\lambda}(a) (\mathcal{U}_{j'}^{\phi\lambda})^{-1}, \quad (6.15)$$

where

$$\left\{ \begin{aligned} \tilde{a}_{jmj'm'}^{\sigma\tau\phi\lambda}(a) &= a_{jmj'm'}^{\phi\lambda}, \\ \tilde{a}_{jmj'm'}^{\sigma\tau\phi\lambda}(\rho a) &= \sigma(-1)^j a_{jmj'm'}^{\phi\lambda}, \\ \tilde{a}_{jmj'm'}^{\sigma\tau\phi\lambda}(\tau a) &= \tau a_{jmj'm'}^{\phi\lambda}, \\ \tilde{a}_{jmj'm'}^{\sigma\tau\phi\lambda}(\rho\tau a) &= \sigma\tau(-1)^j a_{jmj'm'}^{\phi\lambda}, \\ a &\in SL(2\mathbb{C}), \end{aligned} \right. \quad (6.16)$$

and

$$\begin{aligned} \tilde{D}_{jmjqj'm'q'}^{\tau M\lambda}(a) &= \tilde{a}_{jmjqj'm'q'}^{\tau M\lambda}(a) + \\ &+ \sum_{q''q'''} \tilde{U}_{jq''q'''}^{M\lambda} \tilde{a}_{jmqq''j'm'q'''}^{\tau, M, -\lambda}(a) [\tilde{U}_{j'}^{M\lambda}]_{q''q'''}^{-1}, \end{aligned} \quad (6.17)$$

where

$$\left\{ \begin{aligned} \tilde{a}_{jmjqj'm'q'}^{\tau M\lambda}(a) &= \delta_{qq'} a_{jmj'm'}^{qM,\lambda}(a), \\ \tilde{a}_{jmjqj'm'q'}^{\tau M\lambda}(\rho a) &= q^{2M} \delta_{q,-q'} (-1)^{j+M} a_{jmj'm'}^{-qM,\lambda}(a), \\ \tilde{a}_{jmjqj'm'q'}^{\tau M\lambda}(\tau a) &= \tau(-iq)^{2M} \delta_{qq'} a_{jmj'm'}^{qM,\lambda}(a), \\ \tilde{a}_{jmjqj'm'q'}^{\tau M\lambda}(\rho\tau a) &= \tau \delta_{q,-q'} (-i)^{2j} a_{jmj'm'}^{-qM,\lambda}(a), \\ a &\in SL(2\mathbb{C}). \end{aligned} \right. \quad (6.18)$$

These decompositions can be written only if the element a of $\tilde{A}$ has not one of the forms u, su, tu, stu , with $u \in SU(2)$.

7. - THE CROSSED PARTIAL WAVE ANALYSIS AT VANISHING FOUR-
MOMENTUM TRANSFER

In this Section, we treat the crossed partial wave analysis at vanishing four-momentum transfer in the equal mass case, i.e., we expand the function $\sum^0_{j_B m_B j_A m_A}(a)$ defined in Section 4 by means of the matrix elements of the representations of $\tilde{A}$ defined in Section 6. Also in this case it turns out that it is not sufficient to consider an integral containing the matrix elements of the unitary representations of $\tilde{A}$. We overcome this difficulty by means of a procedure perfectly analogous to that used in Section 5 for the case $W < 0$. It seems natural that this procedure can be justified also in this case by means of the multiperipheral model. However we shall not treat this argument in the present paper.

We use the following expansion formula, which is perfectly analogous to Eq. (5.1) (more details are given in the Appendix A)

$$\begin{aligned} \sum^0_{j_B m_B j_A m_A}(a) &= (8i\pi^4)^{-1} \sum_{\sigma\tau} \int_C \mathcal{F}_{j_B j_A}(\sigma, \tau, 0, \lambda) \tilde{\mathcal{D}}^{\sigma\tau 0\lambda}_{j_B m_B j_A m_A}(a) \\ &\cdot (-\lambda^2) d\lambda + (4i\pi^4)^{-1} \sum_{\tau M} \int_C \sum_{q q'} \mathcal{F}_{j_B q j_A q'}(\tau, M, \lambda) \\ &\cdot \tilde{\mathcal{D}}^{\tau M \lambda}_{j_B m_B q j_A m_A q'}(a) (M^2 - \lambda^2) d\lambda. \end{aligned} \quad (7.1)$$

The first term appears only if j_A and j_B are integral and the sum over M , which appears in the second term, has to be extended to all the positive values of M such that $j_A - M$ and $j_B - M$ are non-negative integers. The condition analogous to Eq. (5.4) is

$$\left\{ \begin{aligned} \mathcal{U}_{j_B}^{0\lambda} \mathcal{F}_{j_B j_A}(\sigma, \tau, 0, \lambda) (\mathcal{U}_{j_A}^{0\lambda})^{-1} &= \mathcal{F}_{j_B j_A}(\sigma, \tau, 0, -\lambda), \\ \sum_{q'' q'''} \tilde{\mathcal{U}}_{j_B q q''}^{M\lambda} \mathcal{F}_{j_B q'' j_A q'''}(\tau, M, \lambda) [\tilde{\mathcal{U}}_{j_A}^{M\lambda}]_{q''' q'}^{-1} &= \\ &= \mathcal{F}_{j_B q j_A q'}(\bar{\tau}, M, -\lambda) \end{aligned} \right. \quad (7.2)$$

We assume that the functions $\mathcal{F}$ are meromorphic in the strip

$$-2 \leq \operatorname{Re} \lambda \leq 2 \quad (7.3)$$

and that the path C , which goes from $-i\infty$ to $+i\infty$, is symmetric for $\lambda \rightarrow -\lambda$.

The conditions (4.22) and (4.23) are automatically satisfied by the right-hand side of Eq. (7.1) for $h \in \text{SU}(2)$. They are satisfied also for $h = s$ if the following conditions hold

$$\left\{ \begin{aligned} \mathcal{F}_{j_B j_A}(\sigma, \tau, 0, \lambda) &\neq 0 \\ \text{only if } \sigma = \bar{\pi}_1 \pi_3 (-1)^{j_A} &= \bar{\pi}_4 \pi_2 (-1)^{j_B}, \end{aligned} \right. \quad (7.4)$$

$$\left\{ \begin{aligned} \mathcal{F}_{j_B, q, j_A, -q'}(\tau, M, \lambda) &= \\ &= \bar{\pi}_1 \pi_3 (q')^{2M} (-1)^{j_A + M} \mathcal{F}_{j_B q j_A q'}(\tau, M, \lambda), \\ \mathcal{F}_{j_B, -q, j_A, q'}(\tau, M, \lambda) &= \\ &= \bar{\pi}_4 \pi_2 (-q)^{2M} (-1)^{j_B + M} \mathcal{F}_{j_B q j_A q'}(\tau, M, \lambda). \end{aligned} \right. \quad (7.5)$$

The conditions (4.25) and (4.26), which hold only if certain pairs of particles are of the same kind, are satisfied if it is respectively (note that in this case M must be integral)

$$\begin{cases} \mathcal{F}_{j_B j_A}(\sigma, \tau, 0, \lambda) \neq 0 \text{ only if } \tau = \xi^A (-1)^{j_A} \\ \mathcal{F}_{j_B q j_A q'}(\tau, M, \lambda) \neq 0 \text{ only if } \tau = \xi^A (-1)^{j_A - M} \end{cases} \quad (7.6)$$

$$\begin{cases} \mathcal{F}_{j_B j_A}(\sigma, \tau, 0, \lambda) \neq 0 \text{ only if } \tau = \xi^B (-1)^{j_B} \\ \mathcal{F}_{j_B q j_A q'}(\tau, M, \lambda) \neq 0 \text{ only if } \tau = \xi^B (-1)^{j_B - M} \end{cases} \quad (7.7)$$

In order to investigate the high-energy asymptotic behaviour of the amplitude, it is convenient to use Eqs. (6.15), (6.17) and (7.2) to write Eq. (7.1) in the new form

$$\begin{aligned} \mathcal{T}_{j_B m_B j_A m_A}^0(a) &= (4i\pi^4)^{-1} \sum_{\sigma, \tau} \int_C \mathcal{F}_{j_B j_A}(\sigma, \tau, 0, -\lambda) \\ &\cdot \tilde{a}_{j_B m_B j_A m_A}^{\sigma, \tau, 0, -\lambda}(a) (-\lambda^2) d\lambda + (2i\pi^4)^{-1} \sum_{\tau, M} \int_C \sum_{q, q'} \mathcal{F}_{j_B q j_A q'}(\tau, M, \lambda) \\ &\cdot \tilde{a}_{j_B m_B q j_A m_A q'}^{\tau, M, -\lambda}(a) (M^2 - \lambda^2) d\lambda. \end{aligned} \quad (7.8)$$

We call the poles of the functions $\mathcal{F}_{j_B j_A}(\sigma, \tau, 0, -\lambda)$ and $\mathcal{F}_{j_B q j_A q'}(\tau, M, -\lambda)$ which lie on the left of the integration path "Lorentz poles". We assume that a pole appears in these functions only for definite values of σ , τ and M . We call these parameters the "quantum numbers" of the pole. We assume that the residues of the Lorentz poles can be written in a factorized form and we define them in the following way

$$\lim_{\lambda \rightarrow \lambda_0} [(\lambda - \lambda_0) (2\pi^3)^{-1} \mathcal{F}_{j_B j_A}(\sigma, \tau, -\lambda)] = \gamma_{j_B}^{(B)} \gamma_{j_A}^{(A)} \quad (7.9)$$

or

$$\lim_{\lambda \rightarrow \lambda_0} [(\lambda - \lambda_0) \pi^{-3} f_{j_B q j_A q'}(\bar{\tau}, M, -\lambda)] = \gamma_{j_B q}^{(B)} \gamma_{j_A q'}^{(A)} \quad (7.10)$$

From Eq. (6.14), we see that if we shift the integration path on the left, the contributions to Eq. (7.8) of the Lorentz poles crossed by the path give rise to an asymptotic approximation of the scattering amplitude. From Eqs. (7.3)-(7.10), we have that the contribution of a Lorentz pole at $\lambda = \lambda_0$ with $M = 0$ and given values of σ and τ , has the form

$$\tau_{j_B m_B j_A m_A}^0(a) \sim -\lambda_0^2 \gamma_{j_B}^{(B)} \gamma_{j_A}^{(A)} \tilde{a}_{j_B m_B j_A m_A}^{\sigma, \tau, 0, -\lambda_0}(a) \quad (7.11)$$

and the contribution of a pole with quantum numbers M and τ ($M > 0$) at $\lambda = \lambda_0$ is

$$\tau_{j_B m_B j_A m_A}^0(a) \sim (M^2 - \lambda_0^2) \sum_{q q'} \gamma_{j_B q}^{(B)} \gamma_{j_A q'}^{(A)} \tilde{a}_{j_B m_B q j_A m_A q'}^{\tau, M, -\lambda_0}(a) \quad (7.12)$$

Note that the conditions (7.4)-(7.7) can easily be written in terms of the residues defined by Eqs. (7.9), (7.10). For instance from Eq. (7.5) it follows that

$$\begin{cases} \gamma_{j_A, -1}^{(A)} = \bar{\pi}_1 \pi_3 (-1)^{j_A + M} \gamma_{j_A, 1}^{(A)} \\ \gamma_{j_B, -1}^{(B)} = \bar{\pi}_4 \pi_2 (-1)^{j_B - M} \gamma_{j_B, 1}^{(B)} \end{cases} \quad (7.13)$$

Not all the values of the quantum numbers M, σ, τ can appear in the expansion of the amplitude for a given process. A first limitation comes from the fact that M cannot be greater than j_A and j_B , the maximum value of j_A is $j_1 + j_3$ and the maximum value of j_B is $j_4 + j_2$. It follows that the possible values of M are given by

$$\begin{cases} M = M_{\max}, M_{\max}-1, \dots; & M \geq 0, \\ M_{\max} = \min[(j_1+j_3), (j_4+j_2)]. \end{cases} \quad (7.14)$$

From Eq. (7.4), we obtain the following restrictions on the possible values of σ

$$\begin{cases} \sigma = \pi_1 \pi_3 & \text{if } j_1 = j_3 = 0, \\ \sigma = \pi_4 \pi_2 & \text{if } j_4 = j_2 = 0. \end{cases} \quad (7.15)$$

If the particles (1) and (3) are of the same kind, from Eqs. (7.6) and (7.4) we obtain

$$\begin{cases} \tau = \zeta^A & \text{if } M = 2j_1, \\ \tau = \sigma \zeta^A & \text{if } M = 0. \end{cases} \quad (7.16)$$

If the particles (4) and (2) are of the same kind we obtain in a similar way

$$\begin{cases} \tau = \zeta^B & \text{if } M = 2j_4, \\ \tau = \sigma \zeta^B & \text{if } M = 0. \end{cases} \quad (7.17)$$

If we calculate the spin averaged total cross-section by means of Eq. (4.33), from the remarks given above we have that only Lorentz poles with the following quantum numbers can give a contribution

$$M=0, \quad \sigma=1, \quad \tau = \xi^A = \xi^B. \quad (7.18)$$

We have assumed that particles and antiparticles belong to the same representation of the internal symmetry group, i.e., they are particles of the same kind.

From Eq. (7.2) we see that for every Lorentz pole at $\lambda = \lambda_0$ there is a corresponding "mirror pole" at $\lambda = -\lambda_0$, which does not give a contribution to the asymptotic expansion, but may have interesting consequences.

It is necessary to remark that the functions $\tilde{a}_{jm j' m'}^{\sigma \tau 0 \lambda}(a)$ and $\tilde{a}_{jm q j' m' q'}^{\tau M \lambda}(a)$ may have poles for certain integral values of $\lambda - M$. These poles can give a contribution to the asymptotic expansion which is at most of the order of $\exp(-(M+1)\zeta)$ and we shall not consider them here. If some rather natural conditions (which we shall examine elsewhere) are satisfied, the contributions of these poles cancel each other.

The properties of the Lorentz pole contributions can be more easily understood if we assume that all the velocities are parallel to the z axis. Then the process is described by $\tau_{j_A m_A j_B m_B}^0(a_z(\zeta))$ where ζ is the relative velocity and the quantities m_A and m_B , which must be equal, represent the helicity exchanged in the process. From Eq. (6.14), we see that the asymptotic behaviour of the contribution of a Lorentz pole is of the type

$$\begin{aligned} \tau_{j_A m_A j_B m_B}^0(a_z(\zeta)) &= \\ &= \delta_{m_A m_B} O[\exp[(\lambda - 1 - |m_A| - M)\zeta]] \end{aligned} \quad (7.19)$$

We see that the amplitudes with helicity exchange $\pm M$ dominate at high energy. If the particles (1) and (3) have vanishing mass, it can be shown that a Lorentz pole contributes only to the amplitudes with helicity exchange equal to $\pm M$. This fact can give some information about the physical meaning of the parameter M .

8. -

A CONNECTION BETWEEN THE MATRIX ELEMENTS OF THE
REPRESENTATIONS OF THE GROUP $\hat{A}$ AND OF THE GROUP H

In Ref. ¹³⁾ the following asymptotic expansion has been proved for non-integral values of 2λ

$$a_{jmj'm'}^{M\lambda}(v) \sim 2\pi \sum_{n=0}^{\infty} \eta(\epsilon, \lambda+n) W_{jm}^{M, -\lambda, n} \cdot V_{j'm'}^{M, -\lambda, n} A_{mm'}^{\lambda+n}(v), \quad v \in SU(1,1), \quad (8.1)$$

where the coefficients $W_{jm}^{M\lambda n}$ and $V_{jm}^{M\lambda n}$ have the properties

$$\begin{cases} W_{jm}^{-M, \lambda, n} = (-1)^{j-m+n} W_{jm}^{M\lambda n} \\ V_{jm}^{-M, \lambda, n} = (-1)^{j-m+n} V_{jm}^{M\lambda n} \end{cases}, \quad (8.2)$$

$$\begin{cases} W_{j, -m}^{M\lambda n} = (-1)^{j+M+n} W_{jm}^{M\lambda n} \\ V_{j, -m}^{M\lambda n} = (-1)^{j+M+n} V_{jm}^{M\lambda n} \end{cases}. \quad (8.3)$$

In this Section, we want to derive a similar expansion involving the functions defined by Eqs. (3.25), (6.16) and (6.18). From Eqs. (6.16) and (6.18), using Eqs. (3.14), (4.12) and

$$R_{mm'}^j(\mu_x(\pi)) = (-i)^{2j} \delta_{m, -m'}, \quad (8.4)$$

we obtain

$$\left\{ \begin{aligned} \tilde{a}_{jmj'm'}^{\sigma\tau o\lambda}(v) &= a_{jmj'm'}^{o\lambda}(v), \\ \tilde{a}_{jmj'm'}^{\sigma\tau o\lambda}(v') &= \sigma (-1)^m a_{j,-m,j',m'}^{o\lambda}(v), \\ \tilde{a}_{jmj'm'}^{\sigma\tau o\lambda}(t'v) &= \tau (-1)^j a_{j,-m,j',m'}^{o\lambda}(v), \\ \tilde{a}_{jmj'm'}^{\sigma\tau o\lambda}(v't') &= \sigma\tau (-1)^{j+m} a_{jmj'm'}^{o\lambda}(v), \end{aligned} \right. \quad (8.5)$$

$$v \in SU(1,1),$$

$$\left\{ \begin{aligned} \tilde{a}_{jmjqj'm'q'}^{\tau M\lambda}(v) &= \delta_{qq'} a_{jmj'm'}^{qM,\lambda}(v), \\ \tilde{a}_{jmjqj'm'q'}^{\tau M\lambda}(v') &= \delta_{q,-q'} q^{2M} (-1)^{M-m} a_{j,-m,j',m'}^{-qM,\lambda}(v), \\ \tilde{a}_{jmjqj'm'q'}^{\tau M\lambda}(t'v) &= \delta_{qq'} \tau q^{2M} (-1)^{M+j} a_{j,-m,j',m'}^{qM,\lambda}(v), \\ \tilde{a}_{jmjqj'm'q'}^{\tau M\lambda}(v't') &= \delta_{q,-q'} \tau (-1)^{j-m} a_{jmj'm'}^{-qM,\lambda}(v), \end{aligned} \right. \quad (8.6)$$

$$v \in SU(1,1).$$

If we substitute Eq. (8.1) into Eq. (8.5) and use Eqs. (8.2) and (8.3), we obtain

$$\left\{ \begin{aligned}
 \tilde{a}_{jmjm'}^{\sigma\tau\lambda}(v) &\sim 2\pi \sum_{n=0}^{\infty} \eta(\epsilon, \lambda+n) W_{jm}^{0,-\lambda,n} V_{j'm'}^{0,-\lambda,n} A_{mm'}^{\lambda+n}(v), \\
 \tilde{a}_{jmjm'}^{\sigma\tau\lambda}(\gamma'v) &\sim 2\pi \sum_{n=0}^{\infty} \eta(\epsilon, \lambda+n) \sigma W_{jm}^{0,-\lambda,n} V_{j'm'}^{0,-\lambda,n} A_{-m,m'}^{\lambda+n}(v), \\
 \tilde{a}_{jmjm'}^{\sigma\tau\lambda}(\epsilon'v) &\sim 2\pi \sum_{n=0}^{\infty} \eta(\epsilon, \lambda+n) \tau (-1)^n W_{jm}^{0,-\lambda,n} V_{j'm'}^{0,-\lambda,n} A_{-m,m'}^{\lambda+n}(v), \\
 \tilde{a}_{jmjm'}^{\sigma\tau\lambda}(\gamma'\epsilon'v) &\sim 2\pi \sum_{n=0}^{\infty} \eta(\epsilon, \lambda+n) \sigma \tau (-1)^n W_{jm}^{0,-\lambda,n} V_{j'm'}^{0,-\lambda,n} A_{mm'}^{\lambda+n}(v), \quad (8.7)
 \end{aligned} \right.$$

and, comparing with Eq. (3.25)

$$\begin{aligned}
 \tilde{a}_{jmjm'}^{\sigma\tau\lambda}(h) &\sim 2\pi \sum_{n=0}^{\infty} \eta(\epsilon, \lambda+n) W_{jm}^{0,-\lambda,n} \\
 &\cdot V_{j'm'}^{0,-\lambda,n} \tilde{A}_{mm'}^{\sigma, (-1)^n \tau, \lambda+n}(h), \quad h \in H_-. \quad (8.8)
 \end{aligned}$$

If we substitute Eq. (8.1) into Eq. (8.6) we have

$$\left\{ \begin{aligned}
 \tilde{a}_{jmqj'm'q'}^{\tau M \lambda}(v) &\sim \sum_{n=0}^{\infty} \Phi_1 W_{jm}^{qM, -\lambda, n} V_{j'm'}^{q'M, -\lambda, n} A_{mm'}^{\lambda+n}(v), \\
 \tilde{a}_{jmqj'm'q'}^{\tau M \lambda}(\gamma'v) &\sim \sum_{n=0}^{\infty} \Phi_2 W_{jm}^{qM, -\lambda, n} V_{j'm'}^{q'M, -\lambda, n} A_{-m,m'}^{\lambda+n}(v), \\
 \tilde{a}_{jmqj'm'q'}^{\tau M \lambda}(\epsilon'v) &\sim \sum_{n=0}^{\infty} \Phi_3 W_{jm}^{qM, -\lambda, n} V_{j'm'}^{q'M, -\lambda, n} A_{-m,m'}^{\lambda+n}(v), \\
 \tilde{a}_{jmqj'm'q'}^{\tau M \lambda}(\gamma'\epsilon'v) &\sim \sum_{n=0}^{\infty} \Phi_4 W_{jm}^{qM, -\lambda, n} V_{j'm'}^{q'M, -\lambda, n} A_{mm'}^{\lambda+n}(v), \quad (8.9)
 \end{aligned} \right.$$

where

$$\begin{cases} \Phi_1 = 2\pi \eta(\epsilon, \lambda+m) \delta q q', \\ \Phi_2 = 2\pi \eta(\epsilon, \lambda+m) \delta q, -q' q^{2M} (-1)^{M-m} (-1)^{j+m+n} (-1)^{j+qM+n}, \\ \Phi_3 = 2\pi \eta(\epsilon, \lambda+m) \delta q q' \tau q^{2M} (-1)^{j+q} (-1)^{j+qM+n}, \\ \Phi_4 = 2\pi \eta(\epsilon, \lambda+m) \delta q, -q' \tau (-1)^{j-m} (-1)^{j-m+n}, \end{cases} \quad (8.10)$$

or, after some calculation

$$\begin{cases} \Phi_1 = \pi \eta(\epsilon, \lambda+m) (1+qq'), \\ \Phi_2 = \pi \eta(\epsilon, \lambda+m) (1-qq'), \\ \Phi_3 = \pi \eta(\epsilon, \lambda+m) \tau (-1)^m (1+qq'), \\ \Phi_4 = \pi \eta(\epsilon, \lambda+m) \tau (-1)^m (1-qq'). \end{cases} \quad (8.11)$$

Comparing Eqs. (8.9) and (8.11) with Eq. (3.25), we obtain

$$\begin{aligned} \tilde{A}_{jm q j' m' q'}^{\tau M \lambda}(h) &\sim \pi \sum_{n=0}^{\infty} \eta(\epsilon, \lambda+m) W_{jm}^{qM, -\lambda, m} V_{j' m'}^{q'M, -\lambda, m} \\ &\cdot \left[\tilde{A}_{m m'}^{+1, (-1)^m \tau, \lambda+m}(h) + qq' \tilde{A}_{m m'}^{-1, (-1)^m \tau, \lambda+m}(h) \right], \quad h \in H_-. \end{aligned} \quad (8.12)$$

Eqs. (8.8) and (8.12) contain the result we were looking for. Remember that these equations have been proved for non-integral values of 2λ .

9. - THE DECOMPOSITION OF A LORENTZ POLE CONTRIBUTION INTO
REGGE POLE CONTRIBUTIONS

In the preceding Sections we have considered the scattering amplitude at $W = 0$ as a function defined over the group $\tilde{A}$. However, it is easy to show that if we know the values that this function assumes on the subgroup H_- , we may obtain, by means of the covariance relations (4.17), (4.18) or (4.22), (4.23), all the values which the function assumes on the group $\tilde{A}$. It follows that we may describe completely the scattering amplitude at $W = 0$ by means of a function defined over H_- . Therefore the procedure described in Section 5 can be used also for $W = 0$ [if the masses satisfy Eq. (1.2)].

However, it is necessary to remark that the full covariance property of the scattering amplitude at $W = 0$ considered as a function defined over H_- is given by

$$\begin{aligned} G_{j_B m_B j_A m_A}^0(k_1, h, k_2) &= \sum_{m'_B m'_A} R_{m_B m'_B}^{\pi_4 \pi_2, j_B}(k_1) \\ &\cdot G_{j_B m'_B j_A m'_A}^0(h) R_{m'_A m_A}^{\pi_1 \pi_3, j_A}(k_2), \end{aligned}$$

$$k_1, k_2 \in H_+; \quad h \in H_-; \quad k_1, h, k_2 \in H_-; \quad (9.1)$$

whereas the covariance conditions imposed in the treatment of Section 5 take into account only the case $k_1, k_2 \in K_W^A = K_W^B$ ($W < 0$). It follows for instance that the Regge pole contributions defined by Eq. (5.12) do not necessarily satisfy the condition (9.1), and it may happen that this condition is satisfied only if we consider a family of Regge pole contributions, with the parameters ℓ displaced from one another by integral numbers. This is the situation investigated by Gribov and Volkov²⁷⁾. These additional conditions which appear at $W = 0$ cannot be introduced in an easy and natural way in the formalism of Section 5. Therefore we

prefer to use for $W = 0$ the formalism of Section 7, which takes into account the full covariance conditions, and, then if it is necessary, we translate the results in the language of Section 5. This translation is the aim of the present Section.

If we consider a Lorentz pole contribution of the kind defined by Eq. (7.11), and we use the expansion formula (8.8), we obtain (for non-integral values of 2λ)

$$\begin{aligned} \mathcal{G}_{j_B m_B j_A m_A}^{(0)}(h) &\sim \sum_{n=0}^{\infty} \hat{\beta}_{j_B m_B}^{(B)n} \hat{\beta}_{j_A m_A}^{(A)n} \eta(0, -\lambda + n) \\ &\tilde{A}_{m_B m_A}^{\sigma, (-1)^n \tau, -\lambda + n}(h), \quad h \in H_-, \end{aligned} \quad (9.2)$$

where

$$\begin{cases} \hat{\beta}_{j_A m_A}^{(A)n} = -\lambda^2 2\pi \gamma_{j_A}^{(A)} V_{j_A m_A}^{0 \lambda n} \\ \hat{\beta}_{j_B m_B}^{(B)n} = \gamma_{j_B}^{(B)} W_{j_B m_B}^{0 \lambda n} \end{cases} \quad (9.3)$$

In a similar way, using Eq. (8.12), we can write a Lorentz pole contribution of the kind (7.12) in the form

$$\begin{aligned} \mathcal{G}_{j_B m_B j_A m_A}^{(0)}(h) &\sim \sum_{n=0}^{\infty} \sum_{\sigma} \hat{\beta}_{j_B m_B}^{(B)n\sigma} \hat{\beta}_{j_A m_A}^{(A)n\sigma} \eta(\epsilon, -\lambda + n) \\ &\tilde{A}_{m_B m_A}^{\sigma', (-1)^n \tau, -\lambda + n}(h), \quad \sigma' = (-1)^{2M} \sigma, \end{aligned} \quad (9.4)$$

where

$$\begin{cases} \hat{\beta}_{j_A m_A}^{(A)n\sigma} = \pi (M^2 - \lambda^2) \sum_{q'} \gamma_{j_A q'}^{(A)} V_{j_A m_A}^{q' M, \lambda, n} (q')^{\frac{1-\sigma'}{2}} \\ \hat{\beta}_{j_B m_B}^{(B)n\sigma} = \sum_q \gamma_{j_B q}^{(B)} W_{j_B m_B}^{q M, \lambda, n} q^{\frac{1-\sigma'}{2}} \end{cases} \quad (9.5)$$

The sums over q and q' can be eliminated by means of the condition (7.13) and we obtain

$$\left\{ \begin{aligned} \hat{\beta}_{j_A m_A}^{(A) m \sigma} &= \pi (M^2 - \lambda^2) V_{j_A m_A}^{M \lambda m} \left[1 + \sigma \bar{\pi}_1 \pi_3 (-1)^{M - m_A + m} \right] \chi_{j_A, +1}^{(A)} , \\ \hat{\beta}_{j_B m_B}^{(B) m \sigma} &= W_{j_B m_B}^{M \lambda m} \left[1 + \sigma \bar{\pi}_4 \pi_2 (-1)^{M + m_B + m} \right] \chi_{j_B, +1}^{(B)} . \end{aligned} \right. \quad (9.6)$$

It is clear that a term of the form

$$\tau_{j_B m_B j_A m_A}^0(l_1) \sim \eta(\epsilon, l_0) \hat{\beta}_{j_B m_B}^{(B)} \hat{\beta}_{j_A m_A}^{(A)} \tilde{A}_{m_B m_A}^{\sigma, \tau, -l_0 - 1}(l_1) \quad (9.7)$$

can also be written, using Eq. (4.21), in the form (5.12) if we take

$$\left\{ \begin{aligned} \beta_{m_1 m_3}^{(A)} &= (-1)^{j_1 - m_1} \sum_{j_A m_A} C(j_1, j_3, j_A; m_1, m_3, m_A) \hat{\beta}_{j_A m_A}^{(A)} , \\ \beta_{m_4 m_2}^{(B)} &= (-1)^{j_4 - m_4} \sum_{j_B m_B} C(j_4, j_2, j_B; m_4, m_2, m_B) \hat{\beta}_{j_B m_B}^{(B)} . \end{aligned} \right. \quad (9.8)$$

It follows that Eq. (9.7) represents the contribution of a Regge pole at $l = l_0$ with quantum numbers σ and τ . Therefore Eqs. (9.2) and (9.4) give the decomposition of a Lorentz pole contribution into an asymptotic series of Regge pole contributions.

If we use the symbolic notation $(0, \lambda)^{\sigma \tau}$ or $(M, \lambda)^{\tau}$ for the Lorentz poles and $(\epsilon, \lambda)^{\sigma \tau}$ for the Regge poles, the decompositions proved above can be summarized as follows

$$(0, \lambda)^{\sigma \tau} \rightarrow (0, \lambda-1)^{\sigma \tau} + (0, \lambda-2)^{\sigma, -\tau} + (0, \lambda-3)^{\sigma \tau} + \dots, \quad (9.9)$$

$$\begin{aligned} (M, \lambda)^{\tau} &\rightarrow (\epsilon, \lambda-1)^{+1, \tau} + (\epsilon, \lambda-1)^{-1, \tau} + (\epsilon, \lambda-2)^{+1, -\tau} \\ &+ (\epsilon, \lambda-2)^{-1, -\tau} + (\epsilon, \lambda-3)^{+1, \tau} + (\epsilon, \lambda-3)^{-1, \tau} + \dots, \\ (-1)^{2\epsilon} &= (-1)^{2M}, \end{aligned} \quad (9.10)$$

Note that these families of Regge poles are composed of infinite elements with alternating signature, as required by the argument given by Freedman and Wang¹⁶⁾.

Not all the Regge trajectories mentioned in Eqs. (9.9) and (9.10) must necessarily contribute to a given process to which the original Lorentz pole gives a contribution. In fact it is possible that some of the residues given by Eqs. (9.3) and (9.6) vanish. Sometimes this fact is required by one of the selection rules for the Regge trajectories⁷⁾. For instance in $\pi - N$ scattering the selection rule for the quantum number τ eliminates all the Regge trajectories which correspond to odd values of n .

It is also possible that some of the Regge trajectories generated by a Lorentz pole give a contribution to processes to which the original Lorentz pole cannot contribute (owing to some selection rule). In this case, of course, the contribution of these Regge trajectories must vanish in the forward direction. The simplest point of

view is to consider the Lorentz poles and the complete families of Regge trajectories as independent objects and to assume that any Lorentz pole or Regge trajectory gives a contribution to any process if there is no contrary selection rule.

One should also consider the possibility that a family of Regge trajectories is generated by many Lorentz poles displaced by integral numbers in the λ plane. It is probable that any family of Regge trajectories which satisfies the condition (9.1) required by the conservation of the angular momentum (i.e., the Gribov-Volkov conditions) can be constructed in this way. The hypothesis that a family of Regge trajectories is generated by a single Lorentz pole can be justified only by a dynamical model (for instance the Bethe-Salpeter equation).

Of course the whole picture we are proposing is justified only if it is useful in the very high-energy phenomenology and in the classification of particles and resonances. Therefore we have to find the quantum numbers of the Lorentz poles which generate the known Regge trajectories and to look for some evidence for the other members of the families. If our picture is correct, the trajectories $P, P', \rho, \omega, A_2(R)$, which appear in the expressions for the total cross-sections³⁸⁾, must be generated by a Lorentz pole with quantum numbers given by Eq. (7.18). In this case the families should have the simple structure described by Eq. (9.9). The trajectories which do not contribute to the total cross-sections give rise to more difficult and interesting problems. For instance, the π trajectory could be generated by a Lorentz pole of the type $M = 1, \tau = +1$. In this case the situation is described by Eq. (9.10) and the pion trajectory should cross at vanishing momentum transfer another trajectory with opposite parity. This fact could have interesting consequences for the Regge pole approach to nucleon-nucleon charge exchange scattering.

The fermion Regge trajectories must be generated by a Lorentz pole with half integral M . Therefore in this case the situation is always described by Eq. (9.10). In this way we find again the well-known result that the fermion trajectories are coupled in pairs³⁹⁾⁻⁴²⁾.

10. - MESON-NUCLEON AND NUCLEON-NUCLEON SCATTERING

In order to illustrate the techniques developed in the preceding Sections, we apply them to scattering processes in which only particles with spin 0 or $\frac{1}{2}$ appear. We assume that the decomposition of the amplitude according to the internal quantum numbers of the "exchanged object" has already been done. This decomposition is connected with the purely kinematic analysis we are doing only through the quantities ξ^A and ξ^B defined in Section 4 and given explicitly in terms of the internal quantum numbers in the Appendix B. We assume that charge conjugation is contained in the symmetry group and that the particles (1) and (3), as well as the particles (2) and (4), are of the same kind.

At first, we consider the case $W < 0$ and we obtain some of the results of the usual Regge pole analysis (27), (42)-(46). If the particles (1) and (3) are spinless, from Eqs. (5.6)-(5.9) we see that only Regge poles with the quantum numbers

$$\sigma = 1, \tau = \xi^A, \epsilon = 0 \quad (10.1)$$

can give a contribution. If the particles (1) and (3) have spin $\frac{1}{2}$, from the same equations we see that the poles with $\sigma = 1, \tau = -\xi^A$ cannot appear and, in the other three possible cases, the residues must have the following forms (of course it must be $\epsilon = 0$)

$$\sigma = 1, \tau = \xi^A; \quad \beta_{m_1 m_3}^{(A)} = \beta^{(A)} \sigma_{m_1 m_3}^x + \beta'^{(A)} \sigma_{m_1 m_3}^z; \quad (10.2)$$

$$\sigma = -1, \tau = \xi^A; \quad \beta_{m_1 m_3}^{(A)} = \beta^{(A)} \sigma_{m_1 m_3}^y; \quad (10.3)$$

$$\sigma = -1, \tau = -\xi^A; \quad \beta_{m_1 m_3}^{(A)} = \beta^{(A)} \delta_{m_1 m_3}. \quad (10.4)$$

Similar results can be obtained if the particles (4) and (2) have spin 0 or $\frac{1}{2}$, by means of the substitutions

$$(3) \rightarrow (2), (1) \rightarrow (4), (A) \rightarrow (B). \quad (10.5)$$

If we take into account the identity ^{4), 13)}

$$a_{mm'}^{\ell}(\xi) = a_{-m, -m'}^{\ell}(\xi), \quad (10.6)$$

we obtain from Eqs. (4.28) and (5.12) the following expressions for the Regge pole contributions.

For $j_1 = j_3 = 0, j_2 = j_4 = \frac{1}{2}, \sigma = 1, \tau = \xi^A = \xi^B$ we have

$$\begin{aligned} \tilde{T}_{m_2 m_4}^w(a_x(\xi)) = & \eta(0, \ell) \beta^{(A)} \left[-i \beta^{(B)} \delta_{m_2 m_4} a_{00}^{-\ell-1}(\xi) + \right. \\ & \left. + \beta^{(B)} \sigma_{m_2 m_4}^y a_{10}^{-\ell-1}(\xi) \right], \end{aligned} \quad (10.7)$$

and for $j_1 = j_2 = j_3 = j_4 = \frac{1}{2}$, we obtain

$$\begin{aligned} \tilde{T}_{m_1 m_2 m_3 m_4}^w(a_x(\xi)) = & f_{00}(\xi) \delta_{m_2 m_4} \delta_{m_1 m_3} + \\ & + f_{0y}(\xi) \delta_{m_2 m_4} \sigma_{m_1 m_3}^y + f_{y0}(\xi) \sigma_{m_2 m_4}^y \delta_{m_1 m_3} + \\ & + f_{xx}(\xi) \sigma_{m_2 m_4}^x \sigma_{m_1 m_3}^x + f_{yy}(\xi) \sigma_{m_2 m_4}^y \sigma_{m_1 m_3}^y + \\ & + f_{zz}(\xi) \sigma_{m_2 m_4}^z \sigma_{m_1 m_3}^z, \end{aligned} \quad (10.8)$$

where for $\sigma = 1, \tau = \xi^A = \xi^B$ it is

$$\begin{cases}
f_{00}(\zeta) = -\eta(0, \ell) \beta^{(B)} \beta^{(A)} a_{00}^{-\ell-1}(\zeta), \\
f_{y0}(\zeta) = -i \eta(0, \ell) \beta'^{(B)} \beta^{(A)} a_{10}^{-\ell-1}(\zeta), \\
f_{0y}(\zeta) = i \eta(0, \ell) \beta^{(B)} \beta'^{(A)} a_{01}^{-\ell-1}(\zeta), \\
f_{yy}(\zeta) = -\eta(0, \ell) \beta'^{(B)} \beta'^{(A)} \frac{1}{2} [a_{11}^{-\ell-1}(\zeta) + a_{1,-1}^{-\ell-1}(\zeta)], \\
f_{xx}(\zeta) = -\eta(0, \ell) \beta'^{(B)} \beta'^{(A)} \frac{1}{2} [a_{11}^{-\ell-1}(\zeta) - a_{1,-1}^{-\ell-1}(\zeta)], \\
f_{zz}(\zeta) = 0.
\end{cases} \quad (10.9)$$

for $\sigma = -1$, $\tau = \zeta^A = \zeta^B$, it is

$$\begin{cases}
f_{zz}(\zeta) = \beta^{(B)} \beta^{(A)} \eta(0, \ell) a_{00}^{-\ell-1}(\zeta), \\
f_{00}(\zeta) = f_{y0}(\zeta) = f_{0y}(\zeta) = f_{xx}(\zeta) = f_{yy}(\zeta) = 0,
\end{cases} \quad (10.10)$$

and for $\sigma = -1$, $\tau = -\zeta^A = -\zeta^B$, it is

$$\begin{cases}
f_{xx}(\zeta) = -\eta(0, \ell) \beta^{(B)} \beta^{(A)} \frac{1}{2} [a_{11}^{-\ell-1}(\zeta) + a_{1,-1}^{-\ell-1}(\zeta)], \\
f_{yy}(\zeta) = -\eta(0, \ell) \beta^{(B)} \beta^{(A)} \frac{1}{2} [a_{11}^{-\ell-1}(\zeta) - a_{1,-1}^{-\ell-1}(\zeta)], \\
f_{00}(\zeta) = f_{0y}(\zeta) = f_{y0}(\zeta) = f_{zz}(\zeta) = 0.
\end{cases} \quad (10.11)$$

In the high-energy limit we may use the following asymptotic approximations, which can be derived from Eq. (3.23)

$$\begin{cases}
a_{00}^{-\ell-1}(\zeta) \sim 2^{-\ell} \Gamma(2\ell+1) [\Gamma(\ell+1)]^{-2} (\omega h \zeta)^\ell, \\
a_{01}^{-\ell-1}(\zeta) \sim -2^{-\ell} \Gamma(2\ell+1) [\Gamma(\ell+1)]^{-2} (\omega h \zeta)^\ell, \\
a_{10}^{-\ell-1}(\zeta) \sim -2^{-\ell} \Gamma(2\ell+1) [\Gamma(\ell) \Gamma(\ell+2)]^{-1} (\omega h \zeta)^\ell, \\
\frac{1}{2} [a_{11}^{-\ell-1}(\zeta) + a_{1,-1}^{-\ell-1}(\zeta)] \sim 2^{-\ell} \Gamma(2\ell+1) [\Gamma(\ell) \Gamma(\ell+2)]^{-1} (\omega h \zeta)^\ell, \\
\frac{1}{2} [a_{11}^{-\ell-1}(\zeta) - a_{1,-1}^{-\ell-1}(\zeta)] \sim 2^{-\ell} \Gamma(2\ell+1) [\Gamma(\ell+1) \Gamma(\ell+2)]^{-1} (\omega h \zeta)^{\ell-1}.
\end{cases}$$

(10.12)

Now we consider the Lorentz pole contributions. If it is $j_1 = j_3 = 0$, from Eqs. (7.14)-(7.16), we have that the Lorentz poles which can give a contribution have the following quantum numbers

$$M=0, \quad \sigma=1, \quad \tau = \xi^A. \quad (10.13)$$

If it is $j_1 = j_3 = \frac{1}{2}$, from Eqs. (7.4), (7.13), (7.14) and (7.16), we see that there are three possible sets of quantum numbers and that the corresponding residues must have the forms

$$\begin{cases} M=0, \quad \sigma=1, \quad \tau = \xi^A, \\ \gamma_{j_A}^{(A)} = \gamma^{(A)} \delta_{0j_A}, \end{cases} \quad (10.14)$$

$$\begin{cases} M=0, \quad \sigma=-1, \quad \tau = -\xi^A, \\ \gamma_{j_A}^{(A)} = \gamma^{(A)} \delta_{1j_A}, \end{cases} \quad (10.15)$$

$$\begin{cases} M=1, \quad \tau = \xi^A, \\ \gamma_{j_A q}^{(A)} = 2^{-\frac{1}{2}} \gamma^{(A)} \delta_{1j_A}, \quad q = \pm 1. \end{cases} \quad (10.16)$$

Similar results can be obtained if the particles (2) and (4) have spins 0 or $\frac{1}{2}$, by means of the substitutions (10.5).

If it is $j_1 = j_3 = \frac{1}{2}$, $j_2 = j_4 = 0$, from Eqs. (7.11) and (4.29) we have that the contribution of a Lorentz pole with the quantum numbers $M=0$, $\sigma=1$, $\tau = \xi^A = \xi^B$, has the form

$$\mathcal{T}_{j_A m_A}^0(a_2(\xi)) = -\lambda^2 \gamma^{(B)} \gamma^{(A)} \delta_{0j_A} a_{000}^{0,-\lambda}(\xi), \quad (10.17)$$

$$\tilde{T}_{m_1 m_3}^0(a_z(\zeta)) = i \zeta^{-\frac{1}{2}} \gamma^{(B)} \gamma^{(A)} \delta_{m_1 m_3} a_{000}^{0, -\lambda}(\zeta). \quad (10.18)$$

If it is $j_1 = j_2 = j_3 = j_4 = \frac{1}{2}$, from Eqs. (7.11), (7.12) and (4.29), using the identity 13)

$$a_{-m, j, j'}^{-M, \lambda}(\zeta) = a_{m j j'}^{M \lambda}(\zeta), \quad (10.19)$$

we obtain the following expressions for the contributions of the Lorentz poles. For a pole of the type $M = 0$, $\sigma = 1$, $\tau = \xi^A = \xi^B$ we have

$$\mathcal{T}_{j_B m_B j_A m_A}^0(a_z(\zeta)) = -\lambda^2 \gamma^{(B)} \gamma^{(A)} \delta_{j_B 0} \delta_{j_A 0} a_{000}^{0, -\lambda}(\zeta), \quad (10.20)$$

$$\tilde{T}_{m_1 m_2 m_3 m_4}^0(a_z(\zeta)) = \frac{1}{2} \lambda^2 \gamma^{(B)} \gamma^{(A)} \delta_{m_2 m_4} \delta_{m_1 m_3} a_{000}^{0, -\lambda}(\zeta), \quad (10.21)$$

for a pole of the type $M = 0$, $\sigma = -1$, $\tau = -\xi^A = -\xi^B$ it is

$$\begin{aligned} \mathcal{T}_{j_B m_B j_A m_A}^0(a_z(\zeta)) = \\ -\lambda^2 \gamma^{(B)} \gamma^{(A)} \delta_{j_B 1} \delta_{j_A 1} \delta_{m_B m_A} a_{m_A 11}^{0, -\lambda}(\zeta), \end{aligned} \quad (10.22)$$

$$\begin{aligned} \tilde{T}_{m_1 m_2 m_3 m_4}^0(a_z(\zeta)) = \frac{1}{2} \lambda^2 \gamma^{(B)} \gamma^{(A)} \left[\sigma_{m_2 m_4}^z \sigma_{m_1 m_3}^z \right. \\ \left. a_{011}^{0, -\lambda}(\zeta) + (\sigma_{m_2 m_4}^x \sigma_{m_1 m_3}^x + \sigma_{m_2 m_4}^y \sigma_{m_1 m_3}^y) a_{111}^{0, -\lambda}(\zeta) \right]. \end{aligned} \quad (10.23)$$

and for a pole of the type $M = 1$, $\tau = \xi^A = \xi^B$ we obtain

$$\mathcal{G}_{\dot{\partial}_B m_B \dot{\partial}_A m_A}^0(a_z(\zeta)) = \frac{1}{2} (1-\lambda^2) \gamma^{(B)} \gamma^{(A)}$$

$$\delta_{1\dot{\partial}_B} \delta_{1\dot{\partial}_A} \delta_{m_B m_A} \frac{1}{2} \sum_{q=\pm 1} a_{m_A 11}^{q, -\lambda}(\zeta), \quad (10.24)$$

$$\tilde{T}_{m_1 m_2 m_3 m_4}^0(a_z(\zeta)) = \frac{1}{2} (\lambda^2 - 1) \gamma^{(B)} \gamma^{(A)}$$

$$\left[\sigma_{m_2 m_4}^z \sigma_{m_1 m_3}^z a_{011}^{1, -\lambda}(\zeta) + \left(\sigma_{m_2 m_4}^x \sigma_{m_1 m_3}^x + \sigma_{m_2 m_4}^y \sigma_{m_1 m_3}^y \right) \cdot \frac{1}{2} \left(a_{111}^{1, -\lambda}(\zeta) + a_{111}^{-1, -\lambda}(\zeta) \right) \right]. \quad (10.25)$$

The functions $a_{mjj'}^{\lambda}(\zeta)$ which appear in these formulae can be computed from the general formula given in Ref. ¹³⁾. The result is

$$\left\{ \begin{aligned} a_{000}^{0, -\lambda}(\zeta) &= (2\lambda \sinh \zeta)^{-1} \exp(\lambda \zeta) \sim \lambda^{-1} \exp[\zeta(\lambda-1)], \\ a_{011}^{0, -\lambda}(\zeta) &= 3(2 \sinh \zeta)^{-3} \exp(\lambda \zeta) \left[(\lambda-1) \lambda^{-1} (\lambda+1)^{-1} \exp 2\zeta + \right. \\ &\quad \left. + 2(3-\lambda^2) \lambda^{-1} (\lambda^2-1)^{-1} + (\lambda+1) \lambda^{-1} (\lambda-1)^{-1} \exp(-2\zeta) \right] \sim \\ &\quad \sim 3(\lambda-1) \lambda^{-1} (\lambda+1)^{-1} \exp[\zeta(\lambda-1)], \\ a_{111}^{0, -\lambda}(\zeta) &= a_{011}^{1, -\lambda}(\zeta) = 6(2 \sinh \zeta)^{-3} \exp(\lambda \zeta) \left[\lambda^{-1} (\lambda+1)^{-1} \exp \zeta - \right. \\ &\quad \left. - \lambda^{-1} (\lambda-1)^{-1} \exp(-\zeta) \right] \sim 6 \lambda^{-1} (\lambda+1)^{-1} \exp[\zeta(\lambda-2)], \\ \frac{1}{2} \left[a_{111}^{1, -\lambda}(\zeta) + a_{111}^{-1, -\lambda}(\zeta) \right] &= \frac{3}{2} (2 \sinh \zeta)^{-3} \exp(\lambda \zeta) \cdot \\ &\quad \cdot \left[(\lambda+1)^{-1} \exp 2\zeta + 2(2-\lambda^2) \lambda^{-1} (\lambda^2-1)^{-1} + (\lambda-1)^{-1} \exp(-2\zeta) \right] \sim \\ &\quad \sim \frac{3}{2} (\lambda+1)^{-1} \exp[\zeta(\lambda-1)], \end{aligned} \right. \quad (10.26)$$

The Chew-Frautschi plots given in Fig. 2 show qualitatively the families of Regge trajectories generated by the three kinds of Lorentz poles considered above. Remember that also trajectories connected with Lorentz poles of different kind (for instance with $M = 2$) can give a contribution to non-forward nucleon-nucleon scattering.

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- APPENDIX A -

THE PLANCHEREL FORMULA FOR THE GROUP $\tilde{A}$

The expansion formula (5.1) is a generalization of the formula which permits the expansion of an arbitrary L^2 function defined on the group H . A formula of this kind can be written for all the compact groups and for a wide class of non-compact groups. In general, if we consider a locally compact group H and we call $\hat{H}$ the set of the equivalence classes of irreducible unitary representations of H , we may define the generalized Fourier transform of a function $f(g)$ defined on H in the following way

$$F_{mm'}^\Lambda = \int_H D_{mm'}^\Lambda(g) f(g) dg, \quad (A.1)$$

where dg is the invariant measure over H and $D_{mm'}^\Lambda(g)$ is a representation belonging to the equivalence class Λ ($\Lambda \in \hat{H}$). If certain conditions are satisfied, we can write the following Plancherel formula

$$\int_H |f(g)|^2 dg = \int_{\hat{H}} \sum_{mm'} |F_{mm'}^\Lambda|^2 d\Lambda \quad (A.2)$$

and the inverse formula

$$f(g) = \int_{\hat{H}} \sum_{mm'} \overline{D_{mm'}^\Lambda(g)} F_{mm'}^\Lambda d\Lambda, \quad (A.3)$$

where $d\Lambda$ is the Plancherel measure defined in the set $\hat{H}$.

The Plancherel measure for the group H_- has been given in Ref. 7). From this result it is easy to show that Eq. (5.1) coincides with the inverse formula (A.3) if the integration path C_W is taken along the line $\operatorname{Re} \ell = -\frac{1}{2}$ (47).

We want to show that Eq. (7.1) is related in a similar way to the inverse formula for the generalized Fourier transform on the group $\tilde{A}$.

In order to find the Plancherel measure for the group $\tilde{A}$, we start from the Plancherel formula for the group $SL(2C)$ (35). We define the invariant measure $d'a$ on this group in the following way

$$\int_{SL(2C)} f(a) d'a = (16)^{-1} \int_0^{4\pi} d\mu_1 \int_0^\pi \sin \theta_1 d\theta_1 \int_0^{2\pi} d\gamma_1 \int_0^\pi \sin \theta_2 d\theta_2 \int_0^{2\pi} d\gamma_2$$

$$\int_0^\infty f(\mu_z(\mu_1) \mu_y(\theta_1) \mu_z(\gamma_1) a_z(\zeta) \mu_y(\theta_2) \mu_z(\gamma_2)) (\sinh \zeta)^2 d\zeta \quad (A.4)$$

and we call $d\Lambda'$ the corresponding Plancherel measure. In this case, Λ' stands for the parameters M and λ . From the results given in Ref. 35), we have

$$\int \psi(\Lambda') d\Lambda' = (2\pi^4)^{-1} \sum_{M=-\infty}^{+\infty} \int_0^\infty \psi(M, \lambda) (M^2 + \lambda^2) d\lambda,$$

$$\lambda = i \mu. \quad (A.5)$$

This formula defines the Plancherel measure for $SL(2C)$.

The invariant measure da over $\tilde{A}$ can be written in the following way 7)

$$\int_{\tilde{A}} f(a) da = \frac{1}{2} \sum_{i=1}^4 \int_{SL(2C)} f(w_i a) d'a, \quad (A.6)$$

where

$$w_1 = e, w_2 = \Delta, w_3 = t, w_4 = \Delta t. \quad (\text{A.7})$$

If we define the following quantities

$$F_{jmj'm'}^{M\lambda}(w_i) = \int_{SL(2C)} f(w_i a) \mathcal{O}_{jmj'm'}^{M\lambda}(a) d'a, \quad (\text{A.8})$$

and use the Plancherel formula for $SL(2C)$ we obtain

$$\int_{\tilde{A}} |f(a)|^2 da = (4\pi^4)^{-1} \sum_{i=1}^4 \int_0^\infty \sum_{jm} \sum_{j'm'} |F_{jmj'm'}^{M,i\nu}(w_i)|^2 \cdot (M^2 + \nu^2) d\nu. \quad (\text{A.9})$$

The Fourier transform (A.1) on the group $\tilde{A}$ takes the form

$$\begin{cases} \tilde{F}_{jmj'm'}^{\sigma\tau\lambda} = \int_{\tilde{A}} f(a) \tilde{\mathcal{O}}_{jmj'm'}^{\sigma\tau\lambda}(a) da, \\ \tilde{F}_{jmqj'm'q'}^{\tau M\lambda} = \int_{\tilde{A}} f(a) \tilde{\mathcal{O}}_{jmqj'm'q'}^{\tau M\lambda}(a) da, \end{cases} \quad (\text{A.10})$$

and can be written in terms of the quantities (A.8) in the following way

$$\tilde{F}_{jmj'm'}^{\sigma\tau\lambda} = \frac{1}{2} \left[F_{jmj'm'}^{0\lambda}(e) + \sigma(-1)^j F_{jmj'm'}^{0\lambda}(\Delta) + \tau F_{jmj'm'}^{0\lambda}(t) + \sigma\tau(-1)^j F_{jmj'm'}^{0\lambda}(\Delta t) \right], \quad (\text{A.11})$$

$$\tilde{F}_{jmqj'm'q'}^{\tau M\lambda} = \frac{1}{2} \left[\delta_{qq'} F_{jmj'm'}^{qM,\lambda}(e) + \delta_{q,-q'} q^{2M} (-1)^{j+M} \cdot F_{jmj'm'}^{-qM,\lambda}(\Delta) + \delta_{qq'} \tau (-iq)^{2M} F_{jmj'm'}^{qM,\lambda}(t) + \delta_{q,-q'} \tau (-1)^{j+M} F_{jmj'm'}^{-qM,\lambda}(\Delta t) \right]. \quad (\text{A.12})$$

From these equations it follows that

$$\sum_{\sigma\tau} \sum_{j\ m} \sum_{j'\ m'} |\tilde{F}_{j\ m\ j'\ m'}^{\sigma\tau 0\lambda}|^2 = \sum_i \sum_{j\ m} \sum_{j'\ m'} |F_{j\ m\ j'\ m'}^{0\lambda}(w_i)|^2, \quad (\text{A.13})$$

$$\begin{aligned} & \sum_{\tau} \sum_{j\ m\ q} \sum_{j'\ m'\ q'} |\tilde{F}_{j\ m\ q\ j'\ m'\ q'}^{\tau M\lambda}|^2 = \\ & = \frac{1}{2} \sum_{i\ q} \sum_{j\ m} \sum_{j'\ m'} |F_{j\ m\ j'\ m'}^{qM,\lambda}(w_i)|^2, \end{aligned} \quad (\text{A.14})$$

and, comparing with Eq. (A.9) we obtain the following Plancherel formula

$$\begin{aligned} \int_{\tilde{A}} |f(a)|^2 da &= (4\pi^4)^{-1} \sum_{\sigma\tau} \int_0^\infty \sum_{j\ m} \sum_{j'\ m'} |\tilde{F}_{j\ m\ j'\ m'}^{\sigma,\tau,0,i\pi}|^2 r^2 dr + \\ &+ (2\pi^4)^{-1} \sum_{\tau} \sum_{M>0} \int_0^\infty \sum_{j\ m\ q} \sum_{j'\ m'\ q'} |\tilde{F}_{j\ m\ q\ j'\ m'\ q'}^{\tau,M,i\pi}|^2 (r^2 + r'^2) dr. \end{aligned} \quad (\text{A.15})$$

It follows that the Plancherel measure $d\Lambda$ for the group $\tilde{A}$ is defined by

$$\begin{aligned} \int \psi(\lambda) d\Lambda &= (4\pi^4)^{-1} \sum_{\sigma\tau} \int_0^\infty \psi(\sigma,\tau,0,i\pi) r^2 dr + \\ &+ (2\pi^4)^{-1} \sum_{\tau} \sum_{M>0} \int_0^\infty \psi(\tau,M,i\pi) (r^2 + r'^2) dr. \end{aligned} \quad (\text{A.16})$$

In order to obtain Eq. (7.1), we assume that the function $\tau_{j_B m_B j_A m_A}^0(a)$ is integrable and consider its generalized Fourier transform

$$\begin{aligned} & \int_{\tilde{A}} \tau_{j_B m_B j_A m_A}^0(a) \overline{\tilde{D}_{j\ m\ j'\ m'}^{\sigma\tau 0\lambda}(a)} da = \\ & = \int_{SU(2)} \int_{SU(2)} \int_{\tilde{A}} \tau_{j_B m_B j_A m_A}^0(uau') \overline{\tilde{D}_{j\ m\ j'\ m'}^{\sigma\tau 0\lambda}(uau')} da. \end{aligned} \quad (\text{A.17})$$

We have used the invariance of the measure da and the fact that the total measure of $SU(2)$ is one. We may transform the right-hand side of Eq. (A.17) by means of the covariance properties (4.22) and (4.23) and by means of the representation property. Then the elements u and u' appear only as arguments of rotation matrices of the kind $R_{mm'}^j(u)$ and the integrations over $SU(2)$ can be performed using the orthogonality properties of the elements of the rotation matrices³²⁾. The result is

$$\begin{aligned} & \int_{\tilde{A}} \tau_{j_B m_B j_A m_A}^0(a) \overline{\tilde{D}_{j m j' m'}^{\sigma \tau 0 \lambda}(a)} da = \\ & = \delta_{j_B j} \delta_{j_A j'} \delta_{m_B m} \delta_{m_A m'} \mathcal{F}_{j_B j_A}(\sigma, \tau, 0, \lambda), \end{aligned} \quad (\text{A.18})$$

where $\mathcal{F}_{j_B j_A}(\sigma, \tau, 0, \lambda)$ is a certain expression. We prefer to define this function directly by means of Eq. (A.18). In a similar way, we obtain

$$\begin{aligned} & \int_{\tilde{A}} \tau_{j_B m_B j_A m_A}^0(a) \overline{\tilde{D}_{j m q j' m' q'}^{\tau M \lambda}(a)} da = \\ & = \delta_{j_B j} \delta_{j_A j'} \delta_{m_B m} \delta_{m_A m'} \mathcal{F}_{j_B q j_A q'}(\tau, M, \lambda). \end{aligned} \quad (\text{A.19})$$

It is easy to prove that the functions appearing in the right-hand sides of Eqs. (A.18) and (A.19) satisfy the conditions (7.2), (7.4)-(7.7). Inserting Eqs. (A.18), (A.19) and (A.16) in the inverse formula (A.3), we obtain⁴⁷⁾

$$\begin{aligned} \tau_{j_B m_B j_A m_A}^0(a) &= (4\pi^4)^{-1} \sum_{\sigma \tau} \int_0^\infty \mathcal{F}_{j_B j_A}(\sigma, \tau, 0, i\nu) \tilde{D}_{j_B m_B j_A m_A}^{\sigma, \tau, 0, i\nu}(a) \nu^2 d\nu + \\ &+ (2\pi^4)^{-1} \sum_{\tau M} \int_0^\infty \sum_{q q'} \mathcal{F}_{j_B q j_A q'}(\tau, M, i\nu) \tilde{D}_{j_B m_B q j_A m_A q'}^{\tau, M, i\nu}(a) (M^2 + \nu^2) d\nu. \end{aligned} \quad (\text{A.20})$$

If we remark that the integrand is an even function of p and make a change of integration variable, we obtain exactly Eq. (7.1), where the integration path C_0 is the imaginary axis. In order to represent a physical scattering amplitude, it is necessary to deform the integration path for the same reasons as mentioned in Section 5.

- APPENDIX B -

THE INTERNAL QUANTUM NUMBERS

We consider the internal symmetry group of the strong interactions ³¹⁾, which contains the isotopic spin group, the charge conjugation and the gauge transformations connected with the baryon charge and the hypercharge. There are two kinds of unitary irreducible representations of this group :

- (i) the representations of the first kind can be individuated by the isotopic spin I , the baryon number B and the hypercharge Y (B and Y cannot both be zero) ; their dimension is $2(2I+1)$; the internal quantum numbers α [see Eq. (3.10)] in this case stand for the third component of the isotopic spin and another index which distinguishes the particles from the antiparticles ;
- (ii) the representations of the second kind can be individuated by the isotopic spin I and the G parity (in this case it is $B = Y = 0$) ; their dimension is $(2I+1)$ and the index α in this case stands for the third component of the isotopic spin alone.

We call I_i, B_i, Y_i and G_i the parameters which label the representation which describes the particle (i). It is convenient to write the scattering amplitude defined in Section 2 in the following way

$$\sum_{\Gamma} \left(\bigoplus_{\alpha_1 \alpha_2 \alpha_3 \alpha_4}^{\Gamma} M_{m_1 m_2 m_3 m_4}^{\Gamma} (a_1, a_2, a_3, a_4) \right), \quad (B.1)$$

where Γ stands for the quantum numbers (I, B, Y) or (I, G) of the "exchanged object" and the functions $\bigoplus_{\alpha_1 \alpha_2 \alpha_3 \alpha_4}^{\Gamma}$ are built in the usual way by means of the Clebsch-Gordan coefficients for the internal

symmetry group. The amplitudes M^Γ which appear in Eq. (B.1) do not depend on the indices α_i and can be treated by means of the techniques developed in the preceding Sections. We assume that the singularities in the ℓ and in the λ planes are different for different values of Γ , so that we can assign to the Regge poles and to the Lorentz poles well defined values of the quantum numbers (I, B, Y) or (I, G) .

If the particles (1) and (3) are of the same kind, the quantities ξ^A introduced in Section 4 depend on the parameter Γ and can be defined more exactly by means of the following formula

$$\Theta_{\alpha_1 \alpha_2 \alpha_3 \alpha_4}^\Gamma = \xi^A(\Gamma) \Theta_{\alpha_3 \alpha_2 \alpha_1 \alpha_4}^\Gamma. \quad (B.2)$$

If we compute the quantities $\Theta_{\alpha_1 \alpha_2 \alpha_3 \alpha_4}^\Gamma$, the coefficients $\xi^A(\Gamma)$ can easily be obtained. The result is

$$\begin{cases} \xi^A(I, B, Y) = (-1)^{I - \frac{1}{2}Y} & \text{for } B = 2B_1, Y = 2Y_1, \\ \xi^A(I, G) = G(-1)^I & \text{for } B = Y = 0. \end{cases} \quad (B.3)$$

We have taken into account the condition ³¹⁾

$$(-1)^{2I_i} = (-1)^{Y_i}. \quad (B.4)$$

If it is $B_1 = Y_1 = 0$, it can be shown that it must be $G = 1$ and both the Eqs. (B.3) are true.

In a similar way, if the particles (2) and (4) are of the same kind, we obtain

$$\left\{ \begin{array}{l} \xi^B(I, B, Y) = (-1)^{I - \frac{1}{2}Y} \text{ for } B = 2B_2, Y = 2Y_2, \\ \xi^B(I, G) = G(-1)^I \text{ for } B = Y = 0. \end{array} \right.$$

(B.4)

Note that in any case it is $\xi^A = \xi^B$.

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- FIGURE CAPTIONS -

- Figure 1 - Two coupling schemes for the representations of the Poincaré group which describe the particles involved in the process $(1,1)$. The coupling scheme a) gives rise to the usual partial wave expansion and the coupling scheme b) corresponds to the crossed partial wave expansion. In the second case the horizontal line may be interpreted as an "exchanged object" having four-momentum Q .
- Figure 2 - Chew-Frautschi plots showing the families of Regge trajectories generated by the three possible kinds of Lorentz poles which can contribute to nucleon-nucleon scattering. Only the first few members of the families are shown. The dashed lines indicate trajectories which cannot give a contribution to nucleon-nucleon scattering.

a)

b)

FIG.1

$M=0, \sigma=1, \tau=\xi$

$M=0, \sigma=-1, \tau=-\xi$

$M=1, \tau=\xi$

FIG.2