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ASYMPTOTIC ESTIMATE OF FORWARD DISPERSION RELATIONS

IN THE PRESENCE OF REGGE CUTS

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A B S T R A C T

The asymptotic behaviour of forward dispersion relations is studied. An explicit estimate of the ratio $\text{Re } T / \text{Im } T$ is deduced when Regge cuts are present. Its predictions are checked against experiment, with which they are compatible in the present energy range.

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Recent measurements of the ratio $\alpha(E) = \text{Re } T(E,0)/\text{Im } T(E,0)$ of real to imaginary part of forward scattering amplitudes ¹⁾ seem to indicate that the predicted limit $\alpha(E) \rightarrow 0$ is approached quite slowly. Whereas the appropriate tool for computing α is forward dispersion relations ²⁾, a straightforward calculation is prevented (in the asymptotic region) in some cases (like NN collisions) by the uncertainties coming from the contribution of the unphysical cut ^{3),4)} and, in all cases, by the arbitrariness in the extrapolation of total cross-sections for energies beyond the accessible region ^{2),5)} (in fact, recent fits ⁶⁾ seem to indicate that a pure Regge-pole expansion is questionable, and that one should include Regge-cut effects). This makes it desirable to work in a round-about fashion, using only assumptions as general as possible. As a matter of fact, the difficulty here (as compared with the simpler case of standard "asymptotic theorems" ^{7),8)}) is due to the fact that

$$\lim_{E \rightarrow \infty} \log \text{Im } T(E) / \log E = 1 = \underline{\text{integer}} \quad (1)$$

and that, precisely, one would like to detect derivations of $\text{Im } T(E)$ from the pure behaviour $\text{Im } T(E) = (\text{constant}) \times E$, and distinguish whether these deviations are due to other Regge poles or to the presence of cuts. In spite of this, we want to show that, under what essentially amounts to assumptions of smoothness, one can still obtain general results, of which special cases are the current ones ^{7),9)}. In particular, we shall present a general analysis ¹⁰⁾ and an explicit estimate of the asymptotic value of $\alpha(E)$, the last particularly suited to the situation where Regge cuts are present. Its predictions are checked against experiment, showing that the existence of Regge cuts seems to be compatible with experimental results.

We shall consistently use the notation $T_{\pm}(E) = D_{\pm}(E) + iA_{\pm}(E)$, where the $(+/-)$ refer to the direct/crossed channel, for the forward scattering amplitudes. The variables E (lab. energy) and $k = \sqrt{E^2 - m^2}$ (lab. momentum) will be used interchangeably, as $E/k \rightarrow 1$ very quickly.

1. - ASSUMPTIONS

Apart from the validity of forward dispersion relations ¹¹⁾ we shall assume the following

- i) : Total cross-sections do not oscillate at infinity (in the sense of Ref. ⁸⁾), i.e., :

$$\lim_{E \rightarrow \infty} A(uE)/A(E) \text{ exists for } u \text{ near } 1. \quad (2)$$

Then, also from Ref. ⁸⁾, we know that (1) holds ; the explicit value of the limit is taken from experiment.

- ii) : The Pomeranchuk theorem ^{8), 12)} is satisfied in its strong version. By this we mean that the integral

$$I = \int_0^\infty dE [A_+(E) - A_-(E)]/k^2 \quad (3)$$

exists in the Lebesgue sense. If I is different from zero, these assumptions are all we need. If, however, (and this seems to be the actual situation), $I = 0$, extra hypotheses are, of course, necessary.

As a natural choice we have taken

- iii) : the difference $\sigma^{(-)} = \sigma_+ - \sigma_-$ between total cross-sections is also non-oscillating at infinity (this assumption can be dropped to get some results. Cf. Section 4 below).
- iv) : The present behaviour of $\sigma_\pm$ continues smoothly beyond the region of accessible energies.

More precisely, we can state (iv) as follows : as proved in Ref. ¹³⁾, from (i) it can be shown that, if we define

$$\eta_\pm(k) \equiv \frac{k \sigma_\pm(k)}{\int_0^k dk' \sigma_\pm(k')} - 1, \quad (4)$$

then $\eta(k) \rightarrow 0$ as $k \rightarrow \infty$. [Equation (4) will be recognized as a finite energy sum rule]. Plots of $\eta(k)$ show ⁶⁾ that this limit is attained slowly and without oscillation. Therefore, we formulate (iv) as requiring that η be non-oscillating, namely, that the analogue of Eq. (2) holds:

$$\lim_{k \rightarrow \infty} \eta(uk)/\eta(k) \text{ exists} \quad (5)$$

Then the limit $\lim_{k \rightarrow \infty} \log \eta(k)/\log k$ also exists. Again we take its value from experiment, and, since numerical computations ⁶⁾ seem to give it vanishing, we assume it to be zero ¹⁴⁾. Two more preliminary results will be introduced. First, as shown in Ref. ⁸⁾ (Theorem II.1), from (i), (iv), (iii), it follows, respectively,

$$\begin{aligned} \lim_{k \rightarrow \infty} \sigma_{\pm}(uk)/\sigma_{\pm}(k) &= 1 \\ \lim_{k \rightarrow \infty} \eta_{\pm}(uk)/\eta_{\pm}(k) &= 1 \\ \lim_{k \rightarrow \infty} \sigma^{(-)}(uk)/\sigma^{(-)}(k) &= u^{\beta}, \quad \beta < 0 \end{aligned} \quad (6)$$

Second, solving Eq. (4) (cf. also Ref. ¹³⁾, Appendix), we obtain

$$\begin{aligned} \sigma_{\pm}(k) &= [1 + \eta_{\pm}(k)] \sigma_{\pm}^{\circ}(k) \\ \sigma_{\pm}^{\circ}(k) &= (\text{constant}) \times \exp \left\{ \int_1^k dk' \eta_{\pm}(k')/k' \right\} \end{aligned} \quad (7)$$

Clearly, $\sigma^{\circ}/\sigma \rightarrow 1$, as $k \rightarrow \infty$.

4.

2. - NON-VANISHING I

From (i), (ii) we can, as in Ref. 8), deduce that $D(E)/A(E) \simeq iE/2A(E)$.

Then, since $D_{\pm}(E) = D(\pm E)$, $A_{\pm}(E) = \pm A(\pm E)$, the result $\alpha_{+}(E)/\alpha_{-}(E) \rightarrow -1$ follows trivially. It is perhaps worth while to remark that this result is still true in some cases where $\alpha \rightarrow 0$ (as if, e.g., the Froissart bound is saturated and $\sigma \sim \log^2 k$) or where $\alpha \rightarrow \infty$ [as if $\sigma \sim 1/(\log k)^{\delta}$, $\delta > 0$, as some recent fits seem to indicate ⁶⁾].

3. - VANISHING I

In this case the foregoing analysis breaks down and a more refined estimate is necessary. For this, write the usual twice subtracted dispersion relation in the form ¹⁵⁾

$$\frac{D(E)}{k^2} = \frac{1}{\pi} \int \frac{dE'}{k'^2} \left\{ \frac{E' [A_{+}(E') + A_{-}(E')] + E A^{(-)}(E')}{E'^2 - E^2} \right\} \quad (9a)$$

or still, with $A^{(-)} \equiv A_{+} - A_{-}$,

$$\frac{D_{\pm}(E)}{k^2} = \frac{2}{\pi} \int \frac{dE'}{k'^2} \frac{E' A_{\pm}(E')}{E'^2 - E^2} \mp \frac{1}{\pi} \int \frac{dE'}{k'^2} \frac{A^{(-)}(E')}{E' + E} \equiv \Gamma_{\pm} \pm R \quad (9b)$$

We shall study separately the contributions from Γ and R . As to the second,

$$\begin{aligned} R &= \frac{1}{\pi E} \int \frac{dE'}{k'^2} \frac{A^{(-)}(E')}{k'^2} - \frac{1}{E} \frac{1}{\pi} \int \frac{dE'}{k'^2} \frac{E'}{k'^2} \frac{A^{(-)}(E')}{E' + E} \\ &= - \frac{1}{\pi E} \int \frac{dk'}{k'^2} \frac{\sigma^{(-)}(k')}{E' + E} \end{aligned}$$

where we have used the fact that $I = 0$ to eliminate one term, and the change of variables $E' \rightarrow k'$ and the substitution $A^{(-)} = k' \sigma^{(-)}$ in the other term. Multiplying and dividing by $\sigma^{(-)}(k)$, making the change of variables $k' = uk$, and taking the limit ¹⁶⁾ (6), we get

$$R \simeq - \frac{\sigma^{(-)}(k)}{E} \frac{1}{\pi} \int_0^\infty du \frac{u^\beta}{u+1} = \frac{1}{\sin \pi \beta} \frac{\sigma^{(-)}(k)}{E} \quad (10)$$

As to Γ , making obvious changes of variables we get

$$\Gamma = \frac{2}{\pi} \int_0^\infty dk' \frac{\sigma(k')}{k'^2 - k^2}$$

and, decomposing σ as in (7),

$$\Gamma = \Gamma_1 + \Gamma_0 \quad \Gamma_0 = \frac{2}{\pi} \int_0^\infty dk' \frac{\sigma^0(k')}{k'^2 - k^2} \quad \Gamma_1 = \frac{2}{\pi} \int_0^\infty dk' \frac{\eta(k') \sigma^0(k')}{k'^2 - k^2} \quad (11)$$

Multiplying and dividing Γ_0 (resp : Γ_1) by $\sigma^0(k)$ (resp : $\eta(k) \sigma^0(k)$), and letting $k' = uk$,

$$\begin{aligned} \Gamma_0 &= \frac{\sigma^0(k)}{k} \frac{2}{\pi} \int_0^\infty du \frac{\sigma^0(uk)/\sigma^0(k)}{u^2 - 1} \\ \Gamma_1 &= \eta(k) \frac{\sigma^0(k)}{k} \frac{2}{\pi} \int_0^\infty du \frac{\eta(uk) \sigma^0(uk)/\eta(k) \sigma^0(k)}{u^2 - 1} \end{aligned} \quad (12)$$

Using (7), we see that, to first order in $\eta(k)$,

$$\begin{aligned} \sigma^0(uk)/\sigma^0(k) &= \exp \left\{ \eta(k) \int_1^u du' \frac{\eta(u'k)/\eta(k)}{u'} \right\} \\ &\simeq \exp \left\{ \eta(k) \int_1^u du'/u' \right\} = u^{\eta(k)} \end{aligned}$$

where we have used (6) to take the limit $\eta(u'k)/\eta(k)$. Therefore,

$$\Gamma_0 \simeq \frac{\sigma^0(k)}{k} \frac{2}{\pi} \int_0^\infty du \frac{u^{\eta(k)}}{u^2 - 1} = \frac{\sigma^0}{k} \frac{1 - \cos \pi \eta}{\sin \pi \eta} \simeq \frac{\sigma^0(k)}{2k} \pi \eta(k) \quad (13)$$

Hence, substituting this and using again (6), we get

$$\Gamma_1/\Gamma_0 \simeq \frac{4}{\pi^2} \int_0^\infty du \frac{\eta(uk)\sigma^0(uk)/\eta(k)\sigma^0(k)}{u^2-1} \rightarrow \frac{4}{\pi^2} \int_0^\infty du \frac{1}{u^2-1} = 0$$

We can neglect the contribution of Γ_1 as compared to that of Γ_0 . Collecting our results, we get, finally,

Theorem : Under assumptions (i) to (iv), we have

$$\alpha_{\pm}(k) \simeq \frac{1 - \cos \pi \eta_{\pm}(k)}{\sin \pi \eta_{\pm}(k)} \mp \frac{1}{\sin \pi \beta} \frac{\sigma^{(-)}(k)}{\sigma_{\pm}(k)} \quad (14)$$

4. - DISCUSSION

From the preceding analysis, one can at once decide that $I = 0$, in general. In fact, otherwise one would have $\alpha_+/\alpha_- \approx -1$, which is far from the present values ($\sim +1$ for πN , ~ 0 for NN scattering). Moreover, when the value of I can be computed, it is found to be zero (Goldberger's sum rule for πN scattering). Of course, the same conclusion may be obtained invoking the experimental values of $d\sigma_{\text{ch.ex}}/dt|_{t=0}$. Anyway, we have to apply the analysis of Section 3.

Before proceeding with the actual computations, some comments are due to the verification of hypotheses (i) to (iv). Clearly, all of them are satisfied if a generalized Regge-like behaviour (with cuts) of the form ^{7),8),13)} $\sigma(E) \simeq \sum_j c_j E^{\alpha_j} (\log E)^{\epsilon_j} \dots$ holds. More generally, the situation is as follows : assumptions (i), (ii) are classical, and the same is true of (iii) (the last can also be dropped if we are interested only in the values of $\alpha_+ + \alpha_-$), and we shall not

consider them any further. Assumption (iv) seems to be more difficult to justify on theoretical grounds, although it is apparently well satisfied experimentally ⁶⁾. Whereas the fact that $\eta(k)$ is non-oscillating is not likely to rise objections, the special value of the limit is (5) [or, equivalently, (6)] has some implications. In fact, if a pure Regge pole behaviour holds for σ , then $\eta(k) \sim k^{-\beta}$, and $\eta(uk)/\eta(k) \rightarrow u^{-\beta}$, $\log \eta(k)/\log k \rightarrow -\beta$. The analysis presented here breaks down, and one has to apply different techniques. As the results one obtains in this case are discussed in several other places ^{8),9),17)}, we shall not repeat them here. It should be emphasized again, however, that present experimental analysis ⁶⁾ seems to favour assumption (iv), indicating that some deviation off a pure Regge pole behaviour might be present. In this spirit, testing of Eq. (14) amounts to obtain a dispersion-theoretic confirmation of the existence (or non-existence) of Regge cuts in forward amplitudes.

We have checked the predictions of Eq. (14) for πp and for pp scattering. In doing so, only the values of $\alpha_+ + \alpha_-$ have been considered, as, from (14), it is obvious that $\alpha_+ - \alpha_-$ is given here essentially as in the usual case (i.e., by means of the charge-exchange amplitude) and therefore nothing new may occur. We use Eq. (14) as it is, rather than the expression we would have obtained making the analysis for $A^{(+)}$ or $T^{(+)}$ directly (instead of analyzing $A_{\pm}$, $T_{\pm}$ separately) for the following reason: the determination of $\eta_{\pm}$ is a very delicate process ⁶⁾. If we work with $A^{(+)}$, then $\eta^{(+)}$ is obtained through $\sigma^{(+)} = \sigma_+ + \sigma_-$. Therefore to the errors resulting from the (necessary) interpolation one has to perform for computing the integral in (4), one would, if working with $\sigma^{(+)}$, add (unnecessary) errors due to the fact that a further interpolation is needed (since σ_+ , σ_- are not measured at exactly the same energies). It turns out that these errors contribute to a sizable deviation ($\sim 20\%$) off the more exact value (14).

The results of our analysis are presented in the figure, for $p\pi$ scattering. (We have not displayed the corresponding pp values, as the large errors present preclude any clear analysis.) The experimental points are from Ref. ¹⁾. We see that the upper part of the predicted curve

is compatible with the experimental numbers, within the errors ("theoretical" errors can be estimated to be ¹⁸⁾ $\sim 10\%$). However, present experimental data are not sufficient to decide clearly against or for the existence of a cut. If a free hand extrapolation of the theoretical curve is made, we see that deviations from the pure Regge pole result of Refs. ^{1),5)} will become unambiguously relevant from ~ 26 GeV on (the two curves seem to cross at that energy). However, we believe that the present results are very interesting, as they show that replacement of the P, P' poles by a cut improves the agreement (in fact, a Regge pole analysis without P' disagrees badly with experiment). What is perhaps more relevant is that we have shown practically that dispersion relations, by being sensitive to values of σ_{total} above the accessible region, provide a "crystal ball" where effects of Regge cuts can manifest themselves before what one would think to be a high enough energy.

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- 10) This result represents a refinement of the analysis of H. Lehmann - Nuclear Phys. 29, 300 (1962).
- 11) More generally, we can take the results proved from field theory by J. Bros, H. Epstein and V. Glaser - Comm.Math.Phys. 1, 240 (1965), and our analysis will still hold with trivial modifications.

10.

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- 13) J.L. Gervais and F.J. Yndurain - Phys.Rev. 169, 1187 (1968).
- 14) The analysis presented here breaks down if a non-zero limit is obtained. This will be discussed in detail in Section 4.
- 15) The principal value is to be understood in all singular integrals. We are consistently omitting terms due to the poles of $T(E)/(E^2-m^2)$; their inclusion is straightforward.
- 16) We interchange limits and integrals freely, since, for non-oscillating functions, one can obtain majorizations sufficient to justify the use of Lebesgue theorem (cf. Ref. ⁸), Appendices A and C).
- 17) A. Bialas and E. Bialas - Nuovo Cimento 37, 1686 (1965).
- 18) Sources of these errors are :
 - 1 - the arbitrariness in the lower limit of the integral in Eq. (4) ; a change in c from 800 MeV to 200 MeV gives an error of $\sim 5\%$; we have taken $c = 400$ MeV ;
 - 2 - errors in the total cross-sections, again contributing to (4) ;
 - 3 - errors in the determination of the integral in Eq. (4) ; changing from a direct (linear) interpolation to a parabolic one improves the results by a $\sim 7\%$.To the over-all effect of these errors, a purely theoretical (and unknown) one is added, due to the fact that (14) is only true asymptotically.

Figure caption : Comparison of Regge cut predictions with experiment for πp scattering : plot of $\alpha = \alpha_+ + \alpha_-$ vs. momentum.
Isolated points : data from Ref. ¹).
Solid line : Regge cut prediction
Broken line : pure Regge pole prediction of Refs. ¹), ⁵).

FIG. 1