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ON THE GROUP THEORETICAL APPROACH TO
COMPLEX ANGULAR MOMENTUM AND SIGNATURE

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ABSTRACT

The relativistic two-body scattering amplitude is analyzed in terms of irreducible representations of the Poincaré group, which describes "objects" exchanged in the scattering process. These representations are obtained by the decomposition of the direct product of the representations, which describe an incoming and an outgoing particle, respectively. This procedure, which uses only the invariance and unitarity properties of the amplitude, is analogous to the usual partial wave expansion, and we call it "crossed partial wave analysis".

Complex angular momentum appears as a quantum number describing the "exchanged objects". If the TCP invariance is taken into account, also the signature can be introduced in a natural way. The difficulties due to the fact that the representations considered are infinite dimensional are discussed in detail. It is proved that the crossed partial wave analysis can be performed for every invariant amplitude which satisfies the unitarity condition. However, the "crossed partial wave amplitude" has to be considered as a bounded linear functional in a normed space of functions which depend on the quantum numbers which describe the "exchanged objects".

Particles with arbitrary spins and parities are considered. The symmetry with respect to time and space inversions, and with respect to the exchange of identical particles, is taken into account. The role of the internal symmetry group is briefly discussed.

1. INTRODUCTION

In the last years the Regge pole expansion of the scattering amplitude has become an extremely useful tool for the phenomenological investigation of high-energy scattering.

It seems therefore important to investigate, as deeply as possible, the nature and the meaning of the new concepts which appear in Regge theory, that is the complex angular momentum and the signature¹⁾.

One of the most attractive directions of investigation is given by the connection, suggested by Joos²⁾, between the complex angular momentum and the representations of the Poincaré group generated by a space-like four-momentum³⁾. These representations are individuated by the square W of this four-momentum and by a representation of the corresponding little group, which is the three-dimensional Lorentz group⁴⁾. Some of the representations of the little group depend on a complex parameter l which can be identified with the complex angular momentum.

Roughly speaking, the representations of the Poincaré group we are considering, describe something that is exchanged in the scattering process. The invariant W is the square of the four-momentum carried by this "object" (the four-momentum transfer) and the parameter l is something analogous to the spin.

It is possible to perform a decomposition of the scattering amplitude according to the quantum numbers of the "exchanged object". This decomposition is similar to the usual partial wave decomposition which is performed according to the quantum numbers of the state formed by the interacting particles. Therefore we call it the "crossed partial wave decomposition".

An equivalent formalism has been developed independently by Sertorio and Toller⁵⁾ in order to investigate an approximate field theoretical model, i.e. the fixed momentum transfer integral equation for the scattering amplitude suggested by the multiperipheral model⁶⁾. The invariance of this equation with respect to a three-dimensional Lorentz group can be used to reduce the equation into a set of simpler equations for the "crossed partial wave amplitudes". The kernels of these equations depend on the complex parameter l . Also, the symmetry of the equation with respect to a transformation which contains TCP can be taken into account, giving rise to a

second quantum number which has the properties of the signature. This procedure can be considered as a refinement of the treatment of the asymptotic multiperipheral equation given by Bertocchi et al.⁷⁾.

The crossed partial wave decomposition is related to one of the expansions of the scattering amplitude proposed by Vilenkin and Smorodinskii⁸⁾ [see other papers^{9,10)}]. Further developments of the crossed partial wave formalism can be found in other references¹¹⁻¹⁵⁾.

Recently, Boyce¹⁶⁾ has given proof of the equivalence between this formalism and the usual complex angular momentum theory based on the analytic continuation in the energy angular momentum complex planes¹⁾. This proof is based on the strict relation which exists¹⁷⁾ between the representations of the three-dimensional Lorentz group and the local representations of the rotation group¹⁸⁾. Further connections between the representations of these two groups are given in a paper by Holman and Biedenharn¹⁹⁾.

A general discussion of different possible connections between complex angular momentum and group theory has been given recently by Lévy Leblond²⁰⁾.

In spite of all the above-mentioned work, the crossed partial wave formalism has not yet reached a completely satisfactory arrangement. The main difficulty lies in the fact that only amplitudes which have a sufficiently decreasing high-energy behaviour can be expanded in a well-defined way.

One of the objects of this paper is to give a clearer mathematical definition of the crossed partial wave expansion, to indicate the origin of the above-mentioned convergence difficulties, and to suggest a more general formalism which avoids them. We shall show that the origin of these difficulties is the fact that the irreducible representations of the Poincaré group which describe the particles are infinite-dimensional. A complete theory of infinite-dimensional group representations is available only for representations operating on Hilbert spaces²¹⁻²³⁾. Therefore, in Sections 2 and 3 we give a clear definition of the Hilbert spaces involved in the problem and recall some of the available mathematical theorems.

Sections 4 to 7 are concerned with the second object of the present paper, i.e. to enlarge the formalism in order to take into account the

invariance of the scattering amplitude with respect to reflections, and in particular with respect to time reversal. In the crossed partial wave analysis, the time reversal has a much greater importance than in the usual partial wave analysis because it belongs to the little group corresponding to a space-like four-momentum (the four-momentum transfer). However, some caution is required on account of the antiunitary character of the time reversal operator. This problem is treated in Section 4, where we build in a natural way a representation of the complete symmetry group in which also the operations which involve time reversal are represented by unitary operators. As a consequence, in this formalism the TCP operator has eigenvalues, which can be used for the classification of the "exchanged objects". This is the group theoretical origin of the signature.

In Sections 5 and 6 we consider in detail the above-mentioned representations of the symmetry group, and in Section 7 the decomposition of the direct product of two irreducible representations of this kind is performed. A considerable simplification, both formal and conceptual, is obtained by an extensive use of a formalism developed by Moussa and Stora²⁴), which is an application of the general theory of induced representations^{21,22}). In Section 8 the crossed partial wave amplitude is treated in detail. The convergence difficulties are treated by means of an adaptation of a general method which is the basis of the theory of the Fourier transform of generalized functions^{25,26}). A similar method has been used by Gunson^{27,28}) to treat the continuation in the complex angular momentum plane of the usual partial wave amplitudes.

The method consists of considering the amplitude as a functional defined over a space of test functions, performing the partial wave analysis (harmonic analysis in the mathematical language) of the test functions obtaining a space of transformed test functions, and defining the partial wave amplitude as a functional over this new space. It turns out that in our case the space of the test functions can be built in a very natural way, starting from the wave functions of the particles involved in the scattering process.

A norm is defined in the space of the test functions in such a way that every functional which represents a crossed partial wave amplitude is bounded if the original amplitude satisfies the unitarity condition. In Section 9 some important properties of the test functions are derived.

In Section 10 we show that the simplest terms which may appear in the crossed partial wave decomposition are equivalent to Regge pole contributions and to a background integral. Of course, more complicated terms (e.g. Regge cuts) cannot be excluded in this very general formalism. The unitarity bound for the position of Regge poles is given automatically in the present formalism, as a consequence of the properties of the test functions.

In Section 11 we examine the conditions imposed on the generalized partial wave amplitudes by the symmetry of the scattering operator with respect to the exchange of identical particles.

2. THE MATHEMATICAL BASIS OF THE PARTIAL WAVE ANALYSIS

In order to investigate the nature of the crossed partial wave analysis it is necessary to use adequate mathematical tools. As a first step, in this section we recall the mathematical theorems which form the basis of the usual (direct) partial wave analysis. We enunciate those theorems without proof in a form which is the most suitable for our purposes.

Let G be the symmetry group we want to consider. We assume that it does not contain symmetries which involve time reversal and that it has good mathematical properties, i.e. that it is locally compact and "type I"²¹⁻²³). For instance, the connected subgroup of the Poincaré group has these properties³). Let $\hat{G}$ be the set of the equivalence classes of continuous unitary irreducible representations of G , and let $g \rightarrow U_g$ be the unitary representation which operates in the Hilbert space $\mathcal{H}_g$ of the physical states. We apply to this representation the following theorem²¹⁻²³):

Theorem 1:

If G is a type I locally compact group, every continuous unitary representation $g \rightarrow U_g$ of G is equivalent to a direct integral of the form

$$U \simeq \int_{\hat{G}}^{\oplus} U^{\lambda} d\mu(\lambda) , \quad (2.1)$$

where

$$U^\lambda = \sum_{k=1}^{n_\lambda} \oplus D^\lambda, \quad (2.2)$$

D^λ is an irreducible unitary representation belonging to the class λ ($\lambda \in \hat{G}$), and n_λ may take the values 1, 2, ... ∞ . The measure μ in $\hat{G}$ is uniquely determined within equivalence. The decomposition (2.1) is called central or canonical.

If the scattering operator S commutes with all the operators U_g , we can use the following theorem.

Theorem 2:

If

$$U_g = K \left[\int_{\hat{G}}^{\oplus} U_g^\lambda d\mu(\lambda) \right] K^{-1}, \quad K^+ = K^{-1}, \quad (2.3)$$

is the central decomposition of the representation U of the group G , and S is a bounded operator with the property

$$S U_g = U_g S, \quad \forall g \in G, \quad (2.4)$$

then S can be written in the form

$$S = K \left[\int_{\hat{G}}^{\oplus} S^\lambda d\mu(\lambda) \right] K^{-1}, \quad (2.5)$$

where the operators S^λ satisfy the conditions

$$S^\lambda U_g^\lambda = U_g^\lambda S^\lambda, \quad \forall g \in G, \quad \forall \lambda \in \hat{G}. \quad (2.6)$$

This theorem follows from the definition of central decomposition [Ref. 21), p. 635] and from the results of Section 41 of the book by Naimark²⁹⁾.

The structure of the operators S^λ is given by the following theorem.

Theorem 3:

Let us consider a representation of G having the form (2.2) where D^λ is irreducible, and an operator S^λ with the property (2.6). The space $\mathcal{H}^\lambda$ where the operator U_g^λ and S^λ are defined has the form

$$\mathcal{H}^\lambda = \sum_{k=1}^{n_\lambda} \oplus \mathcal{H}_D^\lambda, \quad (2.7)$$

where $\mathcal{H}_D^\lambda$ is the space where the representation D^λ operates. If we take

$$\begin{cases} f = \sum_{k=1}^{n_\lambda} \oplus f_k, \\ f' = \sum_{k=1}^{n_\lambda} \oplus f'_k, \end{cases} \quad \begin{cases} f, f' \in \mathcal{H}^\lambda, \\ f_k, f'_k \in \mathcal{H}_D^\lambda, \end{cases} \quad (2.8)$$

it is

$$S^\lambda f = \sum_{k'=1}^{n_\lambda} \oplus \sum_{k=1}^{n_\lambda} S_{k'k}^\lambda f_k, \quad (2.9)$$

or, in terms of matrix elements

$$(f', S^\lambda f) = \sum_{k=1}^{n_\lambda} \sum_{k'=1}^{n_\lambda} S_{k'k}^\lambda (f'_k, f_k), \quad (2.10)$$

where $S_{k'k}^\lambda$ are complex numbers.

The numbers $S_{k,k'}^\lambda$ are exactly the partial wave amplitudes. If G is the connected subgroup of the Poincaré group, the superscript λ stands for the centre-of-mass energy and the centre-of-mass total angular momentum, while the subscripts k, k' stand for the non-conserved quantum numbers; for instance, the number of particles, their internal quantum numbers, their helicities in the centre-of-mass system³⁰⁾, and other parameters which are needed if there are more than two particles.

In the following we shall consider only processes involving two incoming and two outgoing particles of definite kinds³¹⁾, for instance the process

$$(3) + (4) \rightarrow (1) + (2). \quad (2.11)$$

We call $\mathcal{H}^{(i)}$ ($i = 1, 2, 3, 4$) the Hilbert spaces containing the states which describe one particle of the kind i . In these spaces the irreducible representations $U^{(i)}$ of the group G operate. The initial and the final states are two-particle states belonging to the Hilbert spaces $\mathcal{H}^{(I)}$ and $\mathcal{H}^{(F)}$ defined by the direct products^{21, 22, 32)}

$$\begin{cases} \mathcal{H}^{(I)} = \mathcal{H}^{(3)} \otimes \mathcal{H}^{(4)}, \\ \mathcal{H}^{(F)} = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}. \end{cases} \quad (2.12)$$

These states transform according to the following representations of the group G

$$\begin{cases} U^{(I)} = U^{(3)} \otimes U^{(4)}, \\ U^{(F)} = U^{(1)} \otimes U^{(2)}. \end{cases} \quad (2.13)$$

In the following we shall often consider S as a bounded operator from the space $\mathcal{H}^{(I)}$ to the space $\mathcal{H}^{(F)}$. In this case it is convenient to use the following extension of theorems 2 and 3.

Theorem 4:

If G is a locally compact type I group and $U^{(I)}, U^{(F)}$ are two unitary continuous representations of G which operate in the Hilbert spaces $\mathcal{H}^{(I)}, \mathcal{H}^{(F)}$, it is always possible to write their central decompositions in the form

$$\begin{cases} U_g^{(I)} = K_I \left[\int_{\Omega_I}^{\oplus} U_g^{(I)\lambda} d\mu(\lambda) \right] K_I^{-1}, \\ U_g^{(F)} = K_F \left[\int_{\Omega_F}^{\oplus} U_g^{(F)\lambda} d\mu(\lambda) \right] K_F^{-1}, \end{cases} \quad (2.14)$$

where Ω_I and Ω_F are measurable subsets of $\hat{G}$, and it is, with the notations used in theorem 2,

$$\begin{cases} U^{(I)\lambda} = \sum_{k=1}^{n_{\lambda}^{(I)}} \oplus D^{\lambda}, \\ U^{(F)\lambda} = \sum_{k=1}^{n_{\lambda}^{(F)}} \oplus D^{\lambda}. \end{cases} \quad (2.15)$$

We call $\mathcal{H}^{(I)\lambda}, \mathcal{H}^{(F)\lambda}, \mathcal{H}_D^{\lambda}$ the spaces of the representations $U^{(I)\lambda}, U^{(F)\lambda}, D^{\lambda}$.

If S is a bounded operator from $\mathcal{H}^{(I)}$ to $\mathcal{H}^{(F)}$ with the property

$$S U_g^{(I)} = U_g^{(F)} S, \quad \forall g \in G, \quad (2.16)$$

it can be written in the form

$$S = K_F \left[\int_{\Omega_I \cap \Omega_F}^{\oplus} S^{\lambda} d\mu(\lambda) \right] K_I^{-1}, \quad (2.17)$$

where S^{λ} is a bounded operator from $\mathcal{H}^{(I)\lambda}$ to $\mathcal{H}^{(F)\lambda}$ with the properties

$$\begin{cases} S^\lambda = 0 & \text{if } \lambda \notin \Omega_I \cap \Omega_F, \\ S^\lambda U_g^{(I)\lambda} = U_g^{(F)\lambda} S^\lambda, & \forall g \in G, \forall \lambda \in \Omega_I \cap \Omega_F. \end{cases} \quad (2.18)$$

If it is

$$\begin{cases} f^{(I)} = \sum_{k=1}^{n_\lambda^{(I)}} \oplus f_k^{(I)}, & \begin{cases} f^{(I)} \in \mathcal{H}^{(I)\lambda}, \\ f^{(F)} \in \mathcal{H}^{(F)\lambda}, \end{cases} \\ f^{(F)} = \sum_{k=1}^{n_\lambda^{(F)}} \oplus f_k^{(F)}, & \begin{cases} f_k^{(I)}, f_k^{(F)} \in \mathcal{H}_D^\lambda, \end{cases} \end{cases} \quad (2.19)$$

we have

$$(f^{(F)}, S^\lambda f^{(I)}) = \sum_{k'=1}^{n_\lambda^{(F)}} \sum_{k=1}^{n_\lambda^{(I)}} S_{k'k}^\lambda (f_{k'}^{(F)}, f_k^{(I)}), \quad (2.20)$$

where $S_{k',k}^\lambda$ are complex numbers.

3. THE CROSSED PARTIAL WAVE ANALYSIS

For the definition of crossed partial wave expansion, we need, besides the concepts given in the preceding section, the concepts of adjoint or dual of a Hilbert space and of adjoint representation^{21,22}).

If $\mathcal{H}$ is a Hilbert space, $\overline{\mathcal{H}}$ denotes its adjoint, i.e. the space of the bounded linear functionals defined over $\mathcal{H}$. All these functionals are generated through the scalar product by elements of $\mathcal{H}$ itself. This means that there is a one-to-one antilinear isometric mapping J of $\mathcal{H}$ onto $\overline{\mathcal{H}}$. J is called the canonical antilinear mapping. If we use Dirac's terminology³³), $\mathcal{H}$ is the space of "kets" and $\overline{\mathcal{H}}$ is the space of "bras".

If U is a unitary representation of the group G in the Hilbert space $\mathcal{H}$, its adjoint $\overline{U}$ is a representation of G in the space $\overline{\mathcal{H}}$ defined by

$$\overline{U}_g = J U_g J^{-1}. \quad (3.1)$$

The representation $\bar{U}$ is unitary and it is not necessarily equivalent to U .

The amplitude for the process (2.11) can be described also by means of the following bounded multilinear functional

$$\begin{aligned} \textcircled{H} (Jf^{(1)}, Jf^{(2)}, f^{(3)}, f^{(4)}) &= ([f^{(1)} \otimes f^{(2)}], (S-1)[f^{(3)} \otimes f^{(4)}]) , \\ f^{(i)} &\in \mathcal{H}^{(i)} . \end{aligned} \quad (3.2)$$

This functional is defined over³⁴⁾

$$\overline{\mathcal{H}}^{(1)} \times \overline{\mathcal{H}}^{(2)} \times \mathcal{H}^{(3)} \times \mathcal{H}^{(4)} . \quad (3.3)$$

Following Hadjioannou¹⁵⁾, we define the Hilbert spaces

$$\mathcal{H}^{(A)} = \overline{\mathcal{H}}^{(1)} \otimes \mathcal{H}^{(3)}, \quad \mathcal{H}^{(B)} = \mathcal{H}^{(2)} \otimes \overline{\mathcal{H}}^{(4)}, \quad (3.4)$$

and consider a functional

$$\Phi(f^{(B)}, f^{(A)}) , \quad f^{(A)} \in \mathcal{H}^{(A)}, \quad f^{(B)} \in \mathcal{H}^{(B)}, \quad (3.5)$$

antilinear with respect to the first argument and linear with respect to the second argument (we call it a bilinear form) with the property

$$\Phi(f^{(2)} \otimes Jf^{(4)}, Jf^{(1)} \otimes f^{(3)}) = \textcircled{H} (Jf^{(1)}, Jf^{(2)}, f^{(3)}, f^{(4)}) . \quad (3.6)$$

From this equation, we can compute the value of Φ for any pair of arguments belonging to the linear spaces

$$\begin{cases} \overline{\mathcal{H}}^{(1)} \otimes \mathcal{H}^{(3)} = \mathcal{L}^{(A)} \subset \mathcal{H}^{(A)}, \\ \mathcal{H}^{(2)} \otimes \overline{\mathcal{H}}^{(4)} = \mathcal{L}^{(B)} \subset \mathcal{H}^{(B)}, \end{cases} \quad (3.7)$$

where the symbol $\otimes$ indicates the tensor product between simple linear spaces (see Appendix D).

Clearly the form Φ is defined over $\mathcal{L}^{(A)} \times \mathcal{L}^{(B)}$. If it is bounded, a general theorem [see Ref. 29), p. 90] tells us that it can be extended

to a bounded bilinear form defined over $\mathcal{H}^{(A)} \times \mathcal{H}^{(B)}$, which can be written as

$$\Phi(f^{(B)}, f^{(A)}) = (f^{(B)}, R f^{(A)}), \quad (3.8)$$

where R is a bounded linear operator from $\mathcal{H}^{(A)}$ to $\mathcal{H}^{(B)}$.

If the Hilbert spaces we are considering are finite-dimensional, as happens if the group G is compact (e.g. the isotopic spin group), the bilinear form is certainly bounded and the operator R exists. However, if G contains the Poincaré group, all the Hilbert spaces we have considered are infinite-dimensional and it is no longer possible to prove that Φ is bounded and that the representation (3.8) exists, even with unbounded R . We shall see later that in many physical situations Eq. (3.8) cannot actually be written.

If we define the representations

$$\begin{cases} U^{(A)} = \overline{U^{(1)}} \otimes U^{(3)}, \\ U^{(B)} = U^{(2)} \otimes \overline{U^{(4)}}, \end{cases} \quad (3.9)$$

it is easy to show that the linear spaces $\mathcal{L}^{(A)}$ and $\mathcal{L}^{(B)}$ are invariant under $U^{(A)}$ and $U^{(B)}$, respectively, and that the bilinear form Φ has the following invariance property

$$\Phi(U_g^{(B)} f^{(B)}, U_g^{(A)} f^{(A)}) = \Phi(f^{(B)}, f^{(A)}), \quad \forall g \in G. \quad (3.10)$$

It follows that if the operator R exists, it satisfies the condition

$$R U_g^{(A)} = U_g^{(B)} R, \quad \forall g \in G. \quad (3.11)$$

Then we can use theorem 4 of Section 2 and write

$$\begin{cases} U_g^{(A)} = K_A \left[\int_{\Omega_A}^{\oplus} U_g^{(A)\lambda} d\mu(\lambda) \right] K_A^{-1}, & U^{(A)\lambda} = \sum_{k=1}^{m_\lambda^{(A)}} \oplus D^\lambda, \\ U_g^{(B)} = K_B \left[\int_{\Omega_B}^{\oplus} U_g^{(B)\lambda} d\mu(\lambda) \right] K_B^{-1}, & U^{(B)\lambda} = \sum_{k=1}^{m_\lambda^{(B)}} \oplus D^\lambda, \\ R = K_B \left[\int_{\Omega_A \cap \Omega_B}^{\oplus} R^\lambda d\mu(\lambda) \right] K_A^{-1}. \end{cases} \quad (3.12)$$

If it is

$$\begin{cases} f^{(A)} = K_A \left[\int_{\Omega_A}^{\oplus} f^{(A)\lambda} d\mu(\lambda) \right], & f^{(A)\lambda} = \sum_{k=1}^{m_\lambda^{(A)}} \oplus f_k^{(A)\lambda}, \\ f^{(B)} = K_B \left[\int_{\Omega_B}^{\oplus} f^{(B)\lambda} d\mu(\lambda) \right], & f^{(B)\lambda} = \sum_{k=1}^{m_\lambda^{(B)}} \oplus f_k^{(B)\lambda}, \end{cases} \quad (3.13)$$

from Eqs. (3.8), (3.12), and (3.13) we have

$$\Phi(f^{(B)}, f^{(A)}) = \int_{\Omega_A \cap \Omega_B} (f^{(B)\lambda}, R^\lambda f^{(A)\lambda}) d\mu(\lambda), \quad (3.14)$$

and using Eq. (2.20)

$$\Phi(f^{(B)}, f^{(A)}) = \int_{\Omega_A \cap \Omega_B} \sum_{k, k'} T_{kk'}^\lambda (f_k^{(B)\lambda}, f_{k'}^{(A)\lambda}) d\mu(\lambda), \quad (3.15)$$

where $T_{kk'}^\lambda$ are complex numbers, which we may call the crossed partial wave amplitudes.

We may write Eq. (3.15) in the form

$$\Phi(f^{(B)}, f^{(A)}) = \int_{\Omega_A \cap \Omega_B} \sum_{k, k'} T_{kk'}^\lambda \psi_{kk'}^\lambda d\mu(\lambda), \quad (3.16)$$

where

$$\psi_{kk'}^\lambda = (f_k^{(B)\lambda}, f_{k'}^{(A)\lambda}). \quad (3.17)$$

We may consider the integral in the right-hand side of Eq. (3.16) as a linear functional in the space of the functions $\psi_{kk'}^\lambda$ and write

$$\Phi(f^{(B)}, f^{(A)}) = T(\psi). \quad (3.18)$$

In the general case of physical interest, the bilinear form Φ is not bounded, the operator R does not exist, and Eq. (3.16) cannot be derived. However, we may hope that it is still possible to write Eq. (3.18) where $T(\psi)$ is a linear functional of a more general kind. We shall show in Section 8, in a more general context, that the representation (3.18) of the scattering amplitude can be written in all the physically interesting cases.

4. THE INVARIANCE OF THE AMPLITUDE WITH RESPECT TO SYMMETRY OPERATIONS CONTAINING TIME REVERSAL

We now want to take into account the symmetries which contain time reversal and are represented by antiunitary operators³⁵⁻³⁷). We assume that all the symmetry operators form a co-representation of the group G' . The elements of G' which do not reverse the time direction form a subgroup G , which is the group considered in the preceding sections.

We assume that G' contains the TCP transformation, and for the sake of brevity we call it this element. Then every element of G' has the form g or tg with $g \in G$.

The invariance of the S -operator with respect to an element of G' which contains time reversal is not given by Eq. (2.4), but by

$$U_g S U_g^{-1} = S^* = S^{-1}, \quad g \in tG. \quad (4.1)$$

If we also take into account the fact that U_g in this case is antiunitary, we have

$$(f^{(F)}, S f^{(I)}) = (U_g f^{(I)}, S U_g f^{(F)}) , \quad g \in tG. \quad (4.2)$$

This invariance property connects amplitudes which describe different kinds of reactions and therefore cannot be written in terms of the single functional Θ defined in the preceding section. If we want to take into account time reversal in the crossed partial wave analysis, we have to treat together all the processes which are connected by the substitution

rule³⁸⁾, and it is therefore necessary to introduce a generalization of the functionals Θ and Φ .

Note that if f is a state which contains a particle, $U_t f$ contains the corresponding antiparticle. We call $\mathcal{H}^{(ic)}$ the Hilbert space containing all the states of the form $U_t f$ with $f \in \mathcal{H}^{(i)}$. It may be that $\mathcal{H}^{(ic)}$ and $\mathcal{H}^{(i)}$ are the same space; this certainly happens if G contains an element which transforms particles in the corresponding antiparticles.

Note also that in the Eq. (4.2) written for $g = t$, an incoming particle in the left-hand side becomes an outgoing antiparticle in the right-hand side, and so on. This fact suggests the definition of the following new Hilbert spaces³⁹⁾

$$\tilde{\mathcal{H}}^{(i)} = \mathcal{H}^{(i)} \oplus \overline{\mathcal{H}}^{(ic)} \quad (4.3)$$

which contain elements of the form

$$\tilde{f}^{(i)} = f^{(i)} \oplus J f^{(ic)}, \quad f^{(i)} \in \mathcal{H}^{(i)}, \quad f^{(ic)} \in \mathcal{H}^{(ic)}. \quad (4.4)$$

It is also useful, in order to obtain a more symmetric treatment, to write the process we are considering in the form

$$(3) + (4) \rightarrow (1c) + (2c). \quad (4.5)$$

The six different processes which are connected with the process (4.5) by means of the substitution rule can be described by means of a single multilinear functional $\tilde{\Theta}$ defined on

$$\tilde{\mathcal{H}}^{(1)} \times \tilde{\mathcal{H}}^{(2)} \times \tilde{\mathcal{H}}^{(3)} \times \tilde{\mathcal{H}}^{(4)} \quad (4.6)$$

by means of the formula

$$\begin{aligned} \tilde{\Theta} & \left(f^{(1)} \oplus J f^{(1c)}, f^{(2)} \oplus J f^{(2c)}, f^{(3)} \oplus J f^{(3c)}, f^{(4)} \oplus J f^{(4c)} \right) = \\ & = \left([f^{(1c)} \otimes f^{(2c)}], (S-1)[f^{(3)} \otimes f^{(4)}] \right) \pm \left([f^{(3c)} \otimes f^{(4c)}], (S-1)[f^{(1)} \otimes f^{(2)}] \right) \pm \\ & \pm \left([f^{(1c)} \otimes f^{(3c)}], (S-1)[f^{(2)} \otimes f^{(4)}] \right) \pm \left([f^{(2c)} \otimes f^{(4c)}], (S-1)[f^{(1)} \otimes f^{(3)}] \right) \pm \\ & \pm \left([f^{(3c)} \otimes f^{(2c)}], (S-1)[f^{(1)} \otimes f^{(4)}] \right) \pm \left([f^{(1c)} \otimes f^{(4c)}], (S-1)[f^{(3)} \otimes f^{(2)}] \right), \end{aligned} \quad (4.7)$$

where the minus sign before a given term has to be selected if the number of fermion exchanges necessary to obtain this term from the first term is odd. It is easy to show that, with this choice of the signs, the functional defined by Eq. (4.7) is symmetric for the exchange of the arguments describing two bosons of the same kind, and is antisymmetric for the exchange of two arguments referring to two fermions of the same kind.

In order to express the invariance properties of the functional (4.7), we define the operators $V_g^{(i)}$, which operate in the spaces $\mathcal{H}^{(i)}$ in the following way

$$V_g^{(i)}(f^{(i)} \oplus J f^{(ic)}) = U_g^{(i)} f^{(i)} \oplus J U_g^{(ic)} f^{(ic)}, \quad g \in G \quad (4.8)$$

$$V_g^{(i)}(f^{(i)} \oplus J f^{(ic)}) = (-1)^{2j_i} U_g^{(ic)} f^{(ic)} \oplus J U_g^{(i)} f^{(i)}, \quad g \in tG, \quad (4.9)$$

where j_i is the spin of the particle (i). Note that the operator $V_g^{(i)}$ is unitary for any element $g \in G'$. A simple calculation starting from Eqs. (2.4), (4.2), and (4.7) to (4.9) shows that the invariance of the scattering operator with respect to an element g of G' containing or not the time reversal can be written in the form

$$\begin{aligned} \tilde{\Theta} (V_g^{(1)} \tilde{f}^{(1)}, V_g^{(2)} \tilde{f}^{(2)}, V_g^{(3)} \tilde{f}^{(3)}, V_g^{(4)} \tilde{f}^{(4)}) = \\ = \tilde{\Theta} (\tilde{f}^{(1)}, \tilde{f}^{(2)}, \tilde{f}^{(3)}, \tilde{f}^{(4)}) . \end{aligned} \quad (4.10)$$

In this calculation we see that the factor $(-1)^{2j_i}$ which appears in Eq. (4.9) is necessary in order to cancel the minus signs which have been inserted in Eq. (4.7) in order to obtain the right symmetry properties of the multilinear functional.

From the definitions (4.8) and (4.9), we obtain

$$V_g^{(i)} V_{g'}^{(i)} = V_{gg'}^{(i)}, \quad (4.11)$$

if at least one of the elements g and g' belongs to G and

$$V_g^{(i)} V_{g'}^{(i)} = (-1)^{2j_i} V_{gg'}^{(i)} \quad \text{if } g \in tG, g' \in tG. \quad (4.12)$$

We see from these equations that the operators $V_g^{(i)}$ form a unitary ray representation of the symmetry group, but do not form a unitary representation of the "quantum mechanical symmetry group" G' [which is homomorphic to the symmetry group, but contains more elements³⁶].

However, we note that

$$(-1)^{2j_i} = V_{e'}^{(i)}, \quad (4.13)$$

where $e' \in G$ is an element which commutes with all the elements of G' and represents "a rotation of an angle 2π ". Therefore we have that the operators $V_g^{(i)}$ form a representation of the group $\tilde{G}$ which contains the same elements as G' , but in which the product of two elements containing time reversal differs from the corresponding product in G' for a factor e' .

Now we can follow the procedure described in the preceding section, i.e. we introduce the Hilbert spaces

$$\begin{cases} \tilde{\mathcal{H}}^{(A)} = \tilde{\mathcal{H}}^{(1)} \otimes \tilde{\mathcal{H}}^{(3)}, \\ \tilde{\mathcal{H}}^{(B)} = \tilde{\mathcal{H}}^{(2)} \otimes \tilde{\mathcal{H}}^{(4)}, \end{cases} \quad (4.14)$$

and the bilinear form

$$\tilde{\Phi}(\tilde{f}^{(B)}, \tilde{f}^{(A)}), \quad \begin{cases} \tilde{f}^{(A)} \in \tilde{\mathcal{L}}^{(A)} = \tilde{\mathcal{H}}^{(1)} \otimes \tilde{\mathcal{H}}^{(3)} \subset \tilde{\mathcal{H}}^{(A)}, \\ \tilde{f}^{(B)} \in \tilde{\mathcal{L}}^{(B)} = \tilde{\mathcal{H}}^{(2)} \otimes \tilde{\mathcal{H}}^{(4)} \subset \tilde{\mathcal{H}}^{(B)}, \end{cases} \quad (4.15)$$

with the property

$$\begin{aligned} & \tilde{\Phi}(\tilde{f}^{(2)} \otimes \tilde{f}^{(4)}, \tilde{f}^{(1)} \otimes \tilde{f}^{(3)}) = \\ & = \tilde{\Theta}(\tilde{f}^{(1)}, \tilde{f}^{(2)}, \tilde{f}^{(3)}, \tilde{f}^{(4)}). \end{aligned} \quad (4.16)$$

We also define the following representations of the group $\tilde{G}$

$$\begin{cases} V^{(A)} = V^{(1)} \otimes V^{(3)}, \\ V^{(B)} = \overline{V}^{(2)} \otimes \overline{V}^{(4)}. \end{cases} \quad (4.17)$$

The spaces $\tilde{\mathcal{L}}^{(A)}$ and $\tilde{\mathcal{L}}^{(B)}$ (which contain all the finite sums of vectors of the type $\tilde{f}^{(1)} \otimes \tilde{f}^{(3)}$ and $\tilde{J}\tilde{f}^{(2)} \otimes \tilde{J}\tilde{f}^{(4)}$) are invariant with respect to the representations (4.17) and the bilinear form $\tilde{\Phi}$ satisfies the invariance condition

$$\tilde{\Phi}(V_g^{(B)} \tilde{f}^{(B)}, V_g^{(A)} \tilde{f}^{(A)}) = \tilde{\Phi}(\tilde{f}^{(B)}, \tilde{f}^{(A)}), \quad \forall g \in \tilde{G}. \quad (4.18)$$

If this bilinear form is bounded, we can write a decomposition completely similar to Eq. (3.15). We shall show that in the general case it is always possible to write a formula similar to Eq. (3.18). Of course, the method described in this section is more powerful than the method of Section 3, because it takes into account a larger class of symmetry properties. We call it the "generalized partial wave expansion", as it contains the usual partial wave expansion for the two processes

$$\begin{aligned} (1) + (3) &\rightarrow (2c) + (4c) \\ (2) + (4) &\rightarrow (1c) + (3c) \end{aligned} \quad (4.19)$$

and the crossed partial wave expansion for the other four processes we are considering.

5. THE REPRESENTATIONS OF THE GROUP $\tilde{G}$

In order to study the representations $V_g^{(i)}$ defined in the preceding section, we shall describe the elements of the Hilbert spaces $\tilde{\mathcal{H}}^{(i)}$ by means of suitable wave functions.

As is well known, in defining a wave function for relativistic one-particle states [as the elements of the spaces $\tilde{\mathcal{H}}^{(i)}$ and $\tilde{\mathcal{H}}^{(ic)}$], special care is needed in specifying the meaning of the spin index. For instance, using different specifications, one obtains the amplitudes used

by Wigner³⁾ or the helicity amplitudes defined by Jacob and Wick³⁰⁾. We shall use a formalism developed by Moussa and Stora²⁴⁾, which contains the above-mentioned formalisms as special cases. This formalism is based on Mackey's theory of induced representations^{21,22)} which is a generalization of the method used by Wigner to find the irreducible representations of the Poincaré group. Of course, the Moussa-Stora formalism has to be generalized in order to describe the spaces $\mathcal{H}^{(i)}$ and the representations $V^{(i)}$.

Our group $\tilde{G}$ is the semidirect product of the translation group T and the "homogeneous" group $\tilde{A}$ which contains the group $SL(2C)$ (homomorphic to the connected subgroup of the Lorentz group), the element t , and eventually the space inversion s and some internal symmetries. We call A the subgroup of $\tilde{A}$ which contains all the elements which do not invert the time direction.

If we write the elements of $\tilde{G}$ in the form

$$g = (x, a), \quad g \in \tilde{G}, \quad x \in T, \quad a \in \tilde{A}, \quad (5.1)$$

the composition rule is

$$(x, a)(x', a') = (x + L(a)x', aa'), \quad (5.2)$$

where $L(a)$ is the Lorentz (4×4) matrix which corresponds to the element a of $\tilde{A}$. In the following we shall often write a instead of $(0, a)$.

We call $\varphi_m^{(i)}$ a (non-normalizable) state representing a particle of the kind (i) at rest. The subscript m is the projection of the spin along the z axis; other subscripts may be necessary if internal symmetries are taken into account, but we do not write them explicitly. The four-momentum of this state is⁴⁰⁾

$$P^{0(i)} = (M_i, 0, 0, 0). \quad (5.3)$$

We call H_+ the corresponding little group defined by

$$h \in H_+ \text{ if } h \in \tilde{A}, \quad L(h)P^{0(i)} = P^{0(i)}. \quad (5.4)$$

This group contains $SU(2)$ (isomorphic to the proper rotation group) and, if they are taken into account, the space inversion and the internal symmetry group. The states $\varphi_m^{(i)}$ have the following transformation property

$$U_{(x,h)}^{(i)} \varphi_m^{(i)} = \exp[i(x, P^{(i)})] \sum_{m'} R_{m'm}^{(i)}(h) \varphi_{m'}^{(i)}, \quad h \in H_+, \quad (5.5)$$

where $R_{m'm}^{(i)}(h)$ is a unitary irreducible representation of the little group H_+ .

In the following part of this section we consider only particles of a given kind and therefore we drop the index (i) . For every four-momentum P with the property $P^2 = M^2$, we choose an element a_P of $\tilde{A}$ with the property

$$L(a_P) P^0 = P, \quad (5.6)$$

and, for $P_t > 0$, we define the states

$$\varphi_m(P) = U_{a_P} \varphi_m. \quad (5.7)$$

Our normalization condition written in terms of these states has the form

$$(\varphi_m(P), \varphi_{m'}(P')) = 2(\vec{P}^2 + M^2)^{\frac{1}{2}} \delta^3(\vec{P} - \vec{P}') \delta_{mm'}, \quad (5.8)$$

and the corresponding completeness property can be written as

$$f = \int \sum_m \varphi_m(P) (\varphi_m(P), f) d\mu_{M^2}^+(P), \quad \forall f \in \mathcal{H}, \quad (5.9)$$

where $\mu_{M^2}^+$ is the invariant measure in the four-momentum space concentrated in the orbit $P^2 = M^2$, $P_t > 0$. It is given by Eq. (A.2) of Appendix A.

If f is a one-particle state, the corresponding Moussa-Stora wave function is a function defined on the group G by means of the formula

$$f_m(g) = (U_g \varphi_m, f), \quad g \in G, \quad f \in \mathcal{H}. \quad (5.10)$$

From Eq. (5.9) it follows that the scalar product can be written in the form

$$(f, f') = \int \sum_m \overline{f_m(a_p)} f'_m(a_p) d\mu_{M^2}^+(p). \quad (5.11)$$

The other properties of these wave functions can be found in Ref. 24), and therefore we treat directly the generalized wave functions which describe vectors of the type

$$\tilde{f} = f \oplus J f^{(c)} \in \tilde{\mathcal{H}} = \mathcal{H} \oplus \overline{\mathcal{H}}^{(c)}. \quad (5.12)$$

They are functions defined on the group $\tilde{G}$ by means of the formulae

$$\begin{cases} \tilde{f}_m(g) = (U_g \varphi_m, f) & \text{if } g \in G, \\ \tilde{f}_m(g) = (f^{(c)}, U_g \varphi_m) & \text{if } g \in tG. \end{cases} \quad (5.13)$$

We define the states $\varphi_m^{(c)}$ (representing antiparticles at rest with the spin projection m along the z axis) by means of the formula

$$\sum_{m'} C_{m'm}^j \varphi_{m'}^{(c)} = U_t \varphi_m, \quad (5.14)$$

where

$$C_{m'm}^j = (-1)^{j-m} \delta_{m, -m'}. \quad (5.15)$$

From the second part of Eq. (5.13) we have

$$\tilde{f}_m(gt) = \sum_{m'} C_{m'm}^j \overline{(U_g \varphi_{m'}^{(c)}, f^{(c)})}, \quad g \in G. \quad (5.16)$$

This equation gives a relation between the generalized wave function with argument gt ($g \in G$) and the complex conjugate Moussa-Stora wave function for the antiparticle state $f^{(c)}$. It is also

$$\tilde{f}_m(tg) = (U_g \varphi_m, [U_t]^{-1} f^{(c)}), \quad g \in G \quad (5.17)$$

and therefore we have

$$\begin{aligned}
 (\tilde{f}, \tilde{f}') &= (f, f') + \overline{(f^{(c)}, f'^{(c)})} = \\
 &= (f, f') + ([U_t]^{-1} f^{(c)}, [U_t]^{-1} f'^{(c)}) = \\
 &= \int \sum_m \left[\overline{\tilde{f}_m(a_p)} \tilde{f}'_m(a_p) + \overline{\tilde{f}_m(ta_p)} \tilde{f}'_m(ta_p) \right] d\mu_{M^2}^+(p) = \quad (5.18) \\
 &= \int \sum_m \overline{\tilde{f}_m(a_p)} \tilde{f}'_m(a_p) d\mu_{M^2}(p),
 \end{aligned}$$

where μ_{M^2} indicates the invariant measure in the space of the four-momenta concentrated on the orbit $P^2 = M^2$ (without any restriction on the sign of P_t); this measure is defined by Eq. (A.3) of Appendix A. We have used the convention

$$t a_p = a_{-p} \quad (5.19)$$

and Eq. (A.4).

The properties of the generalized wave functions are rather similar to the properties of the Moussa-Stora wave functions. From Eqs. (5.5) and (5.13), taking into account the unitary or antiunitary character of the operators U_g , we obtain the following covariance property

$$\begin{aligned}
 \tilde{f}_m(g(x, h)^{-1}) &= \exp[i(x, P^0)] \sum_{m'} R_{mm'}(h) \tilde{f}_{m'}(g), \quad (5.20) \\
 g &\in \tilde{G}, \quad h \in H_+.
 \end{aligned}$$

In order to obtain the transformation properties of the generalized wave function, we note that from Eqs. (5.13) and (4.8) it follows that for $g \in G$

$$\begin{cases} [V_g \tilde{f}]_m(g') = (U_{g'} \Psi_m, U_g f) = (U_{g^{-1}g'} \Psi_m, f), & g' \in G, \\ [V_g \tilde{f}]_m(g') = (U_{g'}^{(c)} f^{(c)}, U_{g'} \Psi_m) = (f^{(c)}, U_{g^{-1}g'} \Psi_m), & g' \in tG, \end{cases} \quad (5.21)$$

and from Eq. (4.9) we have for $g \in tG$

$$\begin{cases} [V_g \tilde{f}]_m(g') = (V_g \varphi_m, U_g^{(c)} f^{(c)}) (-1)^{2j} = \\ = (-1)^{2j} (f^{(c)}, [U_g]^{-1} U_g \varphi_m) = (f^{(c)}, U_{g^{-1}g'} \varphi_m), \quad g' \in G, \\ [V_g \tilde{f}]_m(g') = (U_g f, U_g \varphi_m) = (U_{g^{-1}g'} \varphi_m, f), \quad g' \in tG. \end{cases} \quad (5.22)$$

In the last step of the first Eq. (5.22), the factor $(-1)^{2j}$ disappears because we are using the multiplication rules of the group $\tilde{G}$, whereas the operators U_g form a representation of the group G' .

In all the cases considered we have

$$[V_g \tilde{f}]_m(g') = \tilde{f}_m(g^{-1}g'). \quad (5.23)$$

This is the transformation property of the generalized wave function. The Eqs. (5.18), (5.20), and (5.23) define completely the Hilbert space $\tilde{\mathcal{H}}$ and the representation V as an induced representation^{21,22}.

It is convenient to use wave functions defined over the group $\tilde{A}$ instead of over the whole group $\tilde{G}$. The original wave functions can always be obtained by means of the covariance property (5.20). When written in terms of these reduced wave functions, the properties (5.20) and (5.23) take the form

$$\tilde{f}_m(a h^{-1}) = \sum_{m'} R_{mm'}(h) \tilde{f}_{m'}(a), \quad a \in \tilde{A}, \quad h \in H_+, \quad (5.24)$$

and

$$[V_{(x,a)} \tilde{f}]_m(a') = \exp[i(x, L(a') P^0)] \tilde{f}_m(a^{-1}a') \quad (5.25)$$

The representations described above are not all the unitary irreducible representations of the group $\tilde{G}$. However, from a general theorem by Mackey^{21,22} and from the properties of the group, it follows that all the unitary irreducible representations of $\tilde{G}$ can be obtained as induced representations by means of a slight generalization of the procedure used above, i.e. by dropping the assumption that the four-momentum P^0 has the

form given by Eq. (5.5). Two four-vectors belonging to the same orbit [i.e. which can be transformed one into the other by means of a transformation $L(a)$ with $a \in \tilde{A}$] give rise to the same representations; therefore, it is sufficient to consider a representative four-vector for each orbit. Note that our classification of the orbits is different from the classification given by Wigner³⁾ because we are also considering Lorentz transformations which invert the time direction. There are four classes of orbits characterized by the following representative four-vectors

$$\begin{aligned} \text{a)} \quad (\sqrt{W}, 0, 0, 0) &= Q_W^0 \quad (W > 0) ; \\ \text{b)} \quad (0, 0, 0, \sqrt{-W}) &= Q_W^0 \quad (W < 0) ; \\ \text{c)} \quad (q, 0, 0, q) &= Q_{0q}^0 \quad (q \neq 0 \text{ fixed arbitrarily}); \\ \text{d)} \quad (0, 0, 0, 0) &= Q_0^0. \end{aligned} \quad (5.26)$$

Of course, the little groups corresponding to representative four-momenta belonging to different classes have a different structure and different representations. Only the classes (a) and (b) are of interest for the following investigations. We indicate by H_W and R^{WA} the corresponding little groups and their representations. The index A labels the inequivalent irreducible unitary representations of the same little group. Only the sign of W is of interest in this notation, and we shall often use the notations H_+ , H_- , $R^{(+)\Lambda}$ and $R^{(-)\Lambda}$. The group H_+ has been defined by Eq. (5.4), and the representations $R^{(i)}$ introduced at the beginning of this section should be written with the new notations as $R^{(+)\Lambda_i}$.

We indicate by V^Λ the irreducible representations of $\tilde{G}$ which are generated by the four-momentum Q_W^0 ($W \neq 0$) and by the representation R^{WA} of the little group. They operate in a Hilbert space of functions defined on $\tilde{A}$, which have the properties

$$f_m(a h^{-1}) = \sum_{m'} R_{m m'}^{W \Lambda}(h) f_{m'}(a), \quad a \in \tilde{A}, \quad h \in H_W, \quad (5.27)$$

$$\|f\|^2 = \int \sum_m |f_m(a_Q)|^2 d\mu_w(Q),$$

$$L(a_Q) Q_w^0 = Q, \quad (5.28)$$

and the representation operators are defined by

$$\left[V_{(x,a)}^{W \wedge} f \right]_m(a') = \exp[i(x, L(a') Q_w^0)] f_m(a^{-1} a'). \quad (5.29)$$

6. THE REPRESENTATIONS OF THE LITTLE GROUPS H_+ AND H_-

In order to describe the little groups and their representations, we have to specify the structure of the group $\tilde{A}$. We assume that $\tilde{A}$ contains $SL(2C)$, the TCP operation t , the space inversion s , and some internal symmetries.

We shall use for some elements of $SL(2C)$ the following notations⁴¹⁾

$$\begin{cases} u_z(\mu) = \exp(-\frac{1}{2} i \mu \sigma^z), \\ a_z(\xi) = \exp(\frac{1}{2} \xi \sigma^z), \end{cases} \quad (6.1)$$

where σ^z is a Pauli matrix. $L[u_z(\mu)]$ represents a rotation of an angle μ around the z axis and $L[a_z(\xi)]$ represents a pure Lorentz transformation along the z axis with velocity $\beta = \tanh \xi$. The elements $u_x(\mu)$, $u_y(\mu)$, $a_x(\xi)$, and $a_y(\xi)$ can be defined in a similar way and have a similar meaning. Note that the elements t and s do not belong to $SL(2C)$ and therefore cannot be represented by 2×2 matrices. Clearly it is

$$\begin{cases} L(t)(t, x, y, z) = (-t, -x, -y, -z), \\ L(s)(t, x, y, z) = (t, -x, -y, -z), \end{cases} \quad (6.2)$$

and $L(a) = 1$ if a is an internal symmetry.

We assume that the internal symmetry elements commute with the elements of $SL(2C)$ and that

$$\begin{cases} ta = at, & a \in SL(2C), \\ su = us, & u \in SU(2), \\ sa_z(\xi)s^{-1} = [a_z(\xi)]^{-1}. \end{cases} \quad (6.3)$$

Note that these relations are consistent with Eqs. (6.2). Other properties of the operations s and t have been investigated recently by Lee and Wick³⁷⁾. The result is that if we neglect the weak interactions, it is possible to define a parity operator U_s and a TCP operator U_t which commute with all the internal symmetry operators and have the properties

$$\begin{cases} (U_t)^2 = (-1)^{2j} , \\ (U_s)^2 = (-1)^{2j} , \\ U_s U_t = (-1)^{2j} U_t U_s \end{cases} \quad (6.4)$$

From these equations it follows that [see Eqs. (4.11) to (4.13)]

$$\begin{cases} (V_t)^2 = 1 , \\ (V_s)^2 = (-1)^{2j} = V_{e'} , \\ V_s V_t = (-1)^{2j} V_t V_s = V_{e'} V_t V_s , \end{cases} \quad (6.5)$$

and therefore the multiplication rules of the group $\tilde{A}$ are⁴²⁾

$$\begin{cases} t^2 = e , \\ s^2 = e' , \\ st = e' ts , \end{cases} \quad (6.6)$$

where e denotes the identity and $e' = u_z(2\pi)$.

According to the hypotheses described above (which have an experimental origin) the elements of the internal symmetry group commute with the elements of $SL(2C)$, s and t . In this situation, the internal symmetries can be treated separately, and therefore in the following we shall identify the group $\tilde{A}$ with the group generated by $SL(2C)$, s and t , whose multiplication rules are completely defined by Eqs. (6.3) and (6.6).

The group H_+ contains all the elements of the form u or su where u belongs to the subgroup $SU(2)$ of $SL(2C)$. In this case the index Λ used in the preceding section to label the representations of a little group stands for the spin j and the parity π . As is well known, the representations $R^{(+)\Lambda} = R^{(+)\pi j}$, or more simply $R^{\pi j}$, have the form

$$\begin{cases} R^{\pi j}_{mm'}(u) = R^j_{mm'}(u) , & u \in SU(2) , \\ R^{\pi j}_{mm'}(s) = \pi \delta_{mm'} , \end{cases} \quad (6.7)$$

where $R_{mm}^j(u)$ are the well-known representations of $SU(2)^{43}$ and $\pi = \pm 1$ if j is integral, $\pi = \pm i$ if j is half-integral.

It is easy to show that the elements of the group H_- have one of the forms v , $s'v$, $t'v$, $s't'v$, where v belongs to the subgroup $SU(1,1)$ of $SL(2C)$ and the elements s' and t' are defined by

$$\begin{cases} t' = \mu_x(\pi) t, \\ s' = \mu_y(\pi) s. \end{cases} \quad (6.8)$$

The group $SU(1,1)$ is homomorphic to the connected subgroup of the three-dimensional Lorentz group $O(2,1)$ and contains all the elements of $SL(2C)$ which satisfy the condition

$$v \sigma^z v^\dagger = \sigma^z. \quad (6.9)$$

From Eqs. (6.2) and (6.8) we have:

$$\begin{cases} L(t')(t, x, y, z) = (-t, -x, y, z), \\ L(s')(t, x, y, z) = (t, x, -y, z). \end{cases} \quad (6.10)$$

The multiplication rules for the group H_- are completely determined by the following relations which can easily be obtained from Eqs. (6.3), (6.6), and (6.8)

$$\begin{cases} s' a_x(\xi) s'^{-1} = t' a_x(\xi) t'^{-1} = a_x(\xi), \\ s' \mu_z(\mu) s'^{-1} = t' \mu_z(\mu) t'^{-1} = [\mu_z(\mu)]^{-1}, \end{cases} \quad (6.11)$$

$$\begin{cases} t'^2 = e', \\ s'^2 = e, \\ s' t' = t' s'. \end{cases} \quad (6.12)$$

The irreducible unitary representations of H_- can be constructed starting from the irreducible unitary representations of $SU(1,1)$ given by Bargmann⁴⁾. We use for the representations of $SU(1,1)$ the notations

introduced in Ref. 14), i.e. we indicate by $D^{\epsilon l}$ ($\epsilon = 0, \frac{1}{2}; \text{Re } l = -\frac{1}{2}$) the unitary representations of the principal series of the continuous classes and by $D^{k\pm}$ ($k = \frac{1}{2}, 1, \frac{3}{2} \dots$) the representations of the discrete classes. For the matrix elements of the representation operators we use the expressions⁴⁴⁾

$$\left\{ \begin{array}{l} D_{mm'}^{\epsilon l} (u_z(\mu) a_x(z) u_z(\nu)) = \exp(-i m \mu - i m' \nu) d_{mm'}^{\epsilon l}(\zeta), \\ m, m' = \epsilon, \epsilon \pm 1, \epsilon \pm 2 \dots \end{array} \right. \quad (6.13)$$

$$\left\{ \begin{array}{l} D_{mm'}^{k\pm} (u_z(\mu) a_x(z) u_z(\nu)) = \exp(-i m \mu - i m' \nu) d_{mm'}^{k\pm}(\zeta), \\ m, m' = k, k+1, k+2 \dots \text{ for } D^{k+}, \\ m, m' = -k, -k-1, -k-2 \dots \text{ for } D^{k-}. \end{array} \right. \quad (6.14)$$

The functions $d_{mm'}^{\epsilon l}(\zeta)$ and $d_{mm'}^{k\pm}(\zeta)$ are given explicitly in Ref. 14). We shall use here only the properties

$$\left\{ \begin{array}{l} d_{-m, -m'}^{\epsilon l}(\zeta) = d_{mm'}^{\epsilon l}(\zeta), \\ d_{-m, -m'}^{k+}(\zeta) = (-1)^{m-m'} d_{mm'}^{k-}(\zeta), \\ d_{00}^{\epsilon l}(\zeta) = P_{\epsilon}(\cosh \zeta). \end{array} \right. \quad (6.15)$$

We shall consider two kinds of classes of representations of H_-

- a) The continuous classes. The symbol Λ stands for the four indices $\sigma, \tau, \epsilon, l$, and the representations $R^{(-)\Lambda} = R^{(-)\sigma\tau\epsilon l}$ (which we shall also indicate more simply by $\tilde{D}^{\sigma\tau\epsilon l}$) are given by

$$\left\{ \begin{array}{l} \tilde{D}_{mm'}^{\sigma\tau\epsilon l}(v) = D_{mm'}^{\epsilon l}(v), \quad v \in SU(1,1), \\ \tilde{D}_{mm'}^{\sigma\tau\epsilon l}(\gamma) = \sigma \delta_{m, -m'}, \\ \tilde{D}_{mm'}^{\sigma\tau\epsilon l}(t') = \tau \delta_{m, -m'}, \\ \text{Re } l = -\frac{1}{2}, \text{Im } l > 0; \quad \epsilon = 0, \frac{1}{2}; \\ \sigma = \pm 1, \quad \tau^2 = (-1)^{2\epsilon}. \end{array} \right. \quad (6.16)$$

b) The discrete class. The symbol Λ stands for the two indices π and k and the representations $R^{(-)\Lambda} = R^{(-)\pi k}$ (or more simply $\tilde{D}^{\pi k}$) are given by $[v \in SU(1,1)]$

$$\left\{ \begin{array}{l} \tilde{D}_{mm'}^{\pi k}(v) = \begin{cases} D_{mm'}^{k+}(v) & \text{for } m, m' = k, k+1, \dots \\ D_{mm'}^{k-}(v) & \text{for } m, m' = -k, -k-1, \dots \\ 0 & \text{for other values of } m, m'; \end{cases} \\ \tilde{D}_{mm'}^{\pi k}(v') = \pi (-1)^{m+k} \delta_{m, -m'}, \\ \tilde{D}_{mm'}^{\pi k}(t') = (-1)^{m-k} \delta_{m, -m'}, \\ k = \frac{1}{2}, 1, \frac{3}{2}, \dots; \quad \pi^2 = (-1)^{2k}. \end{array} \right. \quad (6.17)$$

Note that these matrix elements are defined only for $|m|, |m'| = k, k+1, \dots$

It is easy to verify, by means of Eqs. (6.13) to (6.15), that the matrices defined by Eqs. (6.16) and (6.17) are representations of the group H_- . It is also easy to show that these representations are unitary, irreducible, and two by two inequivalent. By means of these representations, in Appendix B we shall derive a Plancherel formula for the group H_- . Therefore we may say that the unitary irreducible representations which we have omitted in the classification given above form a set of vanishing Plancherel measure⁴⁵⁾.

The matrix elements defined by Eq. (6.16) can be continued analytically in the whole complex ℓ plane and they still define representations of H_- , in general non-unitary and for integral values of $\ell - \epsilon$ also non-irreducible. From the properties of the representations $D^{\epsilon \ell}$ given in Refs. 11) and 14), we easily obtain the identity

$$(U_m^\ell)^{-1} \tilde{D}_{mm'}^{\sigma \tau \ell} (h) U_{m'}^\ell = \tilde{D}_{mm'}^{\sigma' \tau' \ell'} (h), \quad h \in H_-, \quad (6.18)$$

where

$$\sigma' = (-1)^{2\epsilon} \sigma, \quad \tau' = (-1)^{2\epsilon} \tau = \bar{\tau}, \quad \ell' = -\ell - 1. \quad (6.19)$$

and

$$U_m^{\ell} = \frac{\Gamma(\ell+m+1)}{\Gamma(m-\ell)}, \quad (6.20)$$

and the identity

$$\tilde{D}_{mm'}^{\pi k}(h) = (X_m^{\tau k})^{-1} \tilde{D}_{mm'}^{\sigma \tau \ell}(h) X_{m'}^{\tau k}, \quad (6.21)$$

where

$$\ell = k-1, \quad \pi = \sigma \tau, \quad (-1)^{2\ell} = (-1)^{2k}, \quad (6.22)$$

and

$$X_m^{\tau k} = \begin{cases} \left[\frac{(|m|+k-1)!}{(|m|-k)!} \right]^{\frac{1}{2}} & \text{for } m = k, k+1, \dots \\ \left[\frac{(|m|+k-1)!}{(|m|-k)!} \right]^{\frac{1}{2}} \tau (-1)^{k-m}, & m = -k, -k-1, \dots \end{cases} \quad (6.23)$$

7. THE DECOMPOSITION OF THE DIRECT PRODUCT OF TWO IRREDUCIBLE UNITARY REPRESENTATIONS OF $\tilde{G}$

We now have to describe and to decompose, according to the programme given in Section 4, the representation $V^{(A)}$ and the space $\tilde{\mathcal{H}}^{(A)}$. The representation $V^{(B)}$ and the space $\tilde{\mathcal{H}}^{(B)}$ can be treated in a similar way, as we shall show at the end of this section. We shall follow approximately the method given in Ref. 24), taking into account the modifications required by the presence of the reflections.

We describe the element $\tilde{f}^{(A)} = \tilde{f}^{(1)} \otimes \tilde{f}^{(3)}$ of $\tilde{\mathcal{H}}^{(A)}$ by means of the wave function

$$\tilde{f}_{m_1 m_3}^{(A)}(a_1, a_3) = \tilde{f}_{m_1}^{(1)}(a_1) \tilde{f}_{m_3}^{(3)}(a_3), \quad (7.1)$$

where the wave functions introduced in Section 5 appear in the right-hand side. A general element of $\tilde{\mathcal{H}}^{(A)}$ is described by a wave function $\tilde{f}_{m_1 m_3}^{(A)}(a_1, a_3)$ which satisfies the covariance property

$$\begin{aligned} \tilde{f}_{m_1 m_3}^{(A)}(a_1 h_1^{-1}, a_3 h_3^{-1}) &= \sum_{m'_1 m'_3} R_{m_3 m'_3}^{\pi_3 j_3}(h_3) \cdot \\ &\cdot R_{m_1 m'_1}^{\pi_1 j_1}(h_1) \tilde{f}_{m'_1 m'_3}^{(A)}(a_1, a_3), \quad h_1, h_3 \in H_+. \end{aligned} \quad (7.2)$$

We require also that the integral

$$\|\tilde{f}^{(A)}\|^2 = \iint \sum_{m_1 m_3} |\tilde{f}_{m_1 m_3}^{(A)}(a_{p^{(1)}}, a_{p^{(3)}})|^2 d\mu_{M_1^2}(p^{(1)}) d\mu_{M_3^2}(p^{(3)}), \quad (7.3)$$

which defines the norm, converges.

The representation $V^{(A)}$ operates on these wave functions in the following way

$$\begin{aligned} [V_{(x,a)}^{(A)} \tilde{f}^{(A)}]_{m_1 m_3}(a_1, a_3) &= \exp[i(x, L(a_1)P^{o(1)} + L(a_3)P^{o(3)})] \cdot \\ &\cdot \tilde{f}_{m_1 m_3}^{(A)}(\bar{a}^{-1} a_1, \bar{a}^{-1} a_3). \end{aligned} \quad (7.4)$$

Equations (7.2) to (7.4) describe completely $\tilde{\mathcal{H}}^{(A)}$ and $V^{(A)}$.

The decomposition into irreducible representations can be performed in two steps. The first step is an application of a general method developed by Mackey²²⁾ and is based on a transformation of variables. We define the four-momentum

$$Q = L(a_1)P^{o(1)} + L(a_3)P^{o(3)}, \quad Q^2 = W. \quad (7.5)$$

The possible values of W are

$$\begin{cases} W \geq (M_1 + M_3)^2 & \text{if } a_1, \bar{a}_3^{-1} \in A, \\ W \leq (M_1 - M_3)^2 & \text{if } a_1, \bar{a}_3^{-1} \in tA. \end{cases} \quad (7.6)$$

Q is the four-momentum transfer in the process (4.5) and the total four-momentum in the processes (4.19). For every possible value of $W \neq 0$, we choose two elements $b_{1W}, b_{2W} \in \tilde{A}$ with the property

$$L(b_{1W}) P^{0(1)} + L(b_{2W}) P^{0(3)} = Q_W^0, \quad (7.7)$$

where Q_W^0 is given by Eq. (5.26). In the following we assume that

$$M_3 \gg M_1, \quad M_2 \gg M_4 \quad (7.8)$$

Then a possible choice of the elements b_{1W} and b_{2W} is

$$\begin{cases} b_{1W} = a_z(\zeta_{1W}), \\ b_{2W} = a_z(\zeta_{2W}), \\ \text{for } W \gg (M_1 + M_3)^2, \end{cases} \quad (7.9)$$

$$\begin{cases} b_{1W} = a_z(\zeta_{1W}) t u_z(\pi), \\ b_{2W} = a_z(\zeta_{2W}), \\ \text{for } W \leq (M_1 - M_3)^2, \end{cases} \quad (7.10)$$

where ζ_{1W} and ζ_{2W} are given as functions of W in Appendix C. It can be shown that any pair a_1, a_3 of elements of $\tilde{A}$ can be written in the form

$$\begin{cases} a_1 = a b_{1W} h_1, \\ a_3 = a b_{2W} h_3, \\ a \in \tilde{A}, \quad h_1, h_3 \in H_+. \end{cases} \quad (7.11)$$

From Eq. (C.17) of Appendix C we have that Eq. (7.3) can be written in the form

$$\|\tilde{f}^{(A)}\|^2 = \int_{\Gamma(M_1, M_3)} \rho(W, M_1, M_3) dW \int d\mu_W(Q). \quad (7.12)$$

$$\int_{H_W} \sum_{m_1, m_3} |\tilde{f}_{m_1, m_3}^{(A)}(a_Q h b_{1W}, a_Q h b_{2W})|^2 dh,$$

where $\Gamma(M_1, M_3)$ is the set of the possible values of W defined by Eqs. (7.6), dh is the invariant measure over the little group H_W (see Appendix A), and a_Q is an arbitrary element of $\tilde{A}$ with the property

$$L(a_Q) Q_W^0 = Q. \quad (7.13)$$

If we define the functions

$$f_{m_1 m_3}^{(A)W}(a) = [P(W, M_1, M_3)]^{\frac{1}{2}} \tilde{f}_{m_1 m_3}^{(A)}(a b_{1W}, a b_{3W}), \quad (7.14)$$

the norm (7.3) can be written in the form

$$\|\tilde{f}^{(A)}\|^2 = \int_{\Gamma(M_1, M_3)} \|f^{(A)W}\|^2 dW, \quad (7.15)$$

where

$$\|f^{(A)W}\|^2 = \int d\mu_W(Q) \int_{H_W} \sum_{m_1, m_3} |f_{m_1 m_3}^{(A)W}(a_Q h)|^2 dh. \quad (7.16)$$

If we call $\mathcal{H}^{(A)W}$ the space of the functions $f_{m_1 m_3}^{(A)W}(a)$ for fixed W and with the norm (7.16), from Eq. (7.15) we see that Eq. (7.14) defines a unitary mapping between the Hilbert spaces $\mathcal{H}^{(A)}$ and

$$\int_{\Gamma(M_1, M_3)}^{\oplus} \mathcal{H}^{(A)W} dW \quad (7.17)$$

It is clear that the transformation (7.4) induces the following transformations in the Hilbert spaces $\mathcal{H}^{(A)W}$

$$\begin{aligned} [V_{(x,a)}^{(A)W} f^{(A)W}]_{m_1 m_3}(a') &= \exp[i(x, L(a') Q_W^0)] \cdot \\ &\cdot f_{m_1 m_3}^{(A)W}(a^{-1} a'). \end{aligned} \quad (7.18)$$

It is necessary to remark that the functions $f_{m_1 m_3}^{(A)W}(a)$ are not arbitrary, but satisfy the following covariance relation which is a consequence of

Eq. (7.2)

$$f_{m_1 m_3}^{(A)W}(a k^{-1}) = \sum_{m'_1 m'_3} R_{m_1 m'_1}^{\pi_1 j_1} (b_{1W}^{-1} k b_{1W}) \cdot R_{m_3 m'_3}^{\pi_3 j_3} (b_{3W}^{-1} k b_{3W}) f_{m'_1 m'_3}^{(A)W}(a), \quad (7.19)$$

$$k \in K_W^{(A)} = b_{1W} H + b_{1W}^{-1} \cap b_{3W} H + b_{3W}^{-1}.$$

If the elements b_{iW} have the form given by Eqs. (7.9) and (7.10), it can be shown that the group $K_W^{(A)}$ is formed by all the elements of the types $u_z(\mu)$ and $s'u_z(\mu)$. Moreover, all these elements commute with b_{iW} . As it is⁴³⁾

$$\begin{cases} R_{m m'}^{\pi j}(u_z(\mu)) = \exp(-i m \mu) \delta_{m m'}, \\ R_{m m'}^{\pi j}(s') = \pi (-1)^{j+m} \delta_{m, -m'}, \end{cases} \quad (7.20)$$

Eq. (7.18) becomes simply

$$\begin{cases} f_{m_1 m_3}^{(A)W}(a u_z(\mu)) = \exp[i(m_1 + m_3)\mu] f_{m_1 m_3}^{(A)W}(a), \\ f_{m_1 m_3}^{(A)W}(a s') = \pi_1 \pi_3 (-1)^{j_1 + m_1 + j_3 + m_3} f_{-m_1, -m_3}^{(A)W}(a). \end{cases} \quad (7.21)$$

Equations (7.16), (7.18), and (7.21) define completely the representations $V^{(A)W}$ as induced representations, and we have proved the direct integral decomposition

$$V^{(A)} \simeq \int_{\Gamma(M_1, M_3)}^{\oplus} V^{(A)W} dW. \quad (7.22)$$

Note that we do not need to consider the case $W = 0$ in all our discussion because this point has a vanishing measure in the direct integrals (7.17) and (7.22).

The representations $V^{(A)W}$ are not yet irreducible, and a further decomposition can be obtained performing the harmonic analysis of

$f_{m_1 m_3}^{(A)W}(ah)$ considered as a function of $h \in H_W$. In other words, we introduce the functions⁴⁶⁾

$$\begin{aligned} \int_{H_W} f_{m_1 m_3}^{(A)W}(ah) R_{m m'}^{W\Lambda}(h) dh &= \\ &= \delta_{m', m_1 + m_3} f_{m_1 m_3 m}^{(A)W\Lambda}(a). \end{aligned} \quad (7.23)$$

The term $\delta_{m', m_1 + m_3}$ appears as a consequence of the first Eq. (7.21) and of the identity

$$R_{m m'}^{W\Lambda}(u_z(\mu)) = \exp(-im\mu) \delta_{m m'}. \quad (7.24)$$

The second Eq. (7.21) gives the following condition

$$\begin{aligned} f_{m_1 m_3 m}^{(A)W\Lambda}(a) &= \int_{H_W} f_{m_1 m_3}^{(A)W}(ah\sigma') R_{m, m_1 + m_3}^{W\Lambda}(h\sigma') dh = \\ &= \pi_1 \pi_3 (-1)^{j_1 + m_1 + j_3 + m_3} \alpha_{m_1 + m_3}(W, \Lambda) f_{-m_1, -m_3, m}^{(A)W\Lambda}(a), \end{aligned} \quad (7.25)$$

where $\alpha_m(W, \Lambda)$ is defined by

$$R_{m m'}^{W\Lambda}(\sigma') = \alpha_{m'}(W, \Lambda) \delta_{m, -m'}. \quad (7.26)$$

From Eqs. (6.16), (6.17), and (7.20) we have

$$\begin{cases} \alpha_m(+, \pi, j) = \pi (-1)^{j-m}, \\ \alpha_m(-, \sigma, \tau, \epsilon, l) = \sigma, \\ \alpha_m(-, \pi, k) = \pi (-1)^{k-m}. \end{cases} \quad (7.27)$$

From Eq. (7.23) we obtain immediately the covariance condition

$$f_{m_1 m_3 m}^{(A)W\Lambda}(ah^{-1}) = \sum_{m'} R_{m m'}^{W\Lambda}(h) f_{m_1 m_3 m'}^{(A)W\Lambda}(a), \quad h \in H_W. \quad (7.28)$$

The Plancherel formula for $\mathbb{H}_W$ (see Appendix B) permits the norm (7.16) to be written in the form

$$\|f^{(A)W}\|^2 = \int d\mu_w(Q) \sum_{m_1, m_3} \int_{\Omega_{m_1, m_3}^{(A)W}} \sum_m |f_{m_1, m_3, m}^{(A)W\Lambda}(a_Q)|^2 d\Lambda, \quad (7.29)$$

where $\Omega_{m_1, m_3}^{(A)W}$ is a subset of $\hat{\mathbb{H}}_W$ which depends on $W, m_1, m_3, j_1, j_3, \pi_1, \pi_3$, and contains all the Λ 's for which the integral (7.23) is defined and not identically vanishing. The structure of this subset is summarized in Table 1. Equation (7.29) can also be written in the form

$$\|f^{(A)W}\|^2 = \sum'_{m_1, m_3} \int_{\Omega_{m_1, m_3}^{(A)W}} \|f_{m_1, m_3}^{(A)W\Lambda}\|^2 d\Lambda, \quad (7.30)$$

where

$$\begin{cases} \|f_{m_1, m_3}^{(A)W\Lambda}\|^2 = 2 \int \sum_m |f_{m_1, m_3, m}^{(A)W\Lambda}(a_Q)|^2 d\mu_w(Q), & m_1, m_3 \neq 0, \\ \|f_{0, 0}^{(A)W\Lambda}\|^2 = \int \sum_m |f_{0, 0, m}^{(A)W\Lambda}(a_Q)|^2 d\mu_w(Q). \end{cases} \quad (7.31)$$

We have used the symmetry relation (7.25), and we have indicated by $\sum'_{m_1, m_3}$ a sum in which the pairs (m_1, m_3) and $(-m_1, -m_3)$ are not considered as distinct and therefore only one of them is taken into account.

If we call $\mathcal{H}_{m_1, m_3}^{(A)W\Lambda}$ the Hilbert space which contains the functions $f_{m_1, m_3, m}^{(A)W\Lambda}(a)$ (with fixed W, Λ, m_1, m_3) which satisfy the condition (7.28) and with the norm given by Eq. (7.31), we see from Eq. (7.30) that Eq. (7.23) defines a unitary mapping between the spaces $\mathcal{H}^{(A)W}$ and

$$\sum'_{m_1, m_3} \oplus \int_{\Omega_{m_1, m_3}^{(A)W}} \mathcal{H}_{m_1, m_3}^{(A)W\Lambda} d\Lambda. \quad (7.32)$$

It is easy to show that the transformation (7.18) induces in the spaces

$\mathcal{H}_{m_1 m_3}^{(A)W\Lambda}$ the transformations

$$\left[V_{(x,a)}^{W\Lambda} f_{m_1 m_3}^{(A)W\Lambda} \right]_m(a') = \exp \left[i(x, L(a') Q_w^0) \right] \cdot f_{m_1 m_3 m}^{(A)W\Lambda}(\bar{a}^{-1} a'). \quad (7.33)$$

Equations (7.28), (7.31), and (7.33) define unitary representations of $\mathcal{G}$ which are identical to the irreducible representations $V^{W\Lambda}$ defined at the end of Section 5. Therefore our decomposition is complete, and the result can be written in the form

$$V^{(A)} \simeq \int_{\Gamma(M_1, M_3)} dW \sum'_{m_1 m_3} \oplus \int_{\Omega_{m_1 m_3}^{(A)W}} V^{W\Lambda} d\Lambda. \quad (7.34)$$

Note that, as $d\Lambda$ is the Plancherel measure for H_W , in Eq. (7.34) we may forget a set of representations $V^{W\Lambda}$ with Λ belonging to a set of vanishing Plancherel measure. In particular, the representation connected with the complementary series of representations of $SU(1,1)$ and the representations of the discrete class with $k = \frac{1}{2}$ do not appear in the direct product decomposition.

The decomposition of the representation $V^{(B)}$ can be performed by means of a procedure completely analogous to the procedure used above for the representation $V^{(A)}$, if we define the wave functions which describe the elements of $\mathcal{H}^{(B)}$ in a suitable way. We associate with the element $\tilde{f}^{(B)} = \tilde{f}^{(2)} \otimes \tilde{f}^{(4)}$ of $\mathcal{H}^{(B)}$ the wave function

$$\tilde{f}_{m_4 m_2}^{(B)}(a_4, a_2) = \sum_{m'_4} C_{m_4 m'_4}^{i_4} \overline{\tilde{f}_{m'_4}^{(4)}(a_4 t)}. \quad (7.35)$$

$$\cdot \sum_{m'_2} C_{m_2 m'_2}^{i_2} \overline{\tilde{f}_{m'_2}^{(2)}(a_2 t)},$$

where the matrices $C_{mm'}^j$, are defined by Eq. (5.15) and have the properties

$$\left\{ \begin{aligned} \sum_{m''} C_{mm''}^j \overline{R_{m''m'}^{\pi j}(h)} &= \sum_{m''} R_{mm''}^{\pi j}(h) C_{m''m'}^j, \quad h \in H_+, \\ \sum_{m''} C_{mm''}^j C_{m'm''}^j &= \delta_{mm'}, \quad C_{mm'}^j = (-1)^{2j} C_{m'm}^j. \end{aligned} \right. \quad (7.36)$$

From Eqs. (7.35), (7.36), and (5.24), using the identity

$$R_{mm'}^{\pi j}(t^{-1}ht) = R_{mm'}^{\pi j}(h), \quad h \in H_+, \quad (7.37)$$

we obtain the covariance property

$$\begin{aligned} \tilde{f}_{m_4 m_2}^{(B)}(a_4 h_4^{-1}, a_2 h_2^{-1}) &= \sum_{m'_4 m'_2} R_{m_4 m'_4}^{\pi_4 j_4}(h_4) \cdot \\ &\cdot R_{m_2 m'_2}^{\pi_2 j_2}(h_2) \tilde{f}_{m'_4 m'_2}^{(B)}(a_4, a_2), \end{aligned} \quad (7.38)$$

which can be obtained from Eq. (7.2) by means of the substitutions

$$(1) \rightarrow (4), \quad (3) \rightarrow (2), \quad (A) \rightarrow (B). \quad (7.39)$$

Of course, the covariance property (7.38) must also hold for wave functions which describe elements of $\tilde{\mathcal{H}}^{(B)}$ of a general type. Also the formula for the norm and the transformation property of the wave functions $\tilde{f}_{m_4 m_2}^{(B)}(a_4, a_2)$ can be obtained from the analogous equations (7.3) and (7.4) for the wave functions $\tilde{f}_{m_1 m_3}^{(A)}(a_1, a_3)$ by means of the substitution (7.39).

It follows that all the treatment given above can be repeated step by step introducing the substitutions (7.39). In particular, we may define the wave functions $f_{m_4 m_2}^{(B)W}(a)$ and $f_{m_4 m_2 m}^{(B)WA}(a)$ which we shall use in the

following sections. The last functions are defined for $W \in \Gamma(M_4, M_2)$, $\Lambda \in \Omega_{m_4 m_2}^{(B)W}$, where the sets $\Omega_{m_4 m_2}^{(B)W}$ are described by a table completely analogous to Table 1.

8. THE DECOMPOSITION OF THE BILINEAR FORM

If the bilinear form $\tilde{\Phi}(\tilde{f}^{(B)}, \tilde{f}^{(A)})$ defined by Eq. (4.16) is bounded and therefore generated by a bounded operator, the discussion given at the end of Section 3 shows that, by means of theorem 4 of Section 2, we may write

$$\tilde{\Phi}(\tilde{f}^{(B)}, \tilde{f}^{(A)}) = \int_{\Gamma} dW \int_{\Omega^W} d\Lambda \sum_{m_1 m_3} \sum_{m_2 m_4} T_{m_1 m_2 m_3 m_4}^{W\Lambda} \cdot \quad (8.1)$$

$$\cdot \int \sum_m \overline{f_{m_4 m_2 m}^{(B)W\Lambda}(a_Q)} f_{m_1 m_3 m}^{(A)W\Lambda}(a_Q) d\mu_W(Q),$$

where

$$\begin{cases} \Gamma = \Gamma(M_1, M_3) \cap \Gamma(M_4, M_2), \\ \Omega^W = \left[\bigcup_{m_1 m_3} \Omega_{m_1 m_3}^{(A)W} \right] \cap \left[\bigcup_{m_4 m_2} \Omega_{m_4 m_2}^{(B)W} \right]. \end{cases} \quad (8.2)$$

If we introduce the functions

$$\Psi_{m_1 m_2 m_3 m_4}^{W\Lambda} = \int \sum_m \overline{f_{m_4 m_2 m}^{(B)W\Lambda}(a_Q)} f_{m_1 m_3 m}^{(A)W\Lambda}(a_Q) d\mu_W(Q), \quad (8.3)$$

we may also write Eq. (8.1) in the form

$$\begin{aligned} \tilde{\Phi}(\tilde{f}^{(B)}, \tilde{f}^{(A)}) &= \\ &= \int_{\Gamma} dW \int_{\Omega^W} d\Lambda \sum_{m_1 m_3} \sum_{m_4 m_2} T_{m_1 m_2 m_3 m_4}^{W\Lambda} \Psi_{m_1 m_2 m_3 m_4}^{W\Lambda}. \end{aligned} \quad (8.4)$$

The functions (8.3) are defined for the values of the arguments which satisfy the condition

$$\Lambda \in \Omega_{m_1 m_3}^{(A)W} \cap \Omega_{m_4 m_2}^{(B)W}. \quad (8.5)$$

From Eq. (7.25) we obtain the property

$$\begin{aligned} \Psi_{m_1 m_2 m_3 m_4}^{W\Lambda} &= \pi_1 \pi_3 (-1)^{j_1+m_1+j_3+m_3} \alpha_{m_1+m_3}(W, \Lambda) \cdot \\ \cdot \Psi_{-m_1, m_2, -m_3, m_4}^{W\Lambda} &= \overline{\pi_4} \overline{\pi_2} (-1)^{j_4+m_4+j_2+m_2} \cdot \\ \cdot \overline{\alpha_{m_4+m_2}(W, \Lambda)} \Psi_{m_1, -m_2, m_3, -m_4}^{W\Lambda}. \end{aligned} \quad (8.6)$$

Also the functions $T_{m_1 m_2 m_3 m_4}^{W\Lambda}$ are defined for values of the arguments which satisfy the condition (8.5). This condition has to be taken into account when we perform the sums over m_i . We should have used in Eqs. (8.1) and (8.4) the sums $\sum_{m_1 m_3}$ and $\sum_{m_4 m_2}$ defined in Section 7, and used in Eqs. (7.30), (7.32), and (7.34), but we prefer to sum over all the possible values of m_i for which the functions to be added are defined and to impose the condition

$$\begin{aligned} T_{m_1 m_2 m_3 m_4}^{W\Lambda} &= \\ &= \overline{\pi_1} \overline{\pi_3} (-1)^{j_1+m_1+j_3+m_3} \overline{\alpha_{m_1+m_3}(W, \Lambda)} T_{-m_1, m_2, -m_3, m_4}^{W\Lambda} = \\ &= \pi_4 \pi_2 (-1)^{j_4+m_4+j_2+m_2} \alpha_{m_4+m_2}(W, \Lambda) T_{m_1, -m_2, m_3, -m_4}^{W\Lambda}. \end{aligned} \quad (8.7)$$

The complex numbers $T_{m_1 m_2 m_3 m_4}^{W\Lambda}$ can be called the "generalized partial wave amplitudes". For $W \geq \max [(M_1+M_3)^2, (M_4+M_2)^2]$, they coincide essentially with the usual partial wave amplitudes for the processes (4.19), and for $W \leq \min [(M_3-M_1)^2, (M_2-M_4)^2]$ they are the crossed partial wave amplitudes for the other four reactions. It is clear that any quantity

regarding the six processes we are considering can be expressed in terms of the generalized partial wave amplitudes defined above.

However, unfortunately, the bilinear form $\tilde{\Phi}$ is in general unbounded, and the formalism given above is not sufficient for the investigation of the physical scattering amplitude. Note that this difficulty does not arise for $W \geq \max [(M_1 + M_3)^2, (M_1 + M_2)^2]$, because the part of the integral (8.1) which corresponds to these values of W gives the part of the bilinear form which describes the processes (4.19), and this part of the bilinear form is generated directly by the scattering operator and is therefore bounded.

In order to obtain a more general formalism, sufficiently powerful for the physical applications, we follow the method outlined in the introduction and at the end of Section 3.

Equation (8.4) shows that the bounded bilinear form $\tilde{\Phi}(\tilde{f}^{(B)}, \tilde{f}^{(A)})$ depends on its arguments only through the function $\psi_{m_1 m_2 m_3 m_4}^{W\Lambda}$ defined by Eq. (8.3). In other words, the scattering amplitude can be described by means of a linear functional $T(\psi)$ in a space of functions of this kind. Now we want to show that this is true also if the bilinear form $\tilde{\Phi}$ is not bounded, but it describes a scattering amplitude which satisfies the unitarity condition for all the six processes we are considering. We shall also define a norm in the space of the functions ψ in such a way that the functional $T(\psi)$ is bounded if the unitarity conditions are satisfied.

In order to obtain the above described results, it is convenient to define some normed linear spaces^{29,47)}. We call $\mathcal{B}$ the linear space defined by the tensor product (see Appendix D)

$$\mathcal{B} = \tilde{\mathcal{H}}^{(1)} \otimes \tilde{\mathcal{H}}^{(2)} \otimes \tilde{\mathcal{H}}^{(3)} \otimes \tilde{\mathcal{H}}^{(4)}, \quad (8.8)$$

where the Hilbert spaces $\tilde{\mathcal{H}}^{(i)}$ are considered as simple linear spaces. The multilinear functional $\tilde{\Theta}$ defined in Section 4 generates a linear functional Ξ defined over the space $\mathcal{B}$ with the property

$$\Xi(\tilde{f}^{(1)} \otimes \tilde{f}^{(2)} \otimes \tilde{f}^{(3)} \otimes \tilde{f}^{(4)}) = \tilde{\Theta}(\tilde{f}^{(1)}, \tilde{f}^{(2)}, \tilde{f}^{(3)}, \tilde{f}^{(4)}) \quad (8.9)$$

For every element $g \in \tilde{G}$ we can define a linear operator V_g in $\mathcal{B}$ with the property

$$V_g(\tilde{f}^{(1)} \circ \tilde{f}^{(2)} \circ \tilde{f}^{(3)} \circ \tilde{f}^{(4)}) = V_g^{(1)} \tilde{f}^{(1)} \circ V_g^{(2)} \tilde{f}^{(2)} \circ V_g^{(3)} \tilde{f}^{(3)} \circ V_g^{(4)} \tilde{f}^{(4)}. \quad (8.10)$$

The properties (8.9) and (8.10) define completely the functional Ξ and the operators V_g . The invariance property of the scattering amplitudes can be written in the form

$$\Xi(V_g \varphi) = \Xi(\varphi), \quad \forall g \in \tilde{G}, \quad \forall \varphi \in \mathcal{B}. \quad (8.11)$$

The space $\mathcal{B}$ can also be defined by means of the tensor product

$$\mathcal{B} = \mathcal{L}^{(24)} \circ \mathcal{L}^{(13)} \quad (8.12)$$

where

$$\mathcal{L}^{(ik)} = \tilde{\mathcal{H}}^{(i)} \circ \tilde{\mathcal{H}}^{(k)} \subset \tilde{\mathcal{H}}^{(i)} \otimes \tilde{\mathcal{H}}^{(k)} = \tilde{\mathcal{H}}^{(ik)}. \quad (8.13)$$

The connection between the functional Ξ and the bilinear form $\tilde{\Phi}$ is

$$\begin{aligned} \tilde{\Phi}(\tilde{f}^{(B)}, \tilde{f}^{(A)}) &= \Xi(J \tilde{f}^{(B)} \circ \tilde{f}^{(A)}), \\ \tilde{f}^{(A)} \in \tilde{\mathcal{L}}^{(A)} &= \mathcal{L}^{(13)}, \quad \tilde{f}^{(B)} \in \tilde{\mathcal{L}}^{(B)}, \quad J \tilde{f}^{(B)} \in \mathcal{L}^{(24)}. \end{aligned} \quad (8.14)$$

We may introduce in the spaces $\mathcal{L}^{(ik)}$ the norm of the Hilbert spaces $\tilde{\mathcal{H}}^{(ik)}$, and in the space $\mathcal{B}$ the "greatest cross norm" which corresponds to the definition (8.12) (see Appendix D). We call this norm

$$\begin{aligned} \|\varphi\|_{13} &= \|\varphi\|_{24}, \quad \varphi \in \mathcal{B}. \quad \text{In a similar way we may define the norms} \\ \|\varphi\|_{12} &= \|\varphi\|_{34}, \quad \|\varphi\|_{14} = \|\varphi\|_{23}. \end{aligned}$$

If we define the following norms for the functional Ξ and for the bilinear form $\tilde{\Phi}$

$$\|\Xi\|_{ik} = \sup \frac{|\Xi(\varphi)|}{\|\varphi\|_{ik}}, \quad (8.15)$$

$$\|\tilde{\Phi}\| = \sup \frac{|\tilde{\Phi}(\tilde{f}^{(B)}, \tilde{f}^{(A)})|}{\|\tilde{f}^{(B)}\| \|\tilde{f}^{(A)}\|}, \quad (8.16)$$

from the result (iii) of Appendix D we have

$$\|\Xi\|_{13} = \|\Xi\|_{24} = \|\tilde{\Phi}\| = \|R\|, \quad (8.17)$$

where R is the operator which generates the bounded bilinear form $\tilde{\Phi}$. In general we may say that if $\|\Xi\|_{ik} < \infty$ the amplitude can be described by means of a bounded operator (with norm $\|\Xi\|_{ik}$) from the space $\mathcal{H}^{(ik)}$ to the space $\overline{\mathcal{H}}^{(rs)}$ (the indices i, k, r, s , are all different). However, in general for a physical amplitude none of the conditions $\|\Xi\|_{ik} < \infty$ is satisfied.

In order to find a condition which is a consequence of the physical unitarity condition, we consider the subspace $\mathcal{B}_{13}$ of $\mathcal{B}$ generated by the vectors of the form

$$\begin{aligned} \varphi = & (f^{(1)} \oplus 0) \circ (0 \oplus J f^{(2c)}) \circ (f^{(3)} \oplus 0) \circ (0 \oplus J f^{(4c)}) + \\ & + (0 \oplus J f^{(1c)}) \circ (f^{(2)} \oplus 0) \circ (0 \oplus J f^{(3c)}) \circ (f^{(4)} \oplus 0). \end{aligned} \quad (8.18)$$

If $\varphi \in \mathcal{B}_{13}$, only the third and the fourth of the six terms of the right-hand side of Eq. (4.7) give a contribution to $\Xi(\varphi)$. But from the unitarity condition it follows that

$$\|S\| \leq 1, \quad (8.19)$$

and

$$\|S-1\| \leq 2. \quad (8.20)$$

Therefore it is easy to show that the contribution of the third and the fourth terms of Eq. (4.7) to the bilinear form $\tilde{\Phi}$ is bounded and its norm is less than two. As a consequence, we have

$$|\Xi(\varphi)| \leq 2 \|\varphi\|_{13}, \quad \varphi \in \mathcal{B}_{13}. \quad (8.21)$$

In a similar way we may define the subspaces $\mathcal{B}_{12}$ and $\mathcal{B}_{14}$, and derive the conditions

$$\begin{cases} |\Xi(\varphi)| \leq 2 \|\varphi\|_{12}, & \varphi \in \mathcal{B}_{12}, \\ |\Xi(\varphi)| \leq 2 \|\varphi\|_{14}, & \varphi \in \mathcal{B}_{14}. \end{cases} \quad (8.22)$$

We may also define a subspace $\mathcal{B}_R$ of $\mathcal{B}$ generated by the direct products of the kinds not considered in the definition of the spaces $\mathcal{B}_{12}$, $\mathcal{B}_{13}$ and $\mathcal{B}_{14}$. It is clear from Eqs. (4.7) and (8.9) that it is

$$\Xi(\varphi) = 0, \quad \varphi \in \mathcal{B}_R. \quad (8.23)$$

This condition is a consequence of the invariance of the scattering amplitude and of the fact that the particles involved are stable. In fact, for $\varphi \in \mathcal{B}_R$ the functional $\Xi(\varphi)$ would describe reactions with a number of incoming particles different from two. It is clear that for $\varphi \in \mathcal{B}_{13}$ the functional $\Xi(\varphi)$ describes the reactions (4.19), and for $\varphi \in \mathcal{B}_{12}$ or $\varphi \in \mathcal{B}_{14}$ it describes the reactions which can be obtained from Eq. (4.19) by means of proper permutation of the indices (i).

The spaces $\mathcal{B}_{12}$, $\mathcal{B}_{13}$, $\mathcal{B}_{14}$, and $\mathcal{B}_R$ are linearly independent, and every vector $\varphi \in \mathcal{B}$ can be uniquely decomposed in the following way:

$$\begin{aligned} \varphi &= \varphi_{12} + \varphi_{13} + \varphi_{14} + \varphi_R, \\ \varphi_{ik} &\in \mathcal{B}_{ik}, \quad \varphi_R \in \mathcal{B}_R. \end{aligned} \quad (8.24)$$

From Eq. (8.23), it follows that the functional $\Xi(\varphi)$ can be considered as a functional in the subspace $\mathcal{B}_P$ defined by the linear sum

$$\mathcal{B}_P = \mathcal{B}_{12} + \mathcal{B}_{13} + \mathcal{B}_{14} . \quad (8.25)$$

Every element of $\mathcal{B}_P$ can be written in the form

$$\varphi = \varphi_{12} + \varphi_{13} + \varphi_{14} , \quad \varphi_{ik} \in \mathcal{B}_{ik} , \quad (8.26)$$

and if we define in the space $\mathcal{B}_P$ the new norm

$$\|\varphi\|_P = \|\varphi_{12}\|_{12} + \|\varphi_{13}\|_{13} + \|\varphi_{14}\|_{14} . \quad (8.27)$$

from Eqs. (8.21) and (8.22) we have

$$|\Xi(\varphi)| \leq 2 \|\varphi\|_P , \quad \varphi \in \mathcal{B}_P . \quad (8.28)$$

It follows that

$$\|\Xi\|_P = \sup_{\varphi \in \mathcal{B}_P} \frac{|\Xi(\varphi)|}{\|\varphi\|_P} \leq 2 . \quad (8.29)$$

This equation is a consequence of Eq. (8.20) and is actually equivalent to it, because it is possible to show that if Eq. (8.29) holds, then Eq. (8.20) must be satisfied by the part of the scattering operator which describes the processes we are considering. In order to obtain a stronger condition equivalent to Eq. (8.19), we call Ξ_0 the functional obtained taking $S = 0$. Then it is easy to show that the condition (8.19) is equivalent to the condition

$$\|\Xi - \Xi_0\|_P \leq 1 . \quad (8.30)$$

In order to exploit the invariance property (8.11) of the functional E , we consider the subspace $\mathcal{A}_P$ of $\mathcal{B}_P$ which contains all the finite sums of vectors of the type

$$\varphi - V_g \varphi, \quad \varphi \in \mathcal{B}_P, \quad g \in \tilde{G}, \quad (8.31)$$

and their limit points with respect to the norm $\|\varphi\|_P$. The space $\mathcal{A}_P$ can also be defined as the space containing all the elements $\varphi \in \mathcal{B}_P$ such that $E(\varphi) = 0$ for all the invariant functionals E with $\|E\|_P < \infty$. We define also the factor space

$$\mathcal{J}_P = \mathcal{B}_P / \mathcal{A}_P, \quad (8.32)$$

which is a normed space (see Appendix D). The elements of $\mathcal{J}_P$ are subsets of $\mathcal{B}_P$, and the invariant linear bounded functionals are constant in these subsets. Conversely, all the bounded linear functionals which are constant in the elements of $\mathcal{J}_P$ are invariant. Therefore, there is a one-to-one correspondence between the invariant bounded linear functionals over $\mathcal{B}_P$ and the bounded linear functionals over $\mathcal{J}_P$:

$$E(\varphi) = T(\psi), \quad \varphi \in \mathcal{B}_P, \quad \varphi \in \psi \in \mathcal{J}_P. \quad (8.33)$$

It follows from the definition of the norm $\|\psi\|_P$ in the space $\mathcal{J}_P$ that (see Appendix D)

$$\|E\|_P = \|T\|_P = \sup \frac{|T(\psi)|}{\|\psi\|_P}. \quad (8.34)$$

We have shown that every physical scattering amplitude can be described by means of a bounded functional $T(\psi)$ in the normed space $\mathcal{J}_P$. In order to complete our programme, we have to show that the elements ψ of $\mathcal{J}_P$ can be represented by means of functions of the type $\psi_{n_1 n_2 n_3 n_4}^{\text{WA}}$. We define the subspaces $\mathcal{A}_{ik}$ (i.e. $\mathcal{A}_{12}$, $\mathcal{A}_{13}$, $\mathcal{A}_{14}$) of $\mathcal{A}_P$ which contain the finite

sums of elements of the type (8.31) and their limit points with respect to the norms $\|\varphi\|_{ik}$, respectively. We define also the factor spaces

$$\mathcal{S}_{ik} = \mathcal{B}_P / \mathcal{A}_{ik}, \quad (8.35)$$

with the corresponding norms $\|\varphi\|_{ik}$, and also the norms $\|T\|_{ik}$ for the functionals $T(\varphi)$ defined in the spaces $\mathcal{S}_{ik}$. The space $\mathcal{A}_{13}$ is the subspace of $\mathcal{B}_P$ which contains all the elements $\varphi \in \mathcal{B}_P$ with the property $E(\varphi) = 0$ for every invariant functional $E(\varphi)$ with $\|E\|_{13} < \infty$. We have shown that every functional of this type corresponds to a bounded bilinear form $\tilde{\Phi}$ which can be written in the form (8.4). For every element of $\mathcal{B}_P$ we can build a function $\psi_{m_1 m_2 m_3 m_4}^{WA}$ which is the sum of a finite number of integrals of the type (8.3), and the value of $E(\varphi)$ is still given by an integral of the type (8.4). As $T_{m_1 m_2 m_3 m_4}^{WA}$ is an arbitrary essentially bounded measurable function, all these integrals vanish if and only if $\psi_{m_1 m_2 m_3 m_4}^{WA} = 0$ almost everywhere (with respect to the measure $dWdA$). Therefore we have that the space $\mathcal{A}_{13}$ is the space containing all the elements $\varphi \in \mathcal{B}_P$ such that the corresponding functions $\psi_{m_1 m_2 m_3 m_4}^{WA}$ vanish almost everywhere. It follows that if we do not consider as distinct those functions which coincide outside a set of vanishing measure, we may associate with every element of $\mathcal{S}_{13}$ a different function $\psi_{m_1 m_2 m_3 m_4}^{WA}$. We may also consider $\mathcal{S}_{13}$ as a function space.

Now in order to avoid convergence difficulties, we consider the space $\mathcal{B}_P^{\circ}$ built in the same way as the space $\mathcal{B}_P$, but starting from wave functions $\tilde{f}_{m_1}^{(i)}(a_1)$ which are continuous and vanish outside a compact set. We may proceed as above and define the spaces $\mathcal{A}_P^{\circ}$, $\mathcal{A}_{ik}^{\circ}$, $\mathcal{S}_P^{\circ}$, $\mathcal{S}_{ik}^{\circ}$. As the wave functions we are considering form dense subspaces of the spaces $\tilde{\mathcal{H}}^{(i)}$, from the results of the Appendix D we have that $\mathcal{B}_P^{\circ}$, $\mathcal{A}_P^{\circ}$, and $\mathcal{A}_{ik}^{\circ}$ are dense in $\mathcal{B}_P$, $\mathcal{A}_P$, and $\mathcal{A}_{ik}$, respectively, and that the spaces $\mathcal{S}_P^{\circ}$, $\mathcal{S}_{ik}^{\circ}$ can be considered as dense subspaces of $\mathcal{S}_P$, $\mathcal{S}_{ik}$. The bounded functionals E and T are completely determined by their restrictions to these subspaces.

If we take

$$\begin{cases} \tilde{f}^{(A)} = \tilde{f}^{(1)} \otimes \tilde{f}^{(3)}, \\ \tilde{f}^{(B)} = J \tilde{f}^{(4)} \otimes J \tilde{f}^{(2)}, \end{cases} \quad (8.36)$$

where the wave functions $\tilde{f}_{m_i}^{(i)}(a_i)$ are continuous and vanish outside a compact set, from Eqs. (7.23) and (8.3) we have

$$\begin{aligned} \Psi_{m_1 m_2 m_3 m_4}^{W\Lambda} &= \int d\mu_W(Q) \sum_m \int_{H_W} \overline{f_{m_4 m_2}^{(B)W}(a_Q h)} R_{m, m_4+m_2}^{W\Lambda}(h) dh \\ &\cdot \int_{H_W} f_{m_1 m_3}^{(A)W}(a_Q h') R_{m, m_1+m_3}^{W\Lambda}(h') dh' = \int d\mu_W(Q) \cdot \end{aligned} \quad (8.37)$$

$$\cdot \int_{H_W} \int_{H_W} \overline{f_{m_4 m_2}^{(B)W}(a_Q h)} f_{m_1 m_3}^{(A)W}(a_Q h') R_{m_4+m_2, m_1+m_3}^{W\Lambda}(h^{-1} h') dh dh',$$

that is

$$\Psi_{m_1 m_2 m_3 m_4}^{W\Lambda} = \int_{H_W} \Psi_{m_1 m_2 m_3 m_4}^W(h) R_{m_4+m_2, m_1+m_3}^{W\Lambda}(h) dh, \quad (8.38)$$

where

$$\begin{aligned} \Psi_{m_1 m_2 m_3 m_4}^W(h) &= \int d\mu_W(Q) \int_{H_W} \overline{f_{m_4 m_2}^{(B)W}(a_Q h')} f_{m_1 m_3}^{(A)W}(a_Q h' h) dh' = \\ &= C(W) \int_{\tilde{A}} \overline{f_{m_2 m_4}^{(B)W}(a)} f_{m_1 m_3}^{(A)W}(a h) da. \end{aligned} \quad (8.39)$$

If we introduce the function

$$\begin{aligned} \Psi_{m_1 m_2 m_3 m_4}(a_1, a_2, a_3, a_4) &= \\ &= \int_{\tilde{A}} \tilde{f}_{m_1}^{(1)}(a a_1) \tilde{f}_{m_2}^{(2)}(a a_2) \tilde{f}_{m_3}^{(3)}(a a_3) \tilde{f}_{m_4}^{(4)}(a a_4) da, \end{aligned} \quad (8.40)$$

we can write, using Eqs. (8.39), (7.1), (7.14) and (7.35)

$$\Psi_{m_1 m_2 m_3 m_4}^w(l_i) = d_{13}(w) \sum_{m'_4, m'_2} C_{m_4, m'_4}^{j_4} C_{m_2, m'_2}^{j_2} \cdot \Psi_{m_1, m'_2, m_3, m'_4}(h b_{1w}, b_{2wt}, h b_{3w}, b_{4wt}) , \quad (8.41)$$

where

$$d_{13}(w) = C(w) \left[p(w, M_1, M_3) p(w, M_4, M_2) \right]^{\frac{1}{2}} . \quad (8.42)$$

Of course, if we want to obtain the function Ψ which corresponds to an arbitrary element of $\mathcal{B}_P^0$, we have to consider a finite sum of integrals of the type (8.40). If we take into account the condition

$$\sum_i L(a_i) P^{o(i)} = 0 , \quad (8.43)$$

from the covariance properties (5.24) we have that the function

$$\left[\sum_{m_1, m_2} \sum_{m_3, m_4} \left| \Psi_{m_1, m_2, m_3, m_4}(a_1, a_2, a_3, a_4) \right|^2 \right]^{\frac{1}{2}} = \Psi(w_{12}, w_{13}, w_{14}) \quad (8.44)$$

depends only on the Mandelstam variables defined by

$$\begin{cases} w_{12} = w_{34} = (L(a_1) P^{o(1)} + L(a_2) P^{o(2)})^2 , \\ w_{13} = w_{24} = (L(a_1) P^{o(1)} + L(a_3) P^{o(3)})^2 , \\ w_{14} = w_{23} = (L(a_1) P^{o(1)} + L(a_4) P^{o(4)})^2 , \end{cases} \quad (8.45)$$

which satisfy the well-known relation

$$w_{12} + w_{13} + w_{14} = \sum_i M_i^2 . \quad (8.46)$$

If we define the function

$$\Psi^W(h) = \left[\sum_{m_1, m_2} \sum_{m_3, m_4} |\Psi_{m_1, m_2, m_3, m_4}^W(h)|^2 \right]^{\frac{1}{2}}, \quad (8.47)$$

from Eq. (8.41) we have

$$\Psi^W(h) = \alpha_{13}(W) \Psi(W_{12}, W, W_{14}). \quad (8.48)$$

If we use the Plancherel formula for the group H_W and use Eqs. (8.38), (8.47), and (8.48), we see that the function $\psi_{m_1 m_2 m_3 m_4}^{WA}$ vanishes almost everywhere if, and only if, the function $\Psi(W_{12}, W_{13}, W_{14})$ vanishes almost everywhere. Actually, if $\varphi \in \mathcal{D}$, these functions are continuous, and if they vanish almost everywhere then they vanish everywhere. Therefore we have that the condition

$$\Psi(W_{12}, W_{13}, W_{14}) = 0 \quad (8.49)$$

defines the subspace $\mathcal{A}_{13}^0$ of $\mathcal{B}_P^0$. If we permute the indices which label the particles, we have that the same condition defines also the subspaces $\mathcal{A}_{12}^0$ and $\mathcal{A}_{14}^0$. It can easily be shown that it is

$$\mathcal{A}_P^0 = \mathcal{A}_{12}^0 \cap \mathcal{B}_{12}^0 + \mathcal{A}_{13}^0 \cap \mathcal{B}_{13}^0 + \mathcal{A}_{14}^0 \cap \mathcal{B}_{14}^0 \quad (8.50)$$

and it follows that also the subspace $\mathcal{A}_P^0$ coincides with the subspaces $\mathcal{A}_{ik}^0$. It follows immediately that the factor spaces $\mathcal{F}_P^0$ and $\mathcal{F}_{ik}^0$ contain the same elements. Note, however, that the norms in these spaces are different.

In conclusion, we have shown that any scattering amplitude which satisfies the invariance and unitarity conditions can be described completely by means of a bounded linear functional in the normed space $\mathcal{F}_P^0$, whose elements coincide with the elements of $\mathcal{F}_{13}^0$ and can therefore be considered as functions of the kind $\psi_{m_1 m_2 m_3 m_4}^{WA}$.

These results are not trivial and it is easy to find examples in which similar results are not true. If, for example, we take $V_g^{(1)} = V_g^{(3)} = 1$

(one-dimensional representation) and $V^{(2)} = \overline{V}^{(4)}$, where $V^{(2)}$ is a representation which describes a particle (see Section 5), it is easy to show that the set $\Omega_A \cap \Omega_B$ [see Eq. (3.5)] is the void set and therefore there is no non-vanishing functional $E(\varphi)$ with $\|E\|_{13} < \infty$. It follows that the space $\mathcal{S}_{13}$ contains only the zero element. If we consider the functional defined by

$$E(\tilde{f}^{(1)}_0 \tilde{f}^{(2)}_0 \tilde{f}^{(3)}_0 \tilde{f}^{(4)}_0) = (J\tilde{f}^{(1)}, \tilde{f}^{(3)}) (J\tilde{f}^{(2)}, \tilde{f}^{(4)}) \quad (8.51)$$

it is easy to show that it is invariant and that $\|E\|_{12} < \infty$. However, obviously, it cannot be described by means of a linear functional defined in a subspace of $\mathcal{S}_{13}$.

9. PROPERTIES OF THE SPACE $\mathcal{S}_P^0$

We shall not be able to characterize completely the functions which belong to the space $\mathcal{S}_P^0$ and to give an explicit useful formula for the norm $\|\psi\|_P$. However, in this section we shall derive some important properties of these functions and some inequalities which imply the norm $\|\psi\|_P$.

As we have seen in the preceding section, the norm $\|T\|_{13}$ is equal to the norm of the operator R which generates the bilinear form $\tilde{\Phi}$. This operator is given by a direct integral of finite-dimensional operators R^{WA} whose matrix elements are the quantities $T^{WA}_{m_1 m_2 m_3 m_4}$. Therefore we have⁴⁸⁾

$$\|T\|_{13} = \|R\| = \sup \max \|R^{WA}\|, \quad (9.1)$$

and, as the norm of an operator is not greater than its Hilbert Schmidt norm, it is

$$\|R^{WA}\| \leq \left[\sum_{m_1 m_2} \sum_{m_3 m_4} |T^{WA}_{m_1 m_2 m_3 m_4}|^2 \right]^{1/2}. \quad (9.2)$$

Therefore, if we take

$$T^{WA}_{m_1 m_2 m_3 m_4} = (\psi^{WA})^{-1} \overline{\psi^{WA}_{m_1 m_2 m_3 m_4}}, \quad (9.3)$$

where

$$\psi^{w\Lambda} = \left[\sum_{m_1, m_2} \sum_{m_3, m_4} |\psi_{m_1, m_2, m_3, m_4}^{w\Lambda}|^2 \right]^{\frac{1}{2}}, \quad (9.4)$$

it is

$$\|R^{w\Lambda}\| \leq 1, \quad \|T\|_{13} \leq 1. \quad (9.5)$$

It follows that

$$\int_{\Gamma} d\omega \int_{\Omega^w} \psi^{w\Lambda} d\Lambda = |T(\psi)| \leq \|\psi\|_{13}. \quad (9.6)$$

If it is $\psi \in \mathcal{F}_P^0$, it is possible to invert Eq. (8.38) and we obtain

$$\psi_{m_1, m_2, m_3, m_4}^w(h) = \int \psi_{m_1, m_2, m_3, m_4}^{w\Lambda} \overline{R_{m_4+m_2, m_1+m_3}^{w\Lambda}(h)} d\Lambda, \quad (9.7)$$

From the unitarity of the representations $R^{w\Lambda}$, it follows

$$|R_{m, m'}^{w\Lambda}(h)| \leq 1, \quad (9.8)$$

and, using Eqs. (9.4), (9.6), and (8.47), from Eq. (9.7) we have

$$\psi^w(h) \leq \int_{\Omega^w} \psi^{w\Lambda} d\Lambda \leq \omega_{13}(w), \quad (9.9)$$

where

$$\int_{\Gamma} \omega_{13}(w) d\omega \leq \|\psi\|_{13}. \quad (9.10)$$

Using Eq. (8.48), we obtain

$$\alpha_{13}(w_{13}) \psi(w_{12}, w_{13}, w_{14}) \leq \omega_{13}(w_{13}). \quad (9.11)$$

More generally, if we define the functions

$$\alpha_{ik}(\psi_{ik}) = C(\psi_{ik}) \left[\rho(\psi_{ik}, M_i, M_k) \rho(\psi_{ik}, M_r, M_s) \right]^{\frac{1}{2}}, \quad (9.12)$$

(the indices i, k, r, s are all different), permuting the indices in Eqs. (9.10) and (9.11), we obtain two other independent conditions involving the norms $\|\psi\|_{12}$ and $\|\psi\|_{14}$.

If we remark that the spaces B_{ik}^0 are invariant with respect to the operators V_{ij} , we see easily that if $\varphi \in \mathcal{A}_P^0$ and $\varphi = \varphi_{12} + \varphi_{13} + \varphi_{14}$ with $\varphi_{ik} \in B_{ik}^0$, then it is also $\varphi_{12}, \varphi_{13}, \varphi_{14} \in \mathcal{A}_P^0$. It follows that the decomposition

$$\varphi = \varphi_{12} + \varphi_{13} + \varphi_{14}, \quad \varphi \in \mathcal{B}_P^0, \quad \varphi_{ik} \in B_{ik}^0, \quad (9.13)$$

induces the decomposition

$$\psi = \psi_{12} + \psi_{13} + \psi_{14}, \quad \psi \in \mathcal{J}_P^0, \quad \psi_{ik} \in \mathcal{J}_{(ik)}^0, \quad (9.14)$$

of the space $\mathcal{J}_P^0$. We have indicated by $\mathcal{J}_{(ik)}^0$ (not to be confused with the spaces $\mathcal{J}_{ik}^0$ defined in the preceding section) the subspace of $\mathcal{J}_P^0$ which contains all the elements of $\mathcal{J}_P^0$ which contain elements of B_{ik}^0 .

From the definition of norm in a factor space (see Appendix D) and from Eq. (8.27), we have

$$\|\psi\|_P = \inf_{\varphi' \in \mathcal{A}_P^0} \|\varphi + \varphi'\|_P = \inf_{\varphi' \in \mathcal{A}_P^0 \cap B_{12}^0} \|\varphi_{12} + \varphi'\|_{12} + \quad (9.15)$$

$$+ \inf_{\varphi' \in \mathcal{A}_P^0 \cap B_{13}^0} \|\varphi_{13} + \varphi'\|_{13} + \inf_{\varphi' \in \mathcal{A}_P^0 \cap B_{14}^0} \|\varphi_{14} + \varphi'\|_{14},$$

where $\varphi = \varphi_{12} + \varphi_{13} + \varphi_{14}$, ($\varphi_{ik} \in B_{ik}^0$) is an arbitrary element of φ . As it is

$$\inf_{\varphi' \in \mathcal{A}_P^0 \cap B_{ik}^0} \|\varphi_{ik} + \varphi'\|_{ik} \geq \inf_{\varphi' \in \mathcal{A}_P^0} \|\varphi_{ik} + \varphi'\|_{ik} = \|\psi_{ik}\|_{ik}, \quad (9.16)$$

$$\varphi_{ik} \in \psi_{ik},$$

we have

$$\|\Psi\|_P \geq \|\Psi_{12}\|_{12} + \|\Psi_{13}\|_{13} + \|\Psi_{14}\|_{14}, \quad (9.17)$$

where ψ and ψ_{ik} are connected by Eq. (9.14).

From Eqs. (5.13), (8.40), (8.44), and (8.45), we see that $\psi \in \mathcal{J}^{\circ}_{(ik)}$ if, and only if, the corresponding function $\Psi(W_{12}, W_{13}, W_{14})$ has the property

$$\Psi(W_{12}, W_{13}, W_{14}) = 0 \quad \text{for } W_{ik} < \hat{W}_{ik}, \quad (9.18)$$

where

$$\hat{W}_{ik} = \max[(M_i + M_k)^2, (M_r + M_s)^2], \quad (9.19)$$

(i, k, r, s all different). From Eqs. (9.10), (9.11), and (9.17) we can therefore obtain the following inequalities

$$\Psi(W_{12}, W_{13}, W_{14}) \leq \frac{\omega_{ik}(W_{ik})}{\alpha_{ik}(W_{ik})} \quad \text{for } W_{ik} > \hat{W}_{ik} \quad (9.20)$$

where

$$\int_{\hat{W}_{12}}^{\infty} \omega_{12}(W_{12}) dW_{12} + \int_{\hat{W}_{13}}^{\infty} \omega_{13}(W_{13}) dW_{13} + \int_{\hat{W}_{14}}^{\infty} \omega_{14}(W_{14}) dW_{14} \leq \|\Psi\|_P. \quad (9.21)$$

From Eq. (8.48) we have

$$\Psi^W(h) \leq \frac{\alpha_{13}(W)}{\alpha_{ik}(W_{ik})} \omega_{ik}(W_{ik}), \quad W_{ik} > \hat{W}_{ik}, \quad (9.22)$$

where the connection between the variables $W = W_{13}$, h , W_{12} and W_{14} is given by

$$W_{12} = \frac{1}{2}(M_1^2 + M_2^2 + M_3^2 + M_4^2 - W) + \frac{1}{2W}(M_3^2 - M_1^2)(M_2^2 - M_4^2) - \frac{z}{2W} \left[\Delta(W, M_1^2, M_3^2) \Delta(W, M_4^2, M_2^2) \right]^{\frac{1}{2}}, \quad (9.23)$$

$$W_{14} = \frac{1}{2}(M_1^2 + M_2^2 + M_3^2 + M_4^2 - W) - \frac{1}{2W}(M_3^2 - M_1^2)(M_2^2 - M_4^2) + \frac{z}{2W} \left[\Delta(W, M_1^2, M_3^2) \Delta(W, M_4^2, M_2^2) \right]^{\frac{1}{2}}, \quad (9.24)$$

where the symbol $\Delta(a,b,c)$ is defined by Eq. (C.19) of Appendix C, and it is

$$\begin{cases} z = L_{tt}(h), & h \in H_- , \\ z = L_{zz}(h), & h \in H_+ . \end{cases} \quad (9.25)$$

Note that

$$\begin{cases} z(u_z(\mu)h) = z(hu_z(\mu)) = z(h'g) = z(hg') = z(h), \\ z(ax(\zeta)) = \omega h \zeta, \\ z(t'ax(\zeta)) = -\omega h \zeta, \\ z(u_q(\theta)) = \omega \theta. \end{cases} \quad (9.26)$$

It can be shown directly from Eq. (8.45) that

$$\begin{cases} W_{12} \geq \hat{W}_{12} & \text{for } 0 \leq W \leq \min[(M_3 - M_1)^2, (M_2 - M_4)^2], -1 \leq z \leq 1, \\ W_{12} \geq \hat{W}_{12} & \text{for } W \leq 0, z \geq 1, \\ W_{14} \geq \hat{W}_{14} & \text{for } W \leq 0, z \leq -1. \end{cases} \quad (9.27)$$

Now we remark that from Eqs. (8.38), (8.47), (9.4), and (9.8) it follows that

$$\psi^{W1} \leq \int_{H_W} \psi^W(h) dh. \quad (9.28)$$

For $0 < W < \min [(M_3 - M_1)^2, (M_2 - M_4)^2]$, using Eqs. (9.26), (A.7), and (A.8) we obtain

$$\begin{aligned} \psi^{W1} &\leq \frac{1}{2} \int_{-1}^{+1} \frac{d_{13}(W)}{d_{12}(W_{12})} \omega_{12}(W_{12}) dz \leq \\ &\leq \gamma(W) \int \omega_{12}(W_{12}) dW_{12} \leq \gamma(W) \|\psi\|_P, \end{aligned} \quad (9.29)$$

where

$$\gamma(W) = \sup_{-1 \leq z \leq 1} \left[\frac{1}{2} \frac{d_{13}(W)}{d_{12}(W_{12})} \left| \frac{dW_{12}}{dz} \right|^{-1} \right]. \quad (9.30)$$

For $W < 0$ we have in a similar way

$$\begin{aligned} \psi^{W1} &\leq \frac{1}{2} \int_1^{\infty} \frac{d_{13}(W)}{d_{12}(W_{12})} \omega_{12}(W_{12}) dW_{12} + \frac{1}{2} \int_{-\infty}^{-1} \frac{d_{13}(W)}{d_{14}(W_{14})} \omega_{14}(W_{14}) dW_{14} \leq \\ &\leq \gamma_{12}(W) \int \omega_{12}(W_{12}) dW_{12} + \gamma_{14}(W) \int \omega_{14}(W_{14}) dW_{14} \leq \\ &\leq \gamma(W) \|\psi\|_P, \end{aligned} \quad (9.31)$$

where

$$\begin{cases} \gamma(W) = \max [\gamma_{12}(W), \gamma_{14}(W)], \\ \gamma_{12}(W) = \sup_{z \geq 1} \left[\frac{1}{2} \frac{d_{13}(W)}{d_{12}(W_{12})} \left| \frac{dW_{12}}{dz} \right|^{-1} \right], \\ \gamma_{14}(W) = \sup_{z \leq -1} \left[\frac{1}{2} \frac{d_{13}(W)}{d_{14}(W_{14})} \left| \frac{dW_{14}}{dz} \right|^{-1} \right]. \end{cases} \quad (9.32)$$

For $W < 0$ more useful inequalities can be obtained if we remark that the matrix elements $R_{mm'}^{WA}(h) = \tilde{D}_{mm'}^{\sigma\tau\epsilon\ell}(h)$ can be continued analytically in the whole ℓ -plane¹¹⁾. It is possible to show that if all the wave functions are continuous and vanish outside a compact set, the integral (8.38) converges for every complex value of ℓ and defines an entire function of ℓ continuous in the variable W for $W < 0$. We shall find an upper bound for the function $\psi_{m_1 m_2 m_3 m_4}^{W\sigma\tau\epsilon\ell}$ in the set defined by

$$W < 0, \quad -2 \leq \operatorname{Re} \ell \leq 1 \quad (9.33)$$

Under these conditions, from the results of Ref. 11) we have

$$|\tilde{D}_{mm'}^{\sigma\tau\epsilon\ell}(h)| \leq P_{\operatorname{Re} \ell}(|z|) \leq |z|, \quad (9.34)$$

and, proceeding as above, we obtain

$$\psi^{W\sigma\tau\epsilon\ell} \leq \gamma'(W) \|\psi\|_P, \quad (9.35)$$

where

$$\begin{cases} \gamma'(W) = \max[\gamma'_{12}(W), \gamma'_{14}(W)], \\ \gamma'_{12}(W) = \sup_{z \geq 1} \left[\frac{1}{2} \frac{d_{13}(W)}{d_{12}(W/_{12})} |z| \left| \frac{dW_{12}}{dz} \right|^{-1} \right], \\ \gamma'_{14}(W) = \sup_{z \leq -1} \left[\frac{1}{2} \frac{d_{13}(W)}{d_{14}(W/_{14})} |z| \left| \frac{dW_{14}}{dz} \right|^{-1} \right]. \end{cases} \quad (9.36)$$

In Appendix E we show that the functions $\gamma(W)$ and $\gamma'(W)$ are continuous functions of W for

$$W < \min[(M_3 - M_1)^2, (M_2 - M_4)^2], \quad W \neq 0, \quad W \neq W^{T12}, \quad (9.37)$$

$$W \neq W^{T14}.$$

where the quantities $\tilde{W}^{T12}$ and $\tilde{W}^{T14}$ are defined by Eqs. (E.10) and (E.11). It follows that the convergence with respect to the norm $\|\psi\|_P$ implies the uniform convergence of the functions $\psi_{m_1 m_2 m_3 m_4}^{WA}$ and of their analytic continuation in the strip (9.33) in any closed bounded subset of the set (9.37). It is easy to show that when the regular wave functions $\tilde{f}_{m_1}^{(i)}(a_i)$ we are considering converge towards arbitrary L^2 wave functions, the corresponding functions ψ converge with respect to the norm $\|\psi\|_P$ and therefore uniformly in the sets defined above. As the uniform limit of a continuous or analytic function is still a continuous or analytic function, we have that the functions ψ obtained from arbitrary L^2 wave functions can be continued analytically in the strip (9.33) and are continuous functions of W in the set (9.37).

From Eqs. (6.18) we obtain the following property of the functions $\psi_{m_1 m_2 m_3 m_4}^{W\sigma\tau\ell}$

$$\left(U_{m_4+m_2}^{\ell} \right)^{-1} \psi_{m_1 m_2 m_3 m_4}^{W\sigma\tau\ell} U_{m_4+m_2}^{\ell} = \psi_{m_1 m_2 m_3 m_4}^{W\sigma'\tau'\ell'} \quad (9.38)$$

where σ' , τ' and ℓ' are given by Eq. (6.19). From Eq. (6.21) we have

$$\psi_{m_1 m_2 m_3 m_4}^{W\pi k} = \left(X_{m_4+m_2}^{\tau k} \right)^{-1} \psi_{m_1 m_2 m_3 m_4}^{W\sigma\tau\ell\ell} X_{m_4+m_2}^{\tau k} \quad (9.39)$$

where the connections between the various indices are given by Eq. (6.22). If we consider arbitrary wave functions, Eq. (9.39) can be written only for $k = 1, \frac{3}{2}, 2$.

Summing up, we have that the functions $\psi_{m_1 m_2 m_3 m_4}^{WA}$ generated by arbitrary L^2 wave functions have the following properties (we consider only the values of W which are of interest for the crossed partial wave expansion):

- i) they are continuous functions of W in the set (9.37);
- ii) they satisfy the inequalities (9.29) and (9.31);
- iii) they satisfy the condition (8.6);
- iv) for $W < 0$, $W \neq W^{T12}$, $W \neq W^{T14}$, the functions $\psi_{m_1 m_2 m_3 m_4}^{W\sigma\tau\ell}$ can be continued analytically in the strip (9.33) and the analytic continuation is continuous on the boundary of the strip and is a continuous function of W ;

- v) this analytic continuation in the strip (9.33) satisfies the inequality (9.35);
- vi) it satisfies the conditions (9.38) and (9.39) for $k = 1, \frac{3}{2}, 2$.

These conditions are certainly satisfied by the functions belonging to the space $\mathcal{F}_P^0$ and by their limits with respect to the norm $\|\psi\|_P$, but they do not characterize completely these functions. A complete description of the space $\mathcal{F}_P^0$ (or of its completion) and an explicit formula for the norm $\|\psi\|_P$ would be useful results, but they are outside the purposes of this paper. As a consequence, we cannot give a complete classification of the bounded linear functionals $T(\psi)$. However, as we shall show in the next section, these conditions enable us to define certain types of functionals which have a great physical interest.

10. EXAMPLES

In this section we give some examples of linear functionals $T(\psi)$, whose boundedness can be proved by means of the inequalities given in the preceding section. As we are mainly interested in the crossed partial wave analysis we use the function ψ only for $W < W^m = \min [(M_3 - M_1)^2, (M_2 - M_4)^2]$.

As a first example we consider the functional

$$T(\psi) = \int_{-\infty}^{W^m} dW \int_{\Omega^W} \sum_{m_1 m_2 m_3 m_4} T_{m_1 m_2 m_3 m_4}^{W\Lambda} \psi_{m_1 m_2 m_3 m_4}^{W\Lambda} d\Lambda. \quad (10.1)$$

From the inequalities (9.29) and (9.31) we have

$$|T(\psi)| \leq \int_{-\infty}^{W^m} dW \int_{\Omega^W} \|\psi\|_P \chi(W) \left[\sum_{m_1 m_2 m_3 m_4} |T_{m_1 m_2 m_3 m_4}^{W\Lambda}|^2 \right]^{\frac{1}{2}} d\Lambda, \quad (10.2)$$

and therefore

$$\|T\|_P \leq \int_{-\infty}^{W^m} dW \chi(W) \left[\sum_{m_1 m_2 m_3 m_4} |T_{m_1 m_2 m_3 m_4}^{W\Lambda}|^2 \right]^{\frac{1}{2}} d\Lambda. \quad (10.3)$$

Another possibility is to use the analytic continuation of the function $\psi_{m_1 m_2 m_3 m_4}^{W \sigma \tau \epsilon \ell}$. Note, however, that it is not possible to use values of ℓ outside the strip (9.33), because the inequality (9.35) holds only in this strip. It can be shown that if ℓ_0 lies outside the strip, it is possible to find functions belonging to $\mathcal{F}_P^0$ with $\|\psi\|_P \leq 1$ and with $|\psi_{m_1 m_2 m_3 m_4}^{W \sigma \tau \epsilon \ell_0}|$ arbitrarily large. We consider two types of functionals; the first has the form

$$T(\psi) = \int_{-\infty}^0 dW \left(\sum_{m_1 m_2 m_3 m_4} \sum_{C^W} \beta_{m_1 m_2 m_3 m_4}^{W \ell} \psi_{m_1 m_2 m_3 m_4}^{W \sigma \tau \epsilon \ell} \right) d\ell, \quad (10.4)$$

where C^W is an arbitrary integration path (which may depend on W) contained in the strip (9.33) of the complex ℓ plane. From the inequality (9.35) we obtain easily

$$\|T\|_P \leq \int_{-\infty}^0 \gamma'(W) dW \left(\sum_{m_1 m_2 m_3 m_4} \sum_{C^W} |\beta_{m_1 m_2 m_3 m_4}^{W \ell}|^2 \right)^{\frac{1}{2}} |d\ell|. \quad (10.5)$$

Another possible functional has the form

$$T(\psi) = \int_{-\infty}^0 dW \sum_{m_1 m_2 m_3 m_4} \sum_{C^W} \beta_{m_1 m_2 m_3 m_4}^{W \ell} \psi_{m_1 m_2 m_3 m_4}^{W \sigma \tau \epsilon \ell(W)}, \quad (10.6)$$

where $-2 \leq \ell(W) \leq 1$ for $W < 0$. In the same way as before we obtain

$$\|T\|_P \leq \int_{-\infty}^0 \left(\sum_{m_1 m_2 m_3 m_4} \sum_{C^W} |\beta_{m_1 m_2 m_3 m_4}^{W \ell}|^2 \right)^{\frac{1}{2}} \gamma'(W) dW. \quad (10.7)$$

Of course a better knowledge of the space $\mathcal{F}_P^0$ would permit the definition of larger classes of bounded functionals and the derivation of more refined inequalities for their norms.

In order to find the meaning of the functionals (10.1), (10.4), and (10.6), we shall compute their contribution to the scattering amplitude considered as a function of the particle four-momenta. When the particles involved have non-vanishing spin, in order to avoid ambiguities in the definition of the spin indices, it is convenient to define the scattering amplitude as a function of four elements of the group $\tilde{A}$. This is a consistent generalization of the formalism introduced in Section 5 for the wave functions, which is in turn a generalization of the formalism introduced by Moussa and Stora²⁴). We define the scattering amplitude by means of the states $\varphi_m^{(i)}$ introduced in Section 5 in the following way. The function $M_{m_1 m_2 m_3 m_4}(a_1, a_2, a_3, a_4)$ does not vanish only if two arguments (we call them a_r, a_s) contain time reversal and the other two (we call them a_i, a_k) do not reverse the time direction. In this case it is given by

$$\begin{aligned} i(2\pi)^{-2} \delta^4 \left(\sum_j L(a_j) P^{o(j)} \right) M_{m_1 m_2 m_3 m_4}(a_1, a_2, a_3, a_4) = \\ = \pm \left([U_{a_r}^{(r)} \varphi_{m_r}^{(r)} \otimes U_{a_s}^{(s)} \varphi_{m_s}^{(s)}], (S^{-1}) [U_{a_i}^{(i)} \varphi_{m_i}^{(i)} \otimes U_{a_k}^{(k)} \varphi_{m_k}^{(k)}] \right), \end{aligned} \quad (10.8)$$

where the minus sign has to be used if the permutation $(1, 2, 3, 4) \rightarrow (r, s, i, k)$ contains an odd number of fermion exchanges. Then, if we use Eqs. (4.7), (5.13), and the completeness property (5.9), we obtain

$$\begin{aligned} \tilde{\Theta}(\tilde{f}^{(1)}, \tilde{f}^{(2)}, \tilde{f}^{(3)}, \tilde{f}^{(4)}) = i(2\pi)^{-2} \left(\prod_{i=1}^4 \int \sum_{m_i} \sum_{m_i} \delta^4 \left(\sum_i P^{(i)} \right) \right. \\ \left. M_{m_1 m_2 m_3 m_4}(a_{p^{(1)}}, a_{p^{(2)}}, a_{p^{(3)}}, a_{p^{(4)}}) \tilde{f}_{m_1}^{(1)}(a_{p^{(1)}}) \tilde{f}_{m_2}^{(2)}(a_{p^{(2)}}) \right. \\ \left. \cdot \tilde{f}_{m_3}^{(3)}(a_{p^{(3)}}) \tilde{f}_{m_4}^{(4)}(a_{p^{(4)}}) d\mu_{M_1^2}(p^{(1)}) d\mu_{M_2^2}(p^{(2)}) d\mu_{M_3^2}(p^{(3)}) d\mu_{M_4^2}(p^{(4)}) \right). \end{aligned} \quad (10.9)$$

It is clear that the amplitude is defined only if

$$\sum_i L(a_i) P^{o(i)} = 0 \quad (10.10)$$

It satisfies the invariance condition

$$M_{m_1 m_2 m_3 m_4}(aa_1, aa_2, aa_3, aa_4) =$$

$$M_{m_1 m_2 m_3 m_4}(a_1, a_2, a_3, a_4), \quad a \in \tilde{A}, \quad (10.11)$$

and four covariance conditions of the type

$$M_{m_1 m_2 m_3 m_4}(a_1 h_1^{-1}, a_2, a_3, a_4) =$$

$$= \sum_{m'_1} R_{m_1 m'_1}^{\pi_1 \delta_1}(h) M_{m'_1 m_2 m_3 m_4}(a_1, a_2, a_3, a_4), \quad h \in H_+. \quad (10.12)$$

If we use twice Eq. (C.17) and Eq. (A.21), and we perform some of the integrations by means of the δ -function, we can transform Eq. (10.9) in the following way:

$$\tilde{H}(\tilde{f}^{(1)}, \tilde{f}^{(2)}, \tilde{f}^{(3)}, \tilde{f}^{(4)}) = i(2\pi)^{-2} \int \varphi(W, M_1, M_3) \varphi(W, M_4, M_2) \cdot$$

$$\cdot dW \int_{H_W} d\mu_W(q) \int_{H_W} dh \int_{H_W} dh' \sum_{m_1 m_2 m_3 m_4} \sum_{m'_1}^{\Gamma} \tilde{f}_{m_1}^{(1)}(a q h b_{1W}) \cdot$$

$$\cdot \tilde{f}_{m_2}^{(2)}(a q h' b_{2W} t) \tilde{f}_{m_3}^{(3)}(a q h b_{3W}) \tilde{f}_{m_4}^{(4)}(a q h' b_{4W} t) \cdot \quad (10.13)$$

$$\cdot M_{m_1 m_2 m_3 m_4}(h b_{1W}, h' b_{2W} t, h b_{3W}, h' b_{4W} t).$$

If we define the following function

$$T_{m_1 m_2 m_3 m_4}^W(l_1) = i(2\pi)^{-2} [\varphi(W, M_1, M_3) \varphi(W, M_4, M_2)]^{\frac{1}{2}} \cdot$$

$$\cdot \sum_{m'_2 m'_4} C_{m_2 m'_2}^{\delta_2} C_{m_4 m'_4}^{\delta_4} M_{m_1 m'_2 m_3 m'_4}(h b_{1W}, b_{2W} t, h b_{3W}, b_{4W} t), \quad (10.14)$$

and take into account Eqs. (7.1), (7.35), (7.36), and (7.14) we may also write

$$\begin{aligned} \tilde{\Theta}(\tilde{f}^{(1)}, \tilde{f}^{(2)}, \tilde{f}^{(3)}, \tilde{f}^{(4)}) &= \tilde{\Phi}(\tilde{f}^{(B)}, \tilde{f}^{(A)}) = \int_{\Gamma} d\omega \int d\mu_{\omega}(a) \cdot \\ &\cdot \int_{H_{\omega}} dh \int_{H_{\omega}} dh' \sum_{m_1, m_2, m_3, m_4} \overline{f_{m_4, m_2}^{(B)\omega}(a, h')} f_{m_1, m_3}^{(A)\omega}(a, h) \cdot \\ &\cdot T_{m_1, m_2, m_3, m_4}^{\omega}(h'^{-1}h). \end{aligned} \quad (10.15)$$

or, using Eq. (8.39)

$$\begin{aligned} \tilde{\Theta}(\tilde{f}^{(1)}, \tilde{f}^{(2)}, \tilde{f}^{(3)}, \tilde{f}^{(4)}) &= T(\psi) = \\ &= \int_{\Gamma} d\omega \int_{H_{\omega}} dh \sum_{m_1, m_2, m_3, m_4} \psi_{m_1, m_2, m_3, m_4}^{\omega}(h) T_{m_1, m_2, m_3, m_4}^{\omega}(h). \end{aligned} \quad (10.16)$$

If $T_{m_1, m_2, m_3, m_4}^{\omega}(h)$ is an L^2 function of h , we can use the Plancherel formula for the group H_{ω} and we obtain Eq. (10.1) where

$$T_{m_1, m_2, m_3, m_4}^{\omega \Lambda} = \int_{H_{\omega}} T_{m_1, m_2, m_3, m_4}^{\omega}(h) \overline{R_{m_4+m_2, m_1+m_3}^{\omega \Lambda}(h)} dh. \quad (10.17)$$

However, in general for $\omega < 0$, the function $T_{m_1, m_2, m_3, m_4}^{\omega}(h)$ is not L^2 , the integral (10.17) does not converge and the functional $T(\psi)$ cannot be written in the form (10.1). This is another way of seeing the difficulties inherent in the crossed partial wave expansion, which have forced us to define the crossed partial wave amplitude as functional.

If we substitute Eq. (8.38) (or the corresponding equation for complex ℓ) into Eqs. (10.1), (10.4), and (10.6), and exchange the order of integration [this is permitted if the integrals (10.3) and (10.5) converge], we obtain Eq. (10.16) where $T_{m_1, m_2, m_3, m_4}^{\omega}(h)$ is given by

$$T_{m_1, m_2, m_3, m_4}^{\omega}(h) = \int_{\Omega^{\omega}} T_{m_1, m_2, m_3, m_4}^{\omega \Lambda} R_{m_4+m_2, m_1+m_3}^{\omega \Lambda}(h) d\Lambda, \quad (10.18)$$

if $T(\psi)$ is given by Eq. (10.1),

$$T_{m_1 m_2 m_3 m_4}^W(h) = \int_{C^W} \beta_{m_1 m_2 m_3 m_4}^{W, \ell} \tilde{D}_{m_4+m_2, m_1+m_3}^{\sigma, \tau, \epsilon, \ell} (h) d\ell, \quad (10.19)$$

if $T(\psi)$ is given by Eq. (10.4), and by

$$T_{m_1 m_2 m_3 m_4}^W(h) = \beta_{m_1 m_2 m_3 m_4}^W \tilde{D}_{m_4+m_2, m_1+m_3}^{\sigma, \tau, \epsilon, \ell(W)}(h), \quad (10.20)$$

if $T(\psi)$ is given by Eq. (10.6).

In general, the functional $T(\psi)$ can be a sum of three functionals of the three kinds considered. A contribution of the form (10.18) was considered in Ref. 11). As shown in Ref. 16), the contribution to the integral (10.18) due to the continuous series is strictly connected to the background integral in the Regge representations. A term of the form (10.20) is very similar to the contribution of a Regge pole. The function $\ell(W)$ describes the Regge trajectory, the functions $\beta_{m_1 m_2 m_3 m_4}^W$ are connected with the residues, the index ϵ is determined by the spins of the interacting particles, and the other two indices σ and τ are connected with the parity and the signature. We shall consider in detail only the case in which all the particles are spinless. Then the subscripts m_i take only the value 0 and the amplitude depends only on W and z where z is defined by Eq. (9.25) and is connected with the Mandelstam variables W_{12} and W_{14} by means of Eqs. (9.23) and (9.24). In this case, from Table 1 we see that, for $W < 0$, the representations of the discrete classes do not contribute and it must be $\epsilon = 0$, $\sigma = \pi_1 \pi_3 = \pi_4 \pi_2$. From the results of Section 6 and Eq. (9.26) we have

$$T_{0000}^W(h) = T(W, z) = \begin{cases} \beta(W) P_{\ell(W)}(z), & z \geq 1, \\ \beta(W) \tau P_{\ell(W)}(|z|), & z \leq -1. \end{cases} \quad (10.21)$$

The usual form of a Regge pole contribution is of the type

$$T(W, z) = \beta'(W) \left[P_{\ell(W)}(-z) + \tau' P_{\ell(W)}(z) \right], \quad (10.22)$$

where τ' is the signature. It is easy to show, using some properties of the Legendre functions⁴⁹⁾ that the two contributions (10.21) and (10.22) are equivalent if we neglect terms of the order $|z|^{-\ell(W)-1}$ (which can be included in the background integral) and identify the parameter τ with signature τ' .

We remark that in the reactions (4.19) only partial waves with $\pi(-1)^j = \pi_1 \pi_3 = \pi_4 \pi_2$ can appear. This fact suggests identification of our parameter σ with the quantity $\pi(-1)^j$ of the usual Regge formalism, where j is the spin and π is the parity of the particles which lie on the trajectory. A more detailed account of the connections between this formalism and the usual Regge theory will be given elsewhere.

11. CONSEQUENCES OF THE SYMMETRY WITH RESPECT TO THE EXCHANGE OF IDENTICAL PARTICLES

If two of the particles belong to the same representation of the symmetry group, we have to take into account some further conditions, due to the symmetry properties of the functional defined by Eq. (4.7) with respect to the exchange of the arguments which describe two particles of the same kind.

We consider only the case in which the particles (1) and (3) are of the same kind. The case in which the particles (2) and (4) are of the same kind can be treated in a similar way by means of the substitutions (7.39). The other possible cases have not a great interest for the problems we are considering.

Therefore we assume that

$$M_1 = M_3, \quad j_1 = j_3, \quad \pi_1 = \pi_3. \quad (11.1)$$

We define the operator E which operates in the space $\tilde{\mathcal{H}}^{(A)}$ in the following way

$$E(\tilde{f}^{(1)} \otimes \tilde{f}^{(3)}) = \tilde{f}^{(3)} \otimes \tilde{f}^{(1)}. \quad (11.2)$$

From Eqs. (4.7) and (4.16) we have the symmetry property

$$\tilde{\Phi}(\tilde{f}^{(B)}, \tilde{f}^{(A)}) = (-1)^{2j_1} \tilde{\Phi}(\tilde{f}^{(B)}, E\tilde{f}^{(A)}). \quad (11.3)$$

In order to obtain the corresponding symmetry property of the functional $T(\psi)$, we note that it is

$$V_g^{(A)} E = E V_g^{(A)} \quad (11.4)$$

and therefore the operator E can be decomposed into a direct integral according to theorem 2 of Section 2. In terms of wave functions we have

$$[E\tilde{f}^{(A)}]_{m_1 m_3}(a_1, a_3) = \tilde{f}_{m_3 m_1}^{(A)}(a_3, a_1). \quad (11.5)$$

From Eqs. (C.1) and (C.3) of Appendix C, taking into account Eq. (11.1) we have

$$\mathfrak{S}_1 \omega = -\mathfrak{S}_3 \omega, \quad (11.6)$$

and from Eqs. (7.9) and (7.10) we have, after some calculation,

$$\begin{cases} b_1 \omega = \mu_g(-\pi) b_3 \omega \mu_g(\pi), \\ b_3 \omega = \mu_g(-\pi) b_1 \omega \mu_g(\pi), \\ \omega \gg (2M_1)^2, \end{cases} \quad (11.7)$$

$$\begin{cases} b_{1\omega} = t' b_{3\omega} u_y(\pi), \\ b_{3\omega} = t' b_{1\omega} u_y(\pi), \\ \omega < 0. \end{cases} \quad (11.8)$$

If we note that

$$R_{m m'}^{\pi j} (u_y(-\pi)) = (-1)^{j+m'} \delta_{m, -m'}, \quad (11.9)$$

we see that the exchange operator operates on the function defined by Eq. (7.14) in the following way

$$\begin{cases} [E^\omega f^{(A)\omega}]_{m_1 m_3} (a) = (-1)^{m_1 - m_3} f_{-m_3, -m_1}^{(A)\omega} (a u_y(-\pi)), \\ \omega \geq (2M_1)^2 \end{cases} \quad (11.10)$$

$$\begin{cases} [E^\omega f^{(A)\omega}]_{m_1 m_3} (a) = (-1)^{m_1 - m_3} f_{-m_3, -m_1}^{(A)\omega} (a t'), \\ \omega < 0. \end{cases} \quad (11.11)$$

The functions defined by Eq. (7.23) are affected by the exchange operator in the following way, for $\omega \geq (2M_1)^2$, $\Lambda = (\pi, j)$

$$\begin{aligned} [E^{\omega \Lambda} f^{(A)\omega \Lambda}]_{m_1 m_3 m} (a) &= (-1)^{m_1 - m_3} \int_{H_+} f_{-m_3, -m_1}^{(A)\omega} (a h u_y(-\pi)) \cdot \\ &\cdot R_{m, m_1 + m_3}^{\pi j} (h) dh = (-1)^{m_1 - m_3} \int_{H_+} f_{-m_3, -m_1}^{(A)\omega} (a h) \cdot \\ &\cdot (-1)^{j - m_1 - m_3} R_{m, -m_1 - m_3}^{\pi j} (h) dh = \\ &= (-1)^{j - 2j_1} f_{-m_3, -m_1, m}^{(A)\omega \Lambda} (a). \end{aligned} \quad (11.12)$$

The case $W < 0$ can be treated in a similar way by means of Eqs. (6.16) and (6.17). The general result is

$$\cdot [E^{W\Lambda} f^{(A)W\Lambda}]_{m_1 m_3 m} (a) = \chi_{m_1 m_3} (W, \Lambda) f_{-m_3, -m_1, m}^{(A)W\Lambda} (a), \quad (11.13)$$

where

$$\begin{cases} \chi_{m_1 m_3} (+, \pi, j) = (-1)^{j-2j_1}, \\ \chi_{m_1 m_3} (-, \sigma, \tau, \epsilon, l) = \tau (-1)^{m_1 - m_3}, \\ \chi_{m_1 m_3} (-, \pi, k) = (-1)^{k-2j_1}. \end{cases} \quad (11.14)$$

From Eq. (8.3) we obtain the corresponding transformation of the functions $\psi_{m_1 m_2 m_3 m_4}^{W\Lambda}$. If we call E_{13} the exchange operator which operates in the space $\mathcal{F}_P$, we have

$$[E_{13} \psi]_{m_1 m_2 m_3 m_4}^{W\Lambda} = \chi_{m_1 m_3} (W, \Lambda) \psi_{-m_3, m_2, -m_1, m_4}^{W\Lambda}. \quad (11.15)$$

The symmetry property of the functional $T(\psi)$ can be written in the form

$$T(E_{13} \psi) = (-1)^{2j_1} T(\psi). \quad (11.16)$$

If we take into account the internal symmetries, it is convenient to decompose the functions $f_{m_1 m_3 m}^{(A)W\Lambda}(a)$ and $\psi_{m_1 m_2 m_3 m_4}^{W\Lambda}$ in two parts, one symmetric and the other antisymmetric with respect to the exchange of the internal quantum numbers of the particles (1) and (3). Then Eq. (11.16) becomes

$$T(E_{13} \psi) = \xi (-1)^{2j_1} T(\psi), \quad (11.17)$$

where we have to take $\xi = 1$ (or $\xi = -1$) if ψ is symmetric (or antisymmetric) with respect to the exchange of the internal quantum numbers.

We may also write

$$T(\psi) = T(\hat{\psi}) , \quad (11.18)$$

where

$$\hat{\psi} = \frac{1}{2} \left[1 + \xi (-1)^{2j_1} E_{13} \right] \psi \quad (11.19)$$

From Eqs. (8.6) and (11.15) we have that the functions $\psi_{m_1 m_2 m_3 m_4}^{W\Lambda}$ vanish identically for some values of the arguments. We may consider them as functions defined for $\Lambda \in \Omega_{m_1 m_3}^{(A)W\xi} \cap \Omega_{m_4 m_2}^{(B)W}$ instead of for $\Lambda \in \Omega_{m_1 m_3}^{(A)W} \cap \Omega_{m_4 m_2}^{(B)W}$. The sets $\Omega_{m_1 m_3}^{(A)W\xi}$ are described in Table 2, which can be considered as a table of the selection rules for the quantum numbers of the "exchanged objects".

In a similar way we may define the sets $\Omega_{m_4 m_2}^{(B)W\xi}$, which have to be used instead of the sets $\Omega_{m_4 m_2}^{(B)W}$ when the particles (1) and (2) are of the same kind.

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* * *

Table 1

Table of the Λ 's which belong to the set $\Omega_{m_1 m_3}^{(\Lambda)W}$. Note that this set depends on the sign of W , on the indices m_1 and m_3 , on the spins j_1 and j_3 , and on the parities π_1 and π_3 .

	$W > 0$	$W < 0$	
	$\Lambda = (\pi, j)$	Continuous class $\Lambda = (\sigma, \tau, \epsilon, \ell)$ $\text{Re } \ell = -\frac{1}{2}$	Discrete class $\Lambda = (\pi, k)$
$m_1 = m_3 = 0$	$j = 0, 1, 2 \dots$ $\pi = \pi_1 \pi_3 (-1)^{j+j_1+j_3}$	$\epsilon = 0$ $\sigma = \pi_1 \pi_3 (-1)^{j_1+j_3}$ $\tau = \pm 1$	No contribution
$m_1 = -m_3 \neq 0$	$j = 0, 1, 2 \dots$ $\pi = \pm 1$	$\epsilon = 0$ $\sigma = \pm 1$ $\tau = \pm 1$	No contribution
$ m_1 + m_3 = n$ $n = 1, 2, 3 \dots$	$j = n, n+1, \dots$ $\pi = \pm 1$	$\epsilon = 0$ $\sigma = \pm 1$ $\tau = \pm 1$	$k = 1, 2, \dots n$ $\pi = \pm 1$
$ m_1 + m_3 = \frac{1}{2}$	$j = \frac{1}{2}, \frac{3}{2} \dots$ $\pi = \pm i$	$\epsilon = \frac{1}{2}$ $\sigma = \pm 1$ $\tau = \pm i$	No contribution
$ m_1 + m_3 = n$ $n = \frac{3}{2}, \frac{5}{2} \dots$	$j = n, n+1, \dots$ $\pi = \pm i$	$\epsilon = \frac{1}{2}$ $\sigma = \pm 1$ $\tau = \pm i$	$k = \frac{3}{2}, \frac{5}{2}, \dots n$ $\pi = \pm i$

Table 2

Table of the Λ 's which belong to the set $\Omega_{m_1 m_3}^{(\Lambda)W\xi}$ when the particles (1) and (3) are of the same kind.

	$W \geq (2M_1)^2$	$W < 0$	
		Continuous class $\Lambda = (\sigma, \tau, \epsilon, \ell)$ $\text{Re } \ell = -\frac{1}{2}$	Discrete class $\Lambda = (\pi, k)$
$m_1 = m_3 = 0$	$j = \begin{cases} 0, 2, 4, \dots (\xi=+1) \\ 1, 3, 5, \dots (\xi=-1) \end{cases}$ $\pi = \xi$	$\epsilon = 0$ $\tau = \xi$ $\sigma = 1$	No contribution
$m_1 = -m_3 \neq 0$	$j = \begin{cases} 0, 2, 4, \dots (\xi=+1) \\ 1, 3, 5, \dots (\xi=-1) \end{cases}$ $\pi = \pm 1$	$\epsilon = 0$ $\tau = \xi$ $\sigma = \pm 1$	No contribution
$m_1 = m_3 \neq 0$ $ m_1 + m_3 = n$ $n = 1, 2, 3, \dots$	$j = n, n+1, \dots$ $\pi = \xi$	$\epsilon = 0$ $\sigma\tau = \xi$ $\tau = \pm 1$	$k = 1, 2, \dots n$ $\pi = \xi$
$ m_1 \neq m_3 $ $ m_1 + m_3 = n$ $n = 1, 2, 3, \dots$	$j = n, n+1, \dots$ $\pi = \pm 1$	$\epsilon = 0$ $\tau = \pm 1$ $\sigma = \pm 1$	$k = 1, 2, \dots n$ $\pi = \pm 1$

INVARIANT MEASURES

A measure μ in the space of four-momenta is called invariant if it has the property

$$\int f(LP) d\mu(P) = \int f(P) d\mu(P), \quad (\text{A.1})$$

for every Lorentz matrix L . The invariant measure concentrated on an orbit is unique up to a constant factor. If we consider the orbit spanned by the four-momenta which can be assumed by a particle of mass M , the corresponding invariant measure is given by

$$\begin{aligned} \int f(P) d\mu_{M^2}^+(P) &= \int f(P) \delta(P^2 - M^2) \eta(P_t) d^4P = \\ &= \int f([\vec{P}^2 + M^2]^{\frac{1}{2}}, \vec{P}) \frac{1}{2} [\vec{P}^2 + M^2]^{-\frac{1}{2}} d^3\vec{P}, \end{aligned} \quad (\text{A.2})$$

where $\eta(x) = 0$ for $x \leq 0$, $\eta(x) = 1$ for $x > 0$. Of course, this measure is not invariant with respect to the inversion of the time co-ordinate. We call μ_W the measure invariant with respect to the full Lorentz group concentrated on the orbit $P^2 = W$. It is given by

$$\int f(P) d\mu_W(P) = \int f(P) \delta(P^2 - W) d^4P. \quad (\text{A.3})$$

If we take $W = M^2$, the measures defined by Eqs. (A.2) and (A.3) are connected by the formula

$$\int f(P) d\mu_{M^2}(P) = \int [f(P) + f(-P)] d\mu_{M^2}^+(P). \quad (\text{A.4})$$

It is possible to define over the groups $\tilde{A}$, H_+ , H_- and over their subgroups $SL(2C)$, $SU(2)$ and $SU(1,1)$ measures which are both right and left invariant. This means that if G is the group and dg the measure it is

$$\int_G f(g'g) dg = \int_G f(gg') dg = \int_G f(g^{-1}) dg = \int_G f(g) dg, \quad (A.5)$$

for any integrable function $f(g)$.

The invariant measure $d'u$ over $SU(2)$ is given by^{35,43)}

$$\int_{SU(2)} f(u) d'u = (4\pi)^{-2} \int_0^{4\pi} d\mu \int_0^{2\pi} d\nu \int_0^\pi f(u_2(\mu) u_y(\theta) u_2(\nu)) \sin\theta d\theta, \quad (A.6)$$

and the invariant measure du over H_+ can be defined in the following way

$$\int_{H_+} f(u) du = \frac{1}{2} \int_{SU(2)} [f(u) + f(su)] d'u. \quad (A.7)$$

With this normalization, the total measures of $SU(2)$ and of H_+ are 1.

The invariant measure $d'v$ over $SU(1,1)$ is given by⁴⁾

$$\int_{SU(1,1)} f(v) d'v = (4\pi)^{-2} \int_0^{4\pi} d\mu \int_0^{2\pi} d\nu \int_0^\infty f(u_2(\mu) a_x(z) u_2(\nu)) \sinh z dz, \quad (A.8)$$

and the measure dv over H_- can be defined by means of the formula

$$\int_{H_-} f(v) dv = \frac{1}{2} \sum_{i=1}^4 \int_{SU(1,1)} f(w_i' v) d'v, \quad (A.9)$$

where

$$w'_1 = l, \quad w'_2 = \lambda, \quad w'_3 = t, \quad w'_4 = \lambda t. \quad (\text{A.10})$$

The invariant measure $d'a$ over $SL(2\mathbb{C})$ is given by⁵⁰⁾

$$\int_{SL(2\mathbb{C})} f(a) d'a = (16)^{-1} \int_0^{4\pi} d\mu_1 \int_0^\pi \sin \theta_1 d\theta_1 \int_0^{2\pi} d\chi_1 \int_0^\pi \sin \theta_2 d\theta_2 \cdot \\ \cdot \int_0^{2\pi} d\chi_2 \int_0^\infty f(u_z(\mu_1) u_y(\theta_1) u_z(\chi_1) a_z(\zeta) u_y(\theta_2) u_z(\chi_2)) (\sinh \zeta)^2 d\zeta, \quad (\text{A.11})$$

and we can define an invariant measure da over the group $\tilde{A}$ in the following way

$$\int_{\tilde{A}} f(a) da = \frac{1}{2} \sum_{i=1}^4 \int_{SL(2\mathbb{C})} f(w_i a) d'a, \quad (\text{A.12})$$

where

$$w_1 = l, \quad w_2 = \lambda, \quad w_3 = t, \quad w_4 = \lambda t. \quad (\text{A.13})$$

Of course, the total measures of $SU(1,1)$, H_- , $SL(2\mathbb{C})$, and $\tilde{A}$ are infinite and the normalization we have used is arbitrary.

In Appendix C we shall use an important connection between the invariant measures on the orbits and the invariant measures over the above-mentioned groups. If we choose for every four-momentum Q belonging to the orbit

$Q^2 = W$ an element a_Q of $\tilde{A}$ with the property

$$L(a_Q) Q_w^0 = Q, \quad (A.14)$$

where Q_w^0 is defined by Eq. (5.26) and $W \neq 0$, it is easy to show that every element a of $\tilde{A}$ can be decomposed in the following way

$$a = a_Q h, \quad h \in H_w, \quad (A.15)$$

and this decomposition is unique. Let us consider the following integral

$$\int_{H_w} d\mu_w(Q) \int f(a_Q h) dh. \quad (A.16)$$

If we remark that it is

$$[a_{L(a)Q}]^{-1} a a_Q \in H_w, \quad (A.17)$$

and use the invariance properties of the measures, we have

$$\begin{aligned} & \int_{H_w} d\mu_w(Q) \int f(a a_Q h) dh = \\ &= \int_{H_w} d\mu_w(Q) \int f(a_{L(a)Q} h) dh = \\ &= \int_{H_w} d\mu_w(Q) \int f(a_Q h) dh. \end{aligned} \quad (A.18)$$

It follows that the integral (A.16) defines a left invariant measure over $\tilde{A}$ and, as this measure is unique within a constant factor, it must be

$$\int_{H_W} d\mu_W(Q) \int_{\tilde{A}} f(aqh) dh = C(W) \int_{\tilde{A}} f(a) da . \quad (A.19)$$

The factor $C(W)$ can be evaluated by means of a more detailed calculation and the result is

$$C(W) = (2\pi^2)^{-1} |W| . \quad (A.20)$$

We shall also use the following identity, which follows immediately from Eq. (A.3)

$$\int_{-\infty}^{+\infty} dW \int f(Q) d\mu_W(Q) = \int f(Q) d^4 Q . \quad (A.21)$$

* * *

THE PLANCHEREL MEASURE FOR THE GROUPS H_+ AND H_-

If $f(h)$ is a function which is both L^1 and L^2 over H_W with respect to the invariant measure dh , we can define the matrices

$$F_{m m'}^\Lambda = \int_{H_W} f(h) R_{m m'}^{\omega \Lambda}(h) dh \quad (B.1)$$

and the following Plancherel formula holds

$$\int_{H_W} |f(h)|^2 dh = \int_{\hat{H}_W} \sum_{m m'} |F_{m m'}^\Lambda|^2 d\Lambda, \quad (B.2)$$

where $\hat{H}_W$ is the space of the equivalence classes of unitary irreducible representations of H_W (the elements of $\hat{H}_W$ are labelled by the symbol Λ) and $d\Lambda$ is the Plancherel measure over the space $\hat{H}_W$.

As the group H_+ is compact and its total invariant measure is normalized to one, from a general theorem^{21, 23, 35)} we have that the Plancherel measure of the representation $R^{\pi j}$ is equal to its dimension, i.e. to $(2j + 1)$.

The Plancherel measure for H_- can be derived from the Plancherel measure $d\Lambda'$ for the subgroup $SU(1,1)$ which is given by^{4, 11)}

$$\begin{aligned} \int \psi(\Lambda') d\Lambda' &= \sum_{\epsilon} \int_0^{\infty} \psi(\epsilon, \ell) \eta(\epsilon, \ell) d\ell + \\ &+ \sum_k (2k-1) [\psi(k+) + \psi(k-)] \end{aligned} \quad (B.3)$$

where

$$\begin{cases} \ell = i\pi - \frac{1}{2}, \\ \eta(\epsilon, \ell) = (2\ell + 1) \cotg \pi(\ell - \epsilon). \end{cases} \quad (B.4)$$

If we define the matrices

$$F_{mm'}^{\Lambda'}(w'_i) = \int_{SU(1,1)} D_{mm'}^{\Lambda'}(v) f(w'_i v) dv, \quad (B.5)$$

from Eq. (A.9) and from the Plancherel formula for $SU(1,1)$ we have

$$\int_H |f(v)|^2 dv = \frac{1}{2} \sum_{i=1}^4 \left(\sum_{mm'} |F_{mm'}^{\Lambda'}(w'_i)|^2 d\Lambda' \right). \quad (B.6)$$

From the Eqs. (6.16), (6.17), (B.1), and (B.5), we obtain

$$F_{mm'}^{\sigma\tau\epsilon\ell} = \frac{1}{2} \left[F_{mm'}^{\epsilon\ell}(\ell) + \sigma F_{-m,m'}^{\epsilon\ell}(\ell') + \tau F_{-m,m'}^{\epsilon\ell}(\ell') + \sigma\tau F_{mm'}^{\epsilon\ell}(\ell't') \right], \quad (B.7)$$

$$\left\{ \begin{aligned} F_{mm'}^{\pi k} &= \frac{1}{2} \left[F_{mm'}^{k+}(\ell) + \pi F_{mm'}^{k+}(\ell't') \right], \text{ for } m, m' = k, k+1, \dots \\ F_{mm'}^{\pi k} &= \frac{1}{2} \left[F_{mm'}^{k-}(\ell) + \pi F_{mm'}^{k-}(\ell't') \right], \text{ for } -m, -m' = k, k+1, \dots \\ F_{mm'}^{\pi k} &= \frac{1}{2} (-1)^{m-k} \left[F_{-m,m'}^{k+}(\ell') + \pi (-1)^{2k} F_{-m,m'}^{k+}(\ell') \right], \\ &\quad \text{for } -m, m' = k, k+1, \dots \\ F_{mm'}^{\pi k} &= \frac{1}{2} (-1)^{m-k} \left[F_{-m,m'}^{k-}(\ell') + \pi (-1)^{2k} F_{-m,m'}^{k-}(\ell') \right], \\ &\quad \text{for } m, -m' = k, k+1, \dots \end{aligned} \right. \quad (B.8)$$

It follows that

$$\sum_{\sigma \tau} \sum_{mm'} |F_{mm'}^{\sigma \tau \epsilon \ell}|^2 = \sum_{i=1}^4 \sum_{mm'} |F_{mm'}^{\epsilon \ell}(w'_i)|^2, \quad (\text{B.9})$$

and

$$\sum_{\pi} \sum_{mm'} |F_{mm'}^{\pi k}|^2 = \frac{1}{2} \sum_{i=1}^4 \left[\sum_{mm'} |F_{mm'}^{k+}(w'_i)|^2 + \sum_{mm'} |F_{mm'}^{k-}(w'_i)|^2 \right], \quad (\text{B.10})$$

From Eqs. (B.6), (B.9), and (B.10) we obtain the Plancherel formula for H_-

$$\int_{H_-} |f(v)|^2 dv = \sum_{\epsilon, \sigma, \tau} \frac{1}{2} \int_0^\infty \sum_{mm'} |F_{mm'}^{\sigma, \tau, \epsilon, i\nu - \frac{1}{2}}|^2 \cdot \eta(\epsilon, i\nu - \frac{1}{2}) d\nu + \sum_{\pi k} (2k-1) \sum_{mm'} |F_{mm'}^{\pi k}|^2. \quad (\text{B.11})$$

This formula coincides with Eq. (B.2) if we define the Plancherel measure $d\lambda$ in the following way:

$$\int_{\hat{H}_-} \psi(\lambda) d\lambda = \sum_{\epsilon, \sigma, \tau} \frac{1}{2} \int_0^\infty \psi(\sigma, \tau, \epsilon, i\nu - \frac{1}{2}) \eta(\epsilon, i\nu - \frac{1}{2}) d\nu + \sum_{\pi k} (2k-1) \psi(\pi, k). \quad (\text{B.12})$$

* * *

A TRANSFORMATION OF VARIABLES

If we insert Eqs. (7.8) and (7.9) into Eq. (7.5) and take into account the form of the four-vector Q_W^0 given by Eq. (5.26), we obtain a system of equations for the quantities ζ_{1W} and ζ_{3W} . One solution, which we shall adopt in this paper, is the following

$$\left\{ \begin{array}{l} \sinh \zeta_{1W} = \frac{M_1^2 - M_3^2 + W}{2 M_1 \sqrt{-W}}, \\ \sinh \zeta_{3W} = \frac{M_1^2 - M_3^2 - W}{2 M_3 \sqrt{-W}}, \\ W < 0. \end{array} \right. \quad (C.1)$$

$$\left\{ \begin{array}{l} \cosh \zeta_{1W} = \frac{M_3^2 - M_1^2 - W}{2 M_1 \sqrt{W}}, \quad \zeta_{1W} \geq 0, \\ \cosh \zeta_{3W} = \frac{M_3^2 - M_1^2 + W}{2 M_3 \sqrt{W}}, \quad \zeta_{3W} \geq 0, \\ 0 < W \leq (M_3 - M_1)^2. \end{array} \right. \quad (C.2)$$

$$\left\{ \begin{array}{l} \cosh \zeta_{1W} = \frac{W + M_1^2 - M_3^2}{2 M_1 \sqrt{W}}, \quad \zeta_{1W} \leq 0, \\ \cosh \zeta_{3W} = \frac{W - M_1^2 + M_3^2}{2 M_3 \sqrt{W}}, \quad \zeta_{3W} \geq 0, \\ W \geq (M_1 + M_3)^2. \end{array} \right. \quad (C.3)$$

Note that it is

$$\begin{cases} W = M_1^2 + M_3^2 - 2M_1 M_3 \cosh \zeta_W, & W \leq (M_1 - M_3)^2, \\ W = M_1^2 + M_3^2 + 2M_1 M_3 \cosh \zeta_W, & W \geq (M_1 + M_3)^2, \end{cases} \quad (C.4)$$

where

$$\zeta_W = \zeta_{1W} - \zeta_{3W}. \quad (C.5)$$

Now we consider a function $\varphi(a_1, a_3)$ with the property

$$\begin{aligned} \varphi(a_1 h_1, a_3 h_3) &= \varphi(a_1, a_3), \\ h_1, h_3 &\in H_+ \end{aligned} \quad (C.6)$$

For instance, we could take

$$\varphi(a_1, a_3) = \sum_{m_1, m_3} \left| \tilde{f}_{m_1, m_3}^{(A)}(a_1, a_3) \right|^2, \quad (C.7)$$

which has the property (C.6) on account of the covariance property (7.2).

As the measure of Π_+ is 1, from Eq. (A.19) of Appendix A we obtain that the integral

$$I = \iint \varphi(a_{p^{(1)}}, a_{p^{(3)}}) d\mu_{M_1^2}(p^{(1)}) d\mu_{M_3^2}(p^{(3)}), \quad (C.8)$$

where

$$L(a_{p^{(i)}})(M_i, 0, 0, 0) = p^{(i)}, \quad (C.9)$$

can be written in the form

$$I = C(M_1^2) C(M_3^2) \int_{\tilde{A}} da_1 \int_{\tilde{A}} da_3 \varphi(a_1, a_3). \quad (C.10)$$

From the invariance of the measure da we have

$$I = C(M_1^2) C(M_3^2) \int_{\tilde{A}} \psi(a) da, \quad (C.11)$$

where

$$\psi(a) = \int_{\tilde{A}} \varphi(a'a, a') da' \quad (C.12)$$

From Eq. (C.6) it follows that

$$\psi(h_1 a h_3) = \psi(a) \quad , \quad h_1, h_3 \in H_+, \quad (C.13)$$

and therefore using Eqs. (A.11) and (A.12) we have

$$\int_{\tilde{A}} \psi(a) da = 4\pi^3 \int_0^\infty [\psi(a_z(\zeta)) + \psi(ta_z(\zeta))] (\sinh \zeta)^2 d\zeta. \quad (C.14)$$

We note that

$$\begin{aligned} \int_{\tilde{A}} \varphi(al_{1w}, al_{3w}) da &= \varphi(l_{3w}^{-1} l_{1w}) = \\ &= \begin{cases} \psi(a_z(\zeta_w)) & \text{for } w \geq (M_1 + M_3)^2, \\ \psi(ta_z(\zeta_w)) & \text{for } w \leq (M_1 - M_3)^2. \end{cases} \end{aligned} \quad (C.15)$$

If we substitute Eq. (C.15) into Eq. (C.14), we obtain

$$\int_{\tilde{A}} \psi(a) da = 4\pi^3 \int_{\Gamma(M_1, M_3)} (\sinh \zeta_w)^2 \left| \frac{d\zeta_w}{dw} \right| \int_{\tilde{A}} \varphi(al_{1w}, al_{3w}) da, \quad (C.16)$$

where $\Gamma(M_1, M_3)$ is the set defined by Eq. (7.6). Using Eq. (C.11) and again

Eq. (A.19) we have

$$\iint \varphi(a_{p^{(1)}}, a_{p^{(3)}}) d\mu_{M_1^2}(p^{(1)}) d\mu_{M_3^2}(p^{(3)}) =$$

$$= \int_{\Gamma(M_1, M_3)} P(W, M_1, M_3) dW \int_{H_W} d\mu_W(Q) \varphi(a_Q h b_{1W}, a_Q h b_{3W}) dh, \quad (C.17)$$

where

$$P(W, M_1, M_3) = 4\pi^3 (\sinh \zeta_W)^2 \left| \frac{d\zeta_W}{dW} \right| C(M_1^2) C(M_3^2) [C(W)]^{-1} =$$

$$= \frac{1}{2} \pi |W|^{-1} \left[\Delta(W, M_1^2, M_3^2) \right]^{\frac{1}{2}}, \quad (C.18)$$

$$\Delta(a, b, c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc. \quad (C.19)$$

* * *

SOME RESULTS OF THE THEORY OF NORMED SPACES

In this Appendix we summarize, without giving the proofs, some results of the theory of normed spaces, which are used in the text. Some of the proofs can be found in Refs. 29) and 47); the other proofs are simple exercises.

- i) If $\mathcal{A}$ and $\mathcal{B}$ are linear spaces, we may consider the formal expressions of the form

$$\sum_{i=1}^N a_i \circ b_i, \quad a_i \in \mathcal{A}, \quad b_i \in \mathcal{B} \quad (\text{D.1})$$

For each expression of this kind there is a bilinear functional defined in the adjoint spaces $\mathcal{A}^*$ and $\mathcal{B}^*$ by means of the formula

$$F(f, g) = \sum_{i=1}^N f(a_i) g(b_i), \quad f \in \mathcal{A}^*, \quad g \in \mathcal{B}^*. \quad (\text{D.2})$$

The sets formed by all the expressions of the form (D.1) which give rise to the same bilinear functional are the elements of a linear space (which is isomorphic to a subspace of the space of all the bilinear functionals defined over $\mathcal{A}^*$ and $\mathcal{B}^*$). We call this linear space the "tensor product" of $\mathcal{A}$ and $\mathcal{B}$ and indicate it by $\mathcal{A} \circ \mathcal{B}$.

- ii) If $\mathcal{A}$ and $\mathcal{B}$ are normed spaces, a norm defined in the space $\mathcal{A} \circ \mathcal{B}$ is called a "cross norm" if it has the property

$$\|a \circ b\| = \|a\| \|b\|, \quad a \in \mathcal{A}, \quad b \in \mathcal{B}. \quad (\text{D.3})$$

For instance, if $\mathcal{A}$ and $\mathcal{B}$ are Hilbert spaces, the norm in the tensor product $\mathcal{A} \otimes \mathcal{B}$ defined by

$$\left\| \sum_{i=1}^N a_i \otimes b_i \right\|^2 = \sum_{i=1}^N \sum_{i'=1}^N (a_i, a_{i'}) (b_i, b_{i'}), \quad (\text{D.4})$$

is a cross norm. The tensor product $\mathcal{A} \otimes \mathcal{B}$ is the completion of the space $\mathcal{A} \circ \mathcal{B}$ with respect to this norm.

In the following we assume that $\mathcal{A}$ and $\mathcal{B}$ are normed spaces. It is possible to define a norm in the space $\mathcal{A} \circ \mathcal{B}$ in the following way

$$\|\varphi\| = \inf \sum_i \|a_i\| \|b_i\|, \quad \varphi = \sum_i a_i \circ b_i, \quad (\text{D.5})$$

$$\varphi \in \mathcal{A} \circ \mathcal{B}, \quad a_i \in \mathcal{A}, \quad b_i \in \mathcal{B}.$$

The greatest lower bound is taken with respect to all the expressions of the kind (D.1) which represent the element φ . The norm defined by Eq. (D.5) is a cross norm and it is greater than all the other cross norms. For this reason it is called the "greatest cross norm".

- iii) If $\Phi(a, b)$ is a bilinear functional defined for $a \in \mathcal{A}$, $b \in \mathcal{B}$ there is a unique functional $\Xi(\varphi)$ defined for $\varphi \in \mathcal{A} \circ \mathcal{B}$ with the property

$$\Xi(a \circ b) = \Phi(a, b). \quad (\text{D.6})$$

If we define the norms

$$\|\Phi\| = \sup \frac{|\Phi(a, b)|}{\|a\| \|b\|}, \quad (\text{D.7})$$

and

$$\|\Xi\| = \sup \frac{|\Xi(\varphi)|}{\|\varphi\|}, \quad (\text{D.8})$$

where $\|\varphi\|$ is the greatest cross norm, it can be shown that it is

$$\|\Xi\| = \|\Phi\|. \quad (\text{D.9})$$

- iv) If $\mathcal{A}^\circ$ and $\mathcal{B}^\circ$ are dense subspaces of the normed spaces $\mathcal{A}$ and $\mathcal{B}$, respectively, the space $\mathcal{A}^\circ \circ \mathcal{B}^\circ$ (considered as a subspace of $\mathcal{A} \circ \mathcal{B}$) is dense in $\mathcal{A} \circ \mathcal{B}$ with respect to any cross norm.
- v) If $\mathcal{B}$ is a normed space and $\mathcal{A}$ is a closed subspace of $\mathcal{B}$, we may decompose $\mathcal{B}$ into classes in such a way that each class contains all

the elements of B which can be obtained by adding an element of $\mathcal{A}$ to a fixed element of B . These classes form in a natural way a linear space called the factor space $B/\mathcal{A}$. The factor space becomes a normed space if we define the norm in the following way

$$\|\psi\| = \inf_{\varphi \in \psi} \|\varphi\|, \quad (D.10)$$

$$\psi \in B/\mathcal{A}, \quad \varphi \in B.$$

There is a one-to-one correspondence between the linear functionals defined over B which vanish when the argument belongs to $\mathcal{A}$ and the linear functionals defined over $B/\mathcal{A}$. This correspondence is given by

$$\Xi(\varphi) = T(\varphi) \quad \text{if } \varphi \in \psi, \quad (D.11)$$

$$\varphi \in B, \quad \psi \in B/\mathcal{A}.$$

If one of the functionals Ξ and T is bounded, the other is also bounded and their norms are equal

$$\|\Xi\| = \|T\|. \quad (D.12)$$

- vi) If $\mathcal{A}$ and B are the same as above, B° is a dense subspace of B , and $\mathcal{A}^\circ = B^\circ \cap \mathcal{A}$ is dense in $\mathcal{A}$, then every class of $B^\circ/\mathcal{A}^\circ$ is contained in a class of $B/\mathcal{A}$ and a class of $B/\mathcal{A}$ cannot contain two different classes of $B^\circ/\mathcal{A}^\circ$. The correspondence established in this way is a (norm preserving) isomorphism between $B^\circ/\mathcal{A}^\circ$ and a dense subspace of $B/\mathcal{A}$. We may also consider $B^\circ/\mathcal{A}^\circ$ as a dense subspace of $B/\mathcal{A}$.

* * *

PROPERTIES OF THE FUNCTIONS $\gamma(W)$ AND $\gamma'(W)$

From Eqs. (9.12), (A.20), (C.18), (9.23), and (9.24), we have

$$\frac{1}{2} \frac{d_{13}(W)}{d_{12}(W_{12})} \left| \frac{dW_{12}}{dz} \right|^{-1} = |W| \left[\Delta(W, M_1^2, M_3^2) \cdot \Delta(W, M_4^2, M_2^2) \Delta(W_{12}, M_1^2, M_2^2) \Delta(W_{12}, M_3^2, M_4^2) \right]^{-\frac{1}{4}}, \quad (E.1)$$

$$\frac{1}{2} \frac{d_{13}(W)}{d_{14}(W_{14})} \left| \frac{dW_{14}}{dz} \right|^{-1} = |W| \left[\Delta(W, M_1^2, M_3^2) \cdot \Delta(W, M_4^2, M_2^2) \Delta(W_{14}, M_1^2, M_4^2) \Delta(W_{14}, M_2^2, M_3^2) \right]^{-\frac{1}{4}}. \quad (E.2)$$

It is easy to show that it is

$$\begin{cases} \Delta(W_{12}, M_1^2, M_2^2) \Delta(W_{12}, M_3^2, M_4^2) = A_{12}(z), \\ \Delta(W_{14}, M_1^2, M_4^2) \Delta(W_{14}, M_2^2, M_3^2) = A_{14}(z), \end{cases} \quad (E.3)$$

where $A_{12}(z)$ and $A_{14}(z)$ are polynomials of fourth degree with four real zeros which lie outside the physical region. It follows that the minimum of the absolute value of these expressions when z varies in an interval of the physically allowed region is reached on the boundary of this interval. Therefore we have from Eqs. (9.30) and (9.32)

$$\begin{cases} \gamma(W) = |W| \left[\Delta(W, M_1^2, M_3^2) \Delta(W, M_4^2, M_2^2) \right]^{-\frac{1}{4}} \cdot \max \left[|A_{12}(1)|^{-\frac{1}{4}}, |A_{12}(-1)|^{-\frac{1}{4}} \right], \\ 0 < W < \min \left[(M_3 - M_1)^2, (M_2 - M_4)^2 \right], \end{cases} \quad (E.4)$$

$$\left\{ \begin{aligned} \gamma(\omega) &= |\omega| [\Delta(\omega, M_1^2, M_3^2) \Delta(\omega, M_4^2, M_2^2)]^{-\frac{1}{4}} \\ &\cdot \max [|A_{12}(1)|^{-\frac{1}{4}}, |A_{14}(-1)|^{-\frac{1}{4}}], \quad \omega < 0. \end{aligned} \right. \quad (\text{E.5})$$

If we define the polynomials

$$\left\{ \begin{aligned} B_{12}(z^{-1}) &= z^{-4} A_{12}(z), \\ B_{14}(z^{-1}) &= z^{-4} A_{14}(z), \end{aligned} \right. \quad (\text{E.6})$$

we see that they have the same properties as the polynomials used above, and therefore the minimum of their absolute value, when z^{-1} varies in an interval contained in the physically allowed region, is reached on the boundary of this interval. As a consequence, from Eq. (9.36) we have

$$\left\{ \begin{aligned} \gamma'(\omega) &= |\omega| [\Delta(\omega, M_1^2, M_3^2) \Delta(\omega, M_4^2, M_2^2)]^{-\frac{1}{4}} \\ &\cdot \max [|B_{12}(0)|^{-\frac{1}{4}}, |B_{14}(0)|^{-\frac{1}{4}}, |B_{12}(1)|^{-\frac{1}{4}}, |B_{14}(-1)|^{-\frac{1}{4}}], \\ \omega &< 0. \end{aligned} \right. \quad (\text{E.7})$$

Comparing with Eq. (E.5) we see that

$$\gamma'(\omega) = \max [\gamma(\omega), \gamma''(\omega)], \quad (\text{E.8})$$

where

$$\begin{aligned}
 \gamma''(W) &= |W| \left[\Delta(W, M_1^2, M_3^2) \Delta(W, M_4^2, M_2^2) \right]^{-\frac{1}{4}} \\
 &\cdot \max \left[|B_{12}(0)|^{-\frac{1}{4}}, |B_{14}(0)|^{-\frac{1}{4}} \right] = \\
 &= 2 |W|^2 \left[\Delta(W, M_1^2, M_3^2) \Delta(W, M_4^2, M_2^2) \right]^{-\frac{3}{4}}
 \end{aligned} \tag{E.9}$$

The function $\gamma''(W)$ is obviously continuous for $W < 0$. The quantities $A_{12}(z)$ and $A_{14}(z)$ vanish on the boundary of the physical region only if $W_{12} = \hat{W}_{12}$ or, respectively, $W_{14} = \hat{W}_{14}$. This can happen only if W takes the values W^{T12} or W^{T14} given by

$$\begin{cases}
 W^{T12} = (M_3 M_2^2 + M_4 M_1^2) (M_3 + M_4)^{-1} - M_3 M_4, & M_3 + M_4 \geq M_1 + M_2, \\
 W^{T12} = (M_2 M_3^2 + M_1 M_4^2) (M_1 + M_2)^{-1} - M_1 M_2, & M_1 + M_2 \geq M_3 + M_4,
 \end{cases} \tag{E.10}$$

$$\begin{cases}
 W^{T14} = (M_3 M_4^2 + M_2 M_1^2) (M_2 + M_3)^{-1} - M_2 M_3, & M_2 + M_3 \geq M_1 + M_4, \\
 W^{T14} = (M_4 M_3^2 + M_1 M_2^2) (M_1 + M_4)^{-1} - M_1 M_4, & M_1 + M_4 \geq M_2 + M_3.
 \end{cases} \tag{E.11}$$

It follows that the function $\gamma(W)$ is continuous in the set defined by Eq. (9.37).

If all the masses are equal it is

$$\left\{ \begin{array}{l} \gamma(W) = |W - 4M^2|^{-1}, \\ \gamma''(W) = 2|W|^{\frac{1}{2}} |W - 4M^2|^{-\frac{3}{2}}, \\ M_i = M, \quad W < 0, \\ W^{T12} = W^{T14} = 0. \end{array} \right. \quad (\text{E.12})$$

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- 46) The integral (7.23) converges only if the functions $f_{m_1 m_3}^{(A)W}(ah)$ belong to a certain dense subset of $\mathcal{H}^{(A)W}$. However the unitary mapping that Eq. (7.23) establishes between this dense subset and a dense subset of the space (7.32) can be extended to the whole Hilbert spaces.
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