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PHOTOPRODUCTION AND ELECTROPRODUCTION OF PIONS

I. - DISPERSION RELATION THEORY

F.A. Berends ^{*)}, A. Donnachie and D. Weaver ^{**)}

CERN - Geneva

A B S T R A C T

The dispersion relation theory of photo- and electroproduction of pions is reviewed and extended, exploiting the parallel between the two. Analytic expressions are given for the Born terms of the individual electroproduction multipoles and for the important kernels in the integral equations for the photoproduction multipoles. A sum rule is derived for the latter, and a discussion given of the application of the theory to data analysis, the results of which will be presented in a subsequent paper.

^{*)} On leave of absence from Instituut-Lorentz, Leiden, The Netherlands (permanent address).

^{**)} NATO Postdoctoral Fellow 1965-66. On leave of absence from Department of Physics, Tufts University, Medford, Mass. U.S.A. (permanent address).

1. INTRODUCTION

A discussion is given of the fixed- t dispersion relation theory of photoproduction and electroproduction of pions, and of the integral equations for the individual multipole transition amplitudes which are obtained by projection from these dispersion relations. The known results are reviewed and extended, exploiting fully the parallel between photoproduction and electroproduction in the one-photon-exchange approximation. In particular, analytic expressions are obtained for all the important kernels in the integral equations for the multipole amplitudes of photoproduction, and for the contributions of the Born terms to the individual multipoles in electroproduction. (The kernels for the latter have also been evaluated in closed form, but are not presented here because of their considerable length.)

In Section 2 a brief account is given of the important role of photoproduction and electroproduction of pions. The notation and kinematics are discussed in Section 3, and in Section 4 the invariant amplitudes are obtained. The fixed- t dispersion relations for these amplitudes are given in Section 5, their multipole decomposition in Section 6 and the multipole dispersion relations in Section 7. The projected Born terms for the multipoles of electroproduction and of photoproduction are obtained in Section 8 and the important kernels for photoproduction in Section 9. A discussion of the phase condition imposed by unitarity is given in Section 10. In Section 11 a sum-rule for the multipoles is obtained, and finally, in Section 12, there is a brief outline of the application of the theory to a multipole analysis of photoproduction, the results of which will be presented in a subsequent paper.

In the three Appendices are gathered the cross-section and polarization formulae (Appendix A), the Born terms of the invariant amplitudes (Appendix B) and the relation between the multipole description and the helicity description (Appendix C).

2. THE ROLE OF PHOTOPION AND ELECTROPION PRODUCTION

A detailed knowledge of photopion production is of considerable value because of the critical role which photopion production plays in our understanding of many theoretical and practical problems. The most important of these are discussed briefly below.

A. Nucleon isobars

One of the outstanding features of the recent pion-nucleon phase shift analyses¹⁻⁴⁾ has been the demonstration of the existence of hitherto unknown resonances of a rather unconventional type, [S_{31} (~ 1700 MeV), S_{11} (~ 1500 MeV) — and possibly S_{11} (~ 1650 MeV) — P_{11} (~ 1420 MeV), D_{15} (~ 1650 MeV)] as well as confirming the four previously known ones [P_{33} (1240 MeV), F_{37} (~ 1920 MeV), D_{13} (~ 1520 MeV), F_{15} (~ 1680 MeV)]. Except for P_{33} , it is very difficult (and in some cases impossible) to determine accurately the masses and, in particular, the widths of these resonances from a study of their two-body decay modes alone, a detailed investigation of their three- and four-body decay modes being required. This also gives information on the possible symmetry or quark model⁵⁾ classifications of these resonances.

How do these resonances manifest themselves in photopion (and electropion) production? Are some suppressed and some enhanced, as indicated by the quark model⁶⁾? If this is indeed true, then the study of the many particle decay modes of the enhanced resonances may be accomplished most readily in photoproduction and/or electroproduction experiments. It is clearly of importance to determine whether this will be possible or not.

B. The quark model

Knowledge of the photopion production multipoles allows a direct test of any symmetry or quark model classification of the resonances. For example in the quark model the resonances can be incorporated using orbital

angular momentum states of $L = 1$ and $L = 2^5$), and this in turn implies specific properties of the matrix elements for the photoproduction of the resonances⁶⁾ in particular transitions to $S_{11}(1650)$, $P_{11}(1420)$ and $D_{15}(1650)$ are not allowed. To date, the only quantitative test of the quark model in photopion production is at the first resonance^{7,8)} where the process is well documented experimentally and well understood theoretically. [The results at the first resonance are also a consequence of $SU(6)^9$ and $SU(6)_W^{10}$.] The results are ambiguous. The predicted vanishing of the electric quadrupole transition to the final P_{33} pion-nucleon state is observed, but the predicted magnitude of the magnetic dipole transition to the same final state is only 80% of the observed value (which is known to within 2 or 3%) and a simple model of symmetry breaking actually makes the situation worse.

C. Sum rules

A fruitful source of sum rules involving photopion production cross-sections is the equal time commutation relation between charge densities

$$[V_0^i(x_1), V_0^j(x_2)] = if^{ijk} V_0^k(x_1) \delta(x_1 - x_2) \quad (2.1)$$

Making use of that part of Eq. (2.1) which is antisymmetric with respect to the isospin indices i, j , one obtains the sum rule¹¹⁾

$$\left(\frac{\mu_A^V}{2m}\right)^2 + \frac{2}{\pi e^2} \int_0^\infty dE \frac{(2\sigma_{1/2}^V(E) - \sigma_{3/2}^V(E))}{E} = \frac{1}{3} \langle r^2 \rangle_F \quad (2.2)$$

where $\langle r^2 \rangle_F$ is the Dirac charge radius, μ_A^V is the isovector anomalous moment and $\sigma_{1/2}^V(E)$, $\sigma_{3/2}^V(E)$ are the total cross-sections for the absorption of an isovector photon, lab. energy E , in the isospin $1/2$, $3/2$ channels respectively.

Alternatively, making use of the isospin symmetric part of Eq. (2.1), one obtains the sum rule ¹²⁾

$$\left(\frac{\mu_A^P}{2m}\right)^2 + \frac{2}{\pi e^2} \int_0^\infty dE \frac{(\sigma_A(E) - \sigma_P(E))}{E} = 0 \quad (2.3)$$

where σ_P (σ_A) is the total cross-section for the absorption of a circularly polarized photon by a proton polarized with its spin parallel (anti-parallel) to the photon spin. The sum rule Eq. (2.3) is also a consequence of the unsubtracted dispersion relation for the spin-flip part of the forward Compton scattering amplitude ¹³⁾.

If the arguments are extended to commutators between space and time components of vector current densities then one obtains the sum rule ¹²⁾

$$\frac{1}{\pi e^2} \int_0^\infty dE \left\{ (2\sigma_{\frac{1}{2}}^V(E) - \sigma_{\frac{3}{2}}^V(E))_P - (2\sigma_{\frac{1}{2}}^V(E) - \sigma_{\frac{3}{2}}^V(E))_A \right\} = \frac{\mu_T^V}{2m} \quad (2.4)$$

where μ_T^V is to total vector magnetic moment and the cross-sections are those with isovector photons.

A somewhat different approach produces the sum rule ¹⁴⁾

$$G_A = \frac{\epsilon}{2\pi e^2} \int_0^\infty \frac{E dE}{\cos \varphi} \left\{ \sigma_P^{(+)}(E, \cos \varphi) - \sigma_P^{(-)}(E, \cos \varphi) - \sigma_n^{(+)}(E, \cos \varphi) + \sigma_n^{(-)}(E, \cos \varphi) \right\} \quad (2.5)$$

where G_A is the renormalized axial vector β -decay coupling constant, and $\sigma_{p,n}^{(\pm)}$ is the total inelastic cross-section for circularly polarized photons with helicity ± 1 and lab. kinetic energy E , incident on polarized protons (neutrons), φ being the angle between the direction of the photon and the polarization of the nucleon. ϵ is a constant depending on the particular model of elementary particles chosen.

For example $\epsilon = +3$ for the usual Gell-Mann-Zweig¹⁵⁾ three-quark model, and $\epsilon = \pm 1$ for the Matl-Hara¹⁶⁾ four-quark model.

Not all sum rules involve photopion cross-sections; relations involving the amplitude itself can be derived¹⁷⁾

$$\frac{\mathcal{M}'_P + \mathcal{M}_n}{2} + \frac{2m}{g} \frac{1}{\pi} \int \frac{dE}{E} \text{Im} A^s(E) = 0 \quad (2.6)$$

$$\frac{\mathcal{M}'_P - \mathcal{M}_n}{2} + \frac{2m}{g} \frac{1}{\pi} \int \frac{dE}{E} \text{Im} A^v(E) = 0 \quad (2.7)$$

Sum rules Eq. (2.2) and Eq. (2.3), which are derived solely from the commutation relation between "good" components, are reasonably convergent and appear to be moderately well satisfied^{18,13)}, although the situation is not as clear as one would like simply because of the lack of sufficiently precise information on the cross-sections above 500 MeV. Sum rule Eq. (2.5) is derived from the commutation relation between "bad" components, and convergence is doubtful. In sum rule Eq. (2.4), which is derived from the commutation relation between "good" and "bad" components, convergence is still problematical, but reasonably possible (the Pomeranchuk theorem requires that $2\sigma_{1/2}(E) - \sigma_{3/2}(E) \rightarrow 0$ as $E \rightarrow \infty$). In this case it is clear that the integral will certainly not be saturated by the low energy (< 500 MeV) cross-sections, information at much higher energies being essential. If a reliable test can be made of Eq. (2.4), it may well give more insight into the structure of current algebras.

Eqs. (2.6) and (2.7) are also strongly dependent on the high-energy amplitudes. In Eq. (2.6) there is no contribution from transitions to the final P_{33} state, and in Eq. (2.7) preliminary calculation¹⁷⁾ shows an almost complete cancellation of the low-energy

contributions; that is in both cases the sum rule must be explained by the higher energy contributions.

D. Electropion production

This process need only be considered in the one photon exchange approximation. Apart from the possible usefulness of electropion production as a tool for investigating the structure of the nucleon isobars, the electroproduction of a charged pion (for example $e^- + p \rightarrow e^- + n + \pi^+$) can, in principle, provide information on the electromagnetic form factors of the pion and the neutron, and in this latter case without the theoretical problems attendant on a deuteron target.

In photopion production it appears that for photon energies below 500 MeV, the dispersion relation description in the $(\text{Born} + M_1 + {}^{(3)} + E_0 +)$ approximation is good^{8,24)}, contributions from the more massive nucleon isobars (both in the s- and u-channels) and from the exchange of meson isobars (ρ, ω etc.) in the t-channel being negligible. If this is indeed true, the same will hold for electroproduction. It must be checked, however, that there is not a fortuitous cancellation between these contributions in low-energy photoproduction, which requires a detailed evaluation of the photoproduction multipoles. If these effects are small, then one has immediately a no-parameter theory of low-energy π^0 electroproduction, and in the case of π^+ electroproduction on protons the only unknowns are the pion electromagnetic form factor and the neutron charge form factor for small momentum transfers, and they can be found, in principle, by fitting to the experimental data.

E. Compton scattering

In dispersion theory, Compton scattering is given by four contributions¹⁹⁾:

- i) The nucleon Born terms, both direct (s-channel) and crossed (u-channel) arising from single nucleon intermediate states in the s- and u-channels. These are determined by the proton charge.

- ii) The pion Born term, arising from single π^0 exchange in the t-channel. This is given in terms of the $\pi^0 \rightarrow 2\gamma$ decay width and the pion-nucleon coupling constant.
- iii) The rescattering terms, both direct (s-channel) and crossed (u-channel) arising from intermediate states of one nucleon and one (or more) pions in the s- and u-channels. They can be expressed completely in terms of the photoproduction multipoles.
- iv) The two-pion exchange terms in the t-channel, $\gamma + \gamma \rightarrow \pi + \pi \rightarrow N + \bar{N}$ which is given up to an unknown multiplicative constant.

Hence once photoproduction is known, this yields a one-parameter theory of Compton scattering, and this parameter may be obtained by fitting to experiment.

F. Vector meson coupling constants

Since vector meson exchange in the t-channel is allowed in photopion production, information can be obtained on their coupling constants, and in particular on $g_{\rho\pi\gamma}$. Direct experimental measurement of this coupling constant in ρ -decay^{20,55)} and in photoproduction of rho mesons²¹⁾ gives an upper limit which is still far in excess of that obtained from analysis of low-energy photopion production^{8, 24)}, and of that required by SU(3). Techniques similar to those used in obtaining information on the π - π interaction from pion-nucleon scattering^{22, 23)} can be used to characterize $g_{\rho\pi\gamma}$ and in so doing to provide a test of SU(3).

To obtain the required detailed information on photopion production it is necessary to perform a multipole analysis of the experimental data comparable in scope to the recent phase-shift analysis of pion-nucleon scattering¹⁻⁴⁾. Compared to pion-nucleon scattering, however, the experimental situation is rather poor. Polarization data are few, and are isolated both in energy and angle, the statistical errors on differential cross-section measurements are much larger than

in scattering (apart from some notable exceptions in the more recent data) and the systematic errors, particularly normalization errors, are very much larger. The data fall short of the ideal situation in which it would be possible to perform a direct multipole analysis of the data without further information, and to enable such an analysis to be performed at all one must inject as much theory into the system as possible. The theory must follow from rather general principles, be parameter free and be one which is known to be reliable. The only available choice is the dispersion relation theory of photopion production^{8, 24-44}) which has achieved remarkable success at low energies, the most recent calculations showing that all existing experimental data below 450 MeV are consistent with the dispersion relation approach.

3. NOTATION AND KINEMATICS

Let P_1 and P_2 be the four-momenta of the initial and final nucleons, Q the four-momentum of the created pion and K the four-momentum of the real or virtual photon. Then

$$P_1 + K = P_2 + Q \quad (3.1)$$

and

$$P_1^2 = P_2^2 = -m^2 \quad Q^2 = -\mu^2 \quad (3.2)$$

where m is the nucleon mass and μ the pion mass.

The usual Mandelstam kinematical variables are given by

$$\begin{aligned} s &= -(K + P_1)^2 \\ t &= -(K - Q)^2 \\ u &= -(K - P_2)^2 \end{aligned} \quad (3.3)$$

and they satisfy

$$s + t + u = 2m^2 + \mu^2 - K^2 \quad (3.4)$$

In the pion-nucleon centre-of-mass system, let W be the total energy and define

$$\begin{aligned} K &= (\underline{k}, k_0) \\ P_1 &= (-\underline{k}, E_1) \\ Q &= (\underline{q}, q_0) \\ P_2 &= (-\underline{q}, E_2) \end{aligned} \quad (3.5)$$

In the pion-nucleon centre-of-mass system

$$\begin{aligned} S &= W^2 \\ t &= 2 \underline{k} \cdot \underline{q} - 2 k_0 q_0 + u^2 - K^2 \\ u &= -2 \underline{k} \cdot \underline{q} - 2 k_0 E_a + m^2 - K^2 \end{aligned} \quad (3.6)$$

and the scattering angle ϑ is given by

$$\cos \vartheta = (\underline{k} \cdot \underline{q}) / |\underline{k}| |\underline{q}| = x \quad (3.7)$$

In this notation the s-channel is the channel corresponding to the reaction $\gamma + N_1 \rightarrow N_2 + \pi$, the t-channel is the channel corresponding to the reaction $\gamma + \pi \rightarrow \bar{N}_1 + N_2$ and the u-channel (the crossed channel) is the channel corresponding to the reaction $\gamma + \bar{N}_2 \rightarrow \bar{N}_1 + \pi$.

In the case of electropion production one must also introduce the four-momenta K_1 and K_2 respectively for the incident and scattered electron. They have components

$$\begin{aligned} K_1 &= (\underline{k}_1, k_{10}) \\ K_2 &= (\underline{k}_2, k_{20}) \end{aligned} \quad (3.8)$$

The momentum transfer is $K = K_1 - K_2$. The squared "mass" of the virtual photon $-K^2$ is therefore negative.

In the following we will denote the electron mass by m_e and use the notation

$$\begin{aligned} k &= |\underline{k}| & \hat{k} &= \underline{k}/k \\ q &= |\underline{q}| & \hat{q} &= \underline{q}/q \end{aligned}$$

Some useful kinematical relations, which express the energies in terms of W and K^2 are:

$$k_0 = \frac{W^2 - K^2 - m^2}{2W}$$

$$E_1 = \frac{W^2 + m^2 + K^2}{2W}$$

$$E_2 = \frac{W^2 - \cancel{\mu^2} + m^2}{2W}$$

$$q_0 = \frac{W^2 + \cancel{\mu^2} - m^2}{2W}$$

$$E_{1 \pm m} = \frac{(W \pm m)^2 + K^2}{2W}$$

$$E_{2 \pm m} = \frac{(W \pm m)^2 - \cancel{\mu^2}}{2W}$$

4. THE INVARIANT AMPLITUDES

In this Section the transition matrix element for electro- and photoproduction of pions will be defined and decomposed into a set of Lorentz invariant matrices and scalar amplitudes.

The S-matrix is given in terms of the T-matrix by

$$S_{fi} = \delta_{fi} - i(2\pi)^{-2} \delta^4(P_2 + Q - P - K) \sqrt{\frac{m^2}{4k_{20}E_1E_2}} T_{fi} \quad (4.1)$$

in the case of photopion production, and by

$$S_{fi} = \delta_{fi} - i(2\pi)^{-7/2} \delta^4(P_2 + Q + K_2 - P - K_1) \sqrt{\frac{m^2 m_e^2}{2k_{20}E_1E_2k_{10}k_{20}}} T_{fi} \quad (4.2)$$

in the case of electroproduction. In Appendix A the differential cross-section is given in terms of $|T|^2$.

The transition matrix element for photo- and electroproduction is given by

$$T = \epsilon_\mu J_\mu \quad (4.3)$$

where J_μ is the nucleon electromagnetic current^{*)} matrix element

$$J_\mu = {}_{out} \langle P_2 Q | J_\mu | P_1 \rangle_{in} \quad (4.4)$$

and ϵ_μ is either the photon polarization vector (photoproduction) or, in the one photon exchange approximation in electroproduction it is

*) The symbol J_μ is used for both the current and its matrix element.

the lepton electromagnetic current and photon propagator⁴⁵⁾

$$\epsilon_{\mu} = \bar{u}(k_2) \gamma_{\mu} u(k_1) e/k^2 \quad (4.5)$$

The most general Lorentz covariant pseudo four-vector for the nucleon electromagnetic current can be written in terms of the eight pseudo-vectors³³⁾

$$\begin{aligned} N_{\mu}^1 &= i \gamma_5 \gamma_{\mu} \gamma \cdot K \\ N_{\mu}^2 &= 2i \gamma_5 P_{\mu} \\ N_{\mu}^3 &= 2i \gamma_5 Q_{\mu} \\ N_{\mu}^4 &= 2i \gamma_5 K_{\mu} \\ N_{\mu}^5 &= \gamma_5 \gamma_{\mu} \\ N_{\mu}^6 &= \gamma_5 \gamma \cdot K P_{\mu} \\ N_{\mu}^7 &= \gamma_5 \gamma \cdot K K_{\mu} \\ N_{\mu}^8 &= \gamma_5 \gamma \cdot K Q_{\mu} \end{aligned} \quad (4.6)$$

where $P = \frac{1}{2}(P_1 + P_2)$. For the Dirac matrices we use the set as used for instance by Jackson⁴⁶⁾. Thus we have

$$J_{\mu} = \sum_{i=1}^8 B_i(s, t, u, k^2) N_{\mu}^i \quad (4.7)$$

Other pseudovectors can be reduced to linear combinations of the eight in Eq. (4.6) using the Dirac equation and the algebra of the Dirac matrices. None of these reductions introduces any kinematical singularities⁴⁷⁾, so that it is possible to apply the Mandelstam representation to the amplitudes B_i .

The conservation of the current J_{μ} requires that

$$K_{\mu} J_{\mu} = 0 \quad (4.8)$$

and this in turn imposes two conditions on the amplitudes B_i , namely

$$\frac{1}{2}K^2 B_1 + P \cdot K B_2 + Q \cdot K B_3 + K^2 B_4 = 0 \quad (4.9a)$$

and

$$B_5 + P \cdot K B_6 + K^2 B_7 + Q \cdot K B_8 = 0 \quad (4.9b)$$

Using these conditions B_3 and B_5 may be eliminated. This is done in such a way as to introduce the smallest number of kinematical singularities while keeping a parallel between electroproduction and photoproduction. The current, so reduced, has the form

$$\begin{aligned} J_\mu = & \left(N'_\mu - \frac{K^2}{2K \cdot Q} N_\mu^3 \right) B_1 + \left(N_\mu^2 - \frac{P \cdot K}{K \cdot Q} N_\mu^3 \right) B_2 \\ & + \left(N_\mu^4 - \frac{K^2}{K \cdot Q} N_\mu^3 \right) B_4 + \left(N_\mu^6 - P \cdot K N_\mu^5 \right) B_6 \\ & + \left(N_\mu^7 - K^2 N_\mu^5 \right) B_7 + \left(N_\mu^8 - Q \cdot K N_\mu^5 \right) B_8 \end{aligned} \quad (4.10)$$

The current written in this form is manifestly conserved, since on forming the four-vector product $K \cdot J$ it is easily seen from Eqs. (4.6) that the coefficient of each B_i is identically zero.

In both photoproduction and electroproduction the condition $K \cdot \epsilon = 0$ holds. This is obvious in the former case since ϵ_μ is the photon polarization vector, and in the latter case it follows immediately from Eq. (4.5) on noting that $K = K_2 - K_1$ and using the Dirac equation for the lepton spinors. Using this condition and defining

$$F_{\mu\nu} = \epsilon_\mu K_\nu - \epsilon_\nu K_\mu \quad (4.11)$$

one finds that

$$\begin{aligned}
 J_{\mu} \epsilon_{\mu} = & \left[\frac{i \gamma_5}{2} \gamma_{\mu} \gamma_{\nu} F_{\mu\nu} + i \gamma_5 \frac{K_{\mu} Q_{\nu}}{K \cdot Q} F_{\mu\nu} \right] B_1 \\
 & + 2i \gamma_5 \frac{P_{\mu} Q_{\nu}}{K \cdot Q} F_{\mu\nu} B_2 + 2i \gamma_5 \frac{K_{\mu} Q_{\nu}}{K \cdot Q} F_{\mu\nu} B_4 \\
 & - \gamma_5 \gamma_{\mu} P_{\nu} F_{\mu\nu} B_6 + \gamma_5 K_{\mu} \gamma_{\nu} F_{\mu\nu} B_7 \\
 & - \gamma_5 \gamma_{\mu} Q_{\nu} F_{\mu\nu} B_8
 \end{aligned} \tag{4.12}$$

A set of explicitly "gauge invariant" matrices can now be introduced by the definitions

$$\begin{aligned}
 M_1 &= \frac{1}{2} i \gamma_5 \gamma_{\mu} \gamma_{\nu} F_{\mu\nu} \\
 M_2 &= 2i \gamma_5 P_{\mu} (Q - \frac{1}{2} K)_{\nu} F_{\mu\nu} \\
 M_3 &= \gamma_5 \gamma_{\mu} Q_{\nu} F_{\mu\nu} \\
 M_4 &= 2 \gamma_5 \gamma_{\mu} P_{\nu} F_{\mu\nu} - 2m M_1 \\
 M_5 &= i \gamma_5 K_{\mu} Q_{\nu} F_{\mu\nu} \\
 M_6 &= \gamma_5 K_{\mu} \gamma_{\nu} F_{\mu\nu}
 \end{aligned} \tag{4.13}$$

This is the set of matrices used by Denner^{*) 34)}. M_5 and M_6 as well as the K_{ν} term in M_2 are identically zero for photo-production, in which case the first four matrices are those of CGIN⁴⁴⁾.

*) In Fubini et al.⁴⁸⁾ a quantity $M_2' = 2i \gamma_5 P_{\mu} Q_{\nu}$ is used instead of M_2 . The connection is $M_2' = \frac{2Q \cdot K}{t - u^2} M_2 - \frac{2P \cdot K}{t - u^2} M_5$

Inserting Eq. (4.13) into Eq. (4.12) one obtains

$$J_\mu \epsilon_\mu = \left[M_1 + \frac{M_5}{K \cdot Q} \right] B_1 + \left[\frac{2Q \cdot K}{t - u^2} M_2 - \frac{2P \cdot K M}{t - u^2} \right] \frac{B_2}{K \cdot Q} \\ + \frac{2M_5}{Q \cdot K} B_4 - \left[\frac{1}{2} M_4 + m M_1 \right] B_6 + M_6 B_7 - M_3 B_8 \quad (4.14)$$

which can be re-written as

$$J_\mu \epsilon_\mu = \sum_{i=1}^6 A_i(s, t, u, K^2) M_i \quad (4.15)$$

where *)

$$A_1 = B_1 - m B_6 \\ A_2 = \frac{2 B_2}{t - u^2} \\ A_3 = -B_8 \\ A_4 = -\frac{B_6}{2} \\ A_5 = \frac{1}{K \cdot Q} \left[B_1 + 2 B_4 - \frac{2 P \cdot K}{t - u^2} B_2 \right] \\ A_6 = B_7 \quad (4.16)$$

Clearly A_2 and A_5 have kinematical singularities introduced by the current conservation condition.

The decomposition of the invariant amplitudes in isotopic spin space is well known, and has the form

$$A_i = A_i^{(+)} \delta_{\alpha 3} + A_i^{(-)} \frac{1}{2} [\gamma_\alpha, \gamma_3]_- + A_i^{(0)} \gamma_\alpha \quad (i=1, \dots, 6) \quad (4.17)$$

*) Using M_2' one would obtain $A_2' = \frac{B_2}{K \cdot Q}$, $A_5' = \frac{B_1 + 2 B_4}{K \cdot Q}$

where τ_α are the Pauli spin matrices and α is the isotopic spin index of the pion. The form Eq. (4.18) occurs because in isotopic spin space the electromagnetic current behaves like an isoscalar plus the third component of an isovector⁴⁹).

The isovector transition amplitudes $A_i^{(+,-)}$ may be expressed in terms of the amplitudes $A_i^{(1,3)}$ with isospin $\frac{1}{2}, \frac{3}{2}$ in the final state:

$$\begin{aligned} A_i^{(1)} &= A_i^{(+)} + 2 A_i^{(-)} \\ A_i^{(3)} &= A_i^{(+)} - A_i^{(-)} \end{aligned} \quad (4.18)$$

The isoscalar transitions $A_i^{(0)}$ always lead to final states with isospin $\frac{1}{2}$.

The amplitudes for particular photoproduction processes are

$$\begin{aligned} \langle n \pi^+ | T | \gamma p \rangle &= \sqrt{2} [A^{(0)} + A^{(-)}] \\ \langle p \pi^0 | T | \gamma p \rangle &= A^{(0)} + A^{(+)} \\ \langle p \pi^- | T | \gamma n \rangle &= \sqrt{2} [A^{(0)} - A^{(-)}] \end{aligned} \quad (4.19)$$

5. FIXED-t DISPERSION RELATIONS

The amplitudes ^{*)} $A_i(s, t, u)$ are related to the matrix elements of the physical s-, t- and u-channel processes in the following way ^{**)}:

s-channel

$$\langle N(P_2), \pi^\alpha(Q) | T | N(P_1), \gamma(K) \rangle = \bar{u}(P_2) \left[\sum_i A_i(s, t, u) M_i \right] u(P_1) \quad (5.1)$$

t-channel

$$\langle N(P_2), \bar{N}(-P_1) | T | \pi^\alpha(Q), \gamma(K) \rangle = \bar{u}(P_2) \left[\sum_i A_i(s, t, u) M_i \right] v(-P_1) \quad (5.2)$$

u-channel

$$\langle \bar{N}(-P_1), \pi^\alpha(Q) | T | \bar{N}(-P_2), \gamma(K) \rangle = -\bar{v}(-P_1) \left[\sum_i A_i(s, t, u) M_i \right] v(-P_2) \quad (5.3)$$

To obtain the matrix elements for physical processes the momenta of the initial and final states should be taken at physical values. For the s-, t-, and u-channels respectively the physical sets are (K, P_1, Q, P_2) , $(K, -Q, -P_1, P_2)$, and $(K, -P_2, Q, -P_1)$.

Apart from the kinematical singularities which are contained in A_2 and A_5 , the analytic properties of the A_i are derived from a knowledge of the possible intermediate states in each channel, poles and cuts arising from one particle and two or more particle intermediate states respectively.

In the s-channel there is a pole at $s = m^2$ coming from the single-nucleon intermediate state, Fig. 1a (the direct nucleon Born

*) The K^2 dependence of the A_i is suppressed.

**) The symbol $\gamma(K)$ stands for the photon or the electron vertex.

term) and a cut for $s \geq (m + \mu)^2$ coming from the (nucleon + one pion) intermediate state, Fig. 1b (the rescattering term).

This analytic structure is reflected in the u-channel. There is a pole at $u = m^2$ coming from the single nucleon exchange diagram, Fig. 1c (the crossed nucleon Born term) and a cut for $u \geq (m + \mu)^2$ coming from the exchanged (nucleon + one pion) state, Fig. 1d (the crossed rescattering term).

In the t-channel there is a pole at $t = \mu^2$ coming from the single pion exchange diagram, Fig. 1e (the pion Born term) and a cut for $t \geq 4\mu^2$ in the isoscalar part of the amplitude coming from the two-pion exchange diagram, Fig. 1f (ρ exchange term) and for $t \geq 9\mu^2$ in the isovector part of the amplitude coming from the three-pion exchange diagram Fig. 1g (ω, ϕ exchange term).

There are other branch points in each channel at each new threshold for continuum intermediate states, the lowest of these being at $(m + 2\mu)^2$ in the s-channel and in the u-channel, and at $t = 16\mu^2$ (isoscalar part) and $25\mu^2$ (isovector part) in the t-channel. These extra branch points in no way alter the analytic structure of the A_i amplitudes.

Under the exchange of s and u (crossing) the functions $A_i(s, t, u)$ are even or odd. This crossing symmetry can be derived from the assumed C-invariance of the S-matrix for photo- and electroproduction. Using C-invariance*) the matrix elements of the s- and u-channel are related via the equation

$$\langle N(p_2), \pi^\alpha(q) | T | N(p_1), \gamma(k) \rangle = (-1)^\alpha \langle \bar{N}(p_2), \pi^\alpha(q) | T | \bar{N}(p_1), \gamma(k) \rangle \quad (5.4)$$

In terms of the amplitudes A_i Eq. (5.4) becomes [using Eqs. (5.1) and (5.3)]

*) The charge conjugation properties of the states involved in (photo- and electroproduction are

$$C | \pi^\alpha \rangle = (-1)^{\alpha+1} | \pi^\alpha \rangle, \quad C | \gamma \rangle = - | \gamma \rangle, \quad C | N \rangle = | \bar{N} \rangle$$

$$\begin{aligned} \chi^T(a) \bar{u}(p_a) &\leq A_i(s, t, u) M_i(p_i, K, p_a, Q) u(p_i) \chi(i) \\ &= -(-1)^\alpha \chi^T(i) \bar{v}(p_i) \leq A_i(u, t, s) M_i(-p_a, K, -p_i, Q) v(p_a) \chi(a) \end{aligned} \quad (5.5)$$

The u's and v's are related by the C matrix which has the usual properties*): $v = C u^T$, $\bar{v} = -u^T C^{-1}$. Here χ is the nucleon isospinor which acts on the isospin dependence (suppressed) of the A_i . Making the substitutions one finds that $A_i^{(+,0)}$ for $i = 1, 2, 4$ and $A_i^{(-)}$ for $i = 3, 5, 6$ are symmetric under the interchange $s \leftrightarrow u$ and that the other amplitudes are odd, i.e.

$$\begin{aligned} A_i^{(+,0)}(s, t, u) &= A_i^{(+,0)}(u, t, s) & i = 1, 2, 4 \\ A_i^{(-)}(s, t, u) &= A_i^{(-)}(u, t, s) & i = 3, 5, 6 \end{aligned} \quad (5.6)$$

and

$$\begin{aligned} A_i^{(-)}(s, t, u) &= -A_i^{(-)}(u, t, s) & i = 1, 2, 4 \\ A_i^{(+,0)}(s, t, u) &= -A_i^{(+,0)}(u, t, s) & i = 3, 5, 6 \end{aligned}$$

The crossing symmetry properties can be compactly written in matrix form by defining the six-vector $\tilde{A}$ with elements $A_1, \dots, A_6$. Then

$$\tilde{A}(s, t, u) = [\tilde{F}] \tilde{A}(u, t, s) \quad (5.7)$$

where $[\tilde{F}]$ is a diagonal matrix

$$[\tilde{F}] = \tilde{F} \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & -1 & & & \\ & & & 1 & & \\ & & & & -1 & \\ & & & & & -1 \end{bmatrix} \quad (5.8)$$

*) The superscript T indicates the transpose of a matrix and $C = \gamma_2 \gamma_4$.

and the parameter ξ is defined by

$$\xi = +1 \quad \text{for isospin index } +, 0$$

$$\xi = -1 \quad \text{for isospin index } -.$$

The fixed- t dispersion relations that follow from the analytic properties and crossing symmetry³³⁾ can be written in matrix notation both for compactness and convenience in later calculation. One has*)

$$\begin{aligned} \text{Re } \tilde{A}(s, t) = & \left[\frac{1}{s-m^2} + [\xi] \frac{1}{u-m^2} \right] \tilde{\Gamma}(t) + \frac{(1-\xi)}{2} \tilde{\Gamma}_t \\ & + \frac{P}{\pi} \int_{(m+u)^2}^{\infty} ds' \left[\frac{1}{s'-s} + [\xi] \frac{1}{s'-u} \right] \text{Im } \tilde{A}(s', t) \end{aligned} \quad (5.9)$$

The Born vector $\tilde{\Gamma}(t)$ has elements

$$\Gamma_1^{(\pm, 0)} = \frac{g}{2} F_1^{(v, s)}(K^2) \quad (5.10)$$

$$\Gamma_a^{(\pm, 0)} = -\frac{g}{t-u} F_1^{(v, s)}(K^2) \quad (5.11)$$

$$\Gamma_3^{(\pm, 0)} = \Gamma_4^{(\pm, 0)} = -\frac{g}{2} F_2^{(v, s)}(K^2) \quad (5.12)$$

$$\Gamma_5^{(\pm, 0)} = \frac{1}{2} \Gamma_2^{(\pm, 0)} \quad (5.13)$$

$$\Gamma_6^{(\pm, 0)} = 0 \quad (5.14)$$

*) The symbol $P \int$ denotes the principal value integral.

and only the 5th element of $\tilde{\Gamma}_t$ is non-zero with the value

$$\Gamma_t = \frac{2g}{K^2} [F_\pi(K^2) - F_1^{(V)}(K^2)]$$

$F_\pi(K^2)$ is the pion electromagnetic form factor, $F_1^{(V,S)}(K^2)$ is the nucleon electric form factor for isovector and isoscalar states, and $F_2^{(V,S)}(K^2)$ is the anomalous isovector and isoscalar magnetic form factor of the nucleon. They are defined, for example, by the 1-nucleon exchange diagram of Fig. 1a -- the s-channel 1-nucleon Born term -- which has the value

$$g \bar{u}(P_a) \gamma_5 \frac{[i \gamma \cdot (K+P) - m]}{m^2 - s} [F_1(K^2) \gamma_\mu + F_2(K^2) \sigma_{\mu\nu} K_\nu] \epsilon_\mu u(P_i) \quad (5.15)$$

$F_{1,2}$ are related to the proton and neutron form factors by

$$F_{1,2}^{(V)}(K^2) = F_{1,2}^P(K^2) - F_{1,2}^N(K^2) \quad (5.16)$$

$$F_{1,2}^{(S)}(K^2) = F_{1,2}^P(K^2) + F_{1,2}^N(K^2) \quad (5.17)$$

For $K^2 = 0$ (photoproduction) $F_{1,2}$ are the charge and anomalous magnetic moments of the nucleon

$$F_1^{(V,S)}(0) = e \quad (5.18)$$

$$F_2^{(V)}(0) = \frac{e}{2m} (\mu'_p - \mu'_n) \quad (5.19)$$

$$F_2^{(S)}(0) = \frac{e}{2m} (\mu'_p + \mu'_n) \quad (5.20)$$

The constants have the values

$$\frac{e^2}{4\pi} = \frac{1}{137}, \quad \frac{g^2}{4\pi} = 14.4, \quad \mu'_p = 1.79, \quad \mu'_n = -1.91$$

The Born terms of Eq. (5.9) are found by adding together the diagrams in Figs. 1a, c and e and then expanding in terms of the invariants M_i . This is discussed in Appendix B.

6. MULTIPOLE DECOMPOSITION OF THE AMPLITUDES

In order to use the angular momentum properties of photo- and electroproduction it is convenient to express the matrix element in the π -N centre-of-mass system in terms of Pauli matrices and two-component spinors χ which have the z-axis as the axis of quantization. To this end, one introduces a quantity $\mathcal{H}$ by

$$\bar{u}(p_a) \sum_{i=1}^6 A_i M_i u(p_i) = \chi^\dagger(a) \mathcal{H} \chi(1) \quad (6.1)$$

for electroproduction³⁴⁾ and a similar quantity

$$\bar{u}(p_a) \sum_{i=1}^4 A_i M_i u(p_i) = \frac{4\pi W}{m} \chi^\dagger(a) \mathcal{H} \chi(1) \quad (6.2)$$

for photoproduction⁴⁴⁾. The normalization for photoproduction is chosen to agree with the conventions in the literature. The following discussion of $\mathcal{H}$ will be given in parallel for photo- and electroproduction with the different normalization suppressed for the moment.

The matrix $\mathcal{H}$ is defined from Eq. (6.1) to be

$$\begin{aligned} \mathcal{H} = & i \vec{\sigma} \cdot \vec{\epsilon} \mathcal{H}_1 + \vec{\sigma} \cdot \hat{\vec{q}} \vec{\sigma} \cdot (\hat{\vec{k}} \times \vec{\epsilon}) \mathcal{H}_2 + i \vec{\sigma} \cdot \hat{\vec{k}} \hat{\vec{q}} \cdot \vec{\epsilon} \mathcal{H}_3 \\ & + i \vec{\sigma} \cdot \hat{\vec{q}} \hat{\vec{q}} \cdot \vec{\epsilon} \mathcal{H}_4 + i \vec{\sigma} \cdot \hat{\vec{k}} \hat{\vec{k}} \cdot \vec{\epsilon} \mathcal{H}_5 + i \vec{\sigma} \cdot \hat{\vec{q}} \hat{\vec{k}} \cdot \vec{\epsilon} \mathcal{H}_6 \\ & - i \vec{\sigma} \cdot \hat{\vec{q}} \epsilon_0 \mathcal{H}_7 - i \vec{\sigma} \cdot \hat{\vec{k}} \epsilon_0 \mathcal{H}_8 \end{aligned} \quad (6.3)$$

Current conservation [Eq. (4.8)] imposes two conditions on $\mathcal{H}$

$$\mathcal{H}_1 + \hat{\vec{k}} \cdot \hat{\vec{q}} \mathcal{H}_3 + \mathcal{H}_5 - k_0/k \mathcal{H}_8 = 0 \quad (6.4a)$$

$$\hat{\vec{k}} \cdot \hat{\vec{q}} \mathcal{H}_4 + \mathcal{H}_6 - k_0/k \mathcal{H}_7 = 0 \quad (6.4b)$$

One can eliminate $\tilde{H}_7$ and $\tilde{H}_8$ to get

$$\begin{aligned} \tilde{H} = & i \vec{\sigma} \cdot \vec{a} \tilde{H}_1 + \vec{\sigma} \cdot \hat{\vec{k}} \vec{\sigma} \cdot (\hat{\vec{k}} \times \vec{a}) \tilde{H}_2 + i \vec{\sigma} \cdot \hat{\vec{k}} \hat{\vec{k}} \cdot \vec{a} \tilde{H}_3 \\ & + i \vec{\sigma} \cdot \hat{\vec{k}} \hat{\vec{k}} \cdot \vec{a} \tilde{H}_4 + i \vec{\sigma} \cdot \hat{\vec{k}} \hat{\vec{k}} \cdot \vec{a} \tilde{H}_5 + i \vec{\sigma} \cdot \hat{\vec{k}} \hat{\vec{k}} \cdot \vec{a} \tilde{H}_6 \end{aligned} \quad (6.5)$$

where

$$a_\mu = \epsilon_\mu - \epsilon_0/k_0 K_\mu \quad (6.6)$$

This is the choice of Denner³⁴⁾ for electroproduction. Noticing that $a_0 = 0$, i.e. that a has no scalar component one sees that this choice of amplitudes $\tilde{H}_i$ (which is actually a choice of gauge since by the current conservation one can replace ϵ_μ by $\epsilon_\mu - \lambda k_\mu$ for any λ) corresponds to a virtual photon with transverse and longitudinal components.

On the other hand one can eliminate instead $\tilde{H}_5$ and $\tilde{H}_6$ from Eq. (6.3) to get

$$\begin{aligned} \tilde{H} = & i \vec{\sigma} \cdot \vec{b} \tilde{H}_1 + \vec{\sigma} \cdot \hat{\vec{k}} \vec{\sigma} \cdot (\hat{\vec{k}} \times \vec{b}) \tilde{H}_2 + i \vec{\sigma} \cdot \hat{\vec{k}} \hat{\vec{k}} \cdot \vec{b} \tilde{H}_3 \\ & + i \vec{\sigma} \cdot \hat{\vec{k}} \hat{\vec{k}} \cdot \vec{b} \tilde{H}_4 - i \vec{\sigma} \cdot \hat{\vec{k}} b_0 \tilde{H}_7 - i \vec{\sigma} \cdot \hat{\vec{k}} b_0 \tilde{H}_8 \end{aligned} \quad (6.7)$$

where

$$b_\mu = \epsilon_\mu - \hat{\vec{k}} \cdot \hat{\vec{k}} / k K_\mu \quad (6.8)$$

With this choice of gauge, $\vec{b} \cdot \hat{\vec{k}} = 0$, i.e. b_μ has no longitudinal component but $b_0 \neq 0$, so that the virtual photon will have transverse and scalar components. In our further discussion of electroproduction this latter form is chosen so that the amplitudes $\tilde{H}_1, \dots, \tilde{H}_4, \tilde{H}_7$ and $\tilde{H}_8$ are used. This choice is made because $\tilde{H}_7$ and $\tilde{H}_8$ have a simpler angular momentum decomposition than $\tilde{H}_5$ and $\tilde{H}_6$.

and because in numerical calculations the latter may introduce spurious unwanted singularities²⁵).

For photoproduction one still has $a \cdot K = b \cdot K = 0$ since $K^2 = 0$ and the choice of gauge does not affect the completely transverse character of the real photon. Here only the first four terms (with coefficients $\tilde{\gamma}_1, \dots, \tilde{\gamma}_4$) of Eqs. (6.3), (6.5), or (6.7) are present in $\tilde{\gamma}$.

If the left-hand side of Eq. (6.1) is decomposed into Pauli matrices and spinors one can obtain the connection between the amplitudes $\tilde{\gamma}_1, \dots, \tilde{\gamma}_4$, $\tilde{\gamma}_7$ and $\tilde{\gamma}_8$ and the amplitudes $A_1, \dots, A_6$. Writing the $\tilde{\gamma}_i$ in matrix form one gets

$$\tilde{\gamma}(s,t) = [C(s)] [B(s,t)] \tilde{A}(s,t) \quad (6.9a)$$

$$\tilde{A}(s,t) = [B^{-1}(s,t)] [C(s)] \tilde{\gamma}(s,t) \quad (6.9b)$$

where

$$\tilde{\gamma} = \begin{bmatrix} \tilde{\gamma}_1 \\ \tilde{\gamma}_2 \\ \tilde{\gamma}_3 \\ \tilde{\gamma}_4 \\ \tilde{\gamma}_7 \\ \tilde{\gamma}_8 \end{bmatrix} \quad (6.10)$$

and the matrices $[C]$, $[B]$, and $[B^{-1}]$ are

$$[C(s)] = \frac{2m}{(W-m)[(E_1+m)(E_2+m)]^{1/2}} \begin{bmatrix} 1 & & & & & \\ & \frac{(E_1+m)(E_2+m)(W-m)}{q k (W+m)} & & & & \\ & & \frac{(E_1+m)(W-m)}{q k (W+m)} & & & \\ & & & \frac{(E_2+m)}{q^2} & & \\ & & & & \frac{(E_2+m)(W-m)}{q} & \\ & & & & & \frac{(E_1+m)(W-m)}{k} \end{bmatrix} \quad (6.11a)$$

and the matrix $[B(s,t)]$ is given by Eq. (6.11b):

(6.11b)

1	0	$-\frac{K \cdot Q}{W-m}$	$\frac{W-m}{+ \frac{K \cdot Q}{W-m}}$	0	$\frac{K^2}{W-m}$
-1	0	$-\frac{K \cdot Q}{W+m}$	$\frac{W+m}{+ \frac{K \cdot Q}{W+m}}$	0	$\frac{K^2}{W+m}$
0	$\frac{1}{W+m} \left[W^2 - m^2 + \frac{K^2}{2} \right]$	1	-1	$-\frac{K^2}{W+m}$	0
0	$-\frac{1}{W-m} \left[W^2 - m^2 - m^2 + \frac{K^2}{2} \right]$	1	-1	$\frac{K^2}{W-m}$	0
$-(E_1 - m)$	$\frac{1}{2K_0} \left[K^2 (3K \cdot Q + 2K_0 W) - K_0^2 (2(W^2 - m^2) + K^2) \right]$	$K \cdot Q + g_0(W+m)$	$-K \cdot Q - g_0(W-m) + (E_1 - m)(W+m)$	$g_0 K^2 - K_0 K \cdot Q$	$-(E_1 - m)(W+m)$
$E_1 + m$	$-\frac{1}{2K_0} \left[K^2 (3K \cdot Q + 2K_0 W) - K_0^2 (2(W^2 - m^2) + K^2) \right]$	$K \cdot Q + g_0(W+m)$	$-K \cdot Q - g_0(W+m) + (E_1 + m)(W-m)$	$-g_0 K^2 + K_0 K \cdot Q$	$-(E_1 + m)(W-m)$

$B(s,t) =$

$\frac{1}{4W^2k_0}$	$(W-m)[(W+m)^2 - K^2 - 2m \frac{K^2}{ka} (E_{t+m})]$	$-(W+m) \frac{k_0}{E_{t+m}} [W^2 - m^2 + K^2]$	$2m(W+m) \times [K \cdot Q - \frac{K^2}{ka} \frac{Q \cdot k}{k}]$	$2m(W-m) \times [K \cdot Q - \frac{K^2}{ka} \frac{Q \cdot k}{k}]$	$2mK^2 \frac{k_0}{ka}$
$\frac{Ka}{2k_0W(t-u^2)}$	$(W-m)[1 - \frac{(E_{t+m}(W-m))}{ka}]$	$k_0 \frac{(W+m)}{E_{t+m}}$	$(W^2 - m^2) \times [\frac{K \cdot Q}{Ka} - \frac{Q \cdot k}{ka}]$	$(W+m) \times [-\frac{K \cdot Q}{Ka} (W-m) + (W-m) \frac{Q \cdot k}{ka}]$	$(W-m) \frac{k_0}{ka}$
$\frac{1}{4W^2k_0}$	$(W-m)[W+m - (E_{t+m}) \frac{Ka}{ka}]$	$k_0 \frac{(W+m)^2}{E_{t+m}}$	$(W+m) \times [K \cdot Q + 2k_0W - \frac{K^2}{ka} \frac{Q \cdot k}{k}]$	$(W-m) \times [K \cdot Q + 2k_0W - \frac{K^2}{ka} \frac{Q \cdot k}{k}]$	$K^2 \frac{k_0}{ka}$
$[B_{cst}^{-1}] = \frac{1}{4W^2k_0}$	$(W-m)[W+m - (E_{t+m}) \frac{Ka}{ka}]$	$k_0 \frac{(W+m)^2}{E_{t+m}}$	$(W+m) \times [K \cdot Q - \frac{K^2}{ka} \frac{Q \cdot k}{k}]$	$(W-m) \times [K \cdot Q - \frac{K^2}{ka} \frac{Q \cdot k}{k}]$	$K^2 \frac{k_0}{ka}$
$\frac{\beta}{2W^2k_0(t-u^2)}$	$(W-m)[1 - \frac{(E_{t+m}(W-m))}{ka}]$	$k_0 \frac{(W+m)}{E_{t+m}}$	$(W^2 - m^2) \times [\frac{\delta}{\beta} - \frac{Q \cdot k}{ka}]$	$(W^2 - m^2) \times [-\frac{\delta}{\beta} + \frac{Q \cdot k}{ka}]$	$(W-m) \frac{k_0}{ka}$
$\frac{W^2 - m^2}{4W^2k_0}$	$-1 + \frac{(E_{t+m}(W-m))}{ka}$	$\frac{k_0}{E_{t+m}}$	$-\frac{K \cdot Q}{W+m} + (W+m) \frac{Q \cdot k}{ka}$	$-\frac{K \cdot Q}{W+m} + (W-m) \frac{Q \cdot k}{ka}$	$-\frac{k_0}{ka}$

where $\delta = \frac{3}{2} K \cdot Q + k_0 W$ and $\beta = W^2 - m^2 + \frac{K^2}{2}$

If this is multiplied into [B] on page 27 the result is $[B^{-1}][B] = 1$

For photoproduction the matrix $[C]$ must be multiplied by $4\pi W/m$ and all quantities taken at $K^2 = 0$. Of course only the first four rows and columns contribute to photoproduction.

By a lengthy but straightforward calculation⁵⁰⁾ one can connect $\tilde{H}$ with the eigenamplitudes of parity and angular momentum.

For electroproduction there are six types of transitions possible to a π -N final state with angular momentum ℓ and definite parity. They are classified according to the character of the photon, transverse or scalar, and the total angular momentum $J = \ell \pm \frac{1}{2}$ of the final state. The transverse photon states can also be either electric with parity $(-1)^L$ or magnetic with parity $(-1)^{L+1}$ where L is the total orbital angular momentum of the photon. This is shown in Table 1 with the appropriate notation for the six transitions (in photoproduction only the transverse photon states contribute).

TABLE 1

Multipole states for electro- and photoproduction^{*)}

J	L	Parity = $-(-1)^L$	Multipole transition	Notation	Lowest value of ℓ permitted
$\ell + \frac{1}{2}$	$L = J + \frac{1}{2} = \ell + 1$	$(-1)^L$	electric $a^{\ell+1}$	$E_{\ell+}$	0
$\ell - \frac{1}{2}$	$L = J - \frac{1}{2} = \ell - 1$	$(-1)^L$	electric $a^{\ell-1}$	$E_{\ell-}$	2
$\ell + \frac{1}{2}$	$L = J - \frac{1}{2} = \ell$	$(-1)^L$	magnetic a^{ℓ}	$M_{\ell+}$	1
$\ell - \frac{1}{2}$	$L = J + \frac{1}{2} = \ell$	$(-1)^L$	magnetic a^{ℓ}	$M_{\ell-}$	1
$\ell + \frac{1}{2}$	$L = J + \frac{1}{2} = \ell + 1$	$(-1)^L$	scalar $a^{\ell+1}$	$S_{\ell+}$	0
$\ell - \frac{1}{2}$	$L = J - \frac{1}{2} = \ell - 1$	$(-1)^L$	scalar $a^{\ell-1}$	$S_{\ell-}$	1

*) The scalar multipoles are related to longitudinal multipoles³⁴⁾ by $L_{\ell\pm} = (k_0/\bar{k}) S_{\ell\pm}$.

The relation between the multipoles $E_{\ell\pm}$, $M_{\ell\pm}$, $S_{\ell\pm}$ and the amplitudes $\tilde{\gamma}_i$ is

$$\tilde{\gamma}_i(s, t) = \sum_{\ell=0}^{\infty} \begin{bmatrix} G_{\ell}(x) & 0 \\ 0 & H_{\ell}(x) \end{bmatrix} \tilde{M}_{\ell}(s) \quad (6.12)$$

where $\tilde{M}_{\ell}$ is a column vector with elements

$$\tilde{M}_{\ell} = \begin{bmatrix} E_{\ell+} \\ E_{\ell-} \\ M_{\ell+} \\ M_{\ell-} \\ S_{\ell+} \\ S_{\ell-} \end{bmatrix} \quad (6.13)$$

and G_{ℓ} and H_{ℓ} are respectively 4×4 and 2×2 matrices with elements^{*)}

$$G_{\ell} = \begin{bmatrix} P'_{\ell+1} & P'_{\ell-1} & \ell P'_{\ell+1} & (\ell+1) P'_{\ell-1} \\ 0 & 0 & (\ell+1) P'_{\ell} & \ell P'_{\ell} \\ P''_{\ell+1} & P''_{\ell-1} & -P''_{\ell+1} & P''_{\ell-1} \\ -P''_{\ell} & -P''_{\ell} & P''_{\ell} & -P''_{\ell} \end{bmatrix} \quad (6.14a)$$

$$H_{\ell} = \begin{bmatrix} -P'_{\ell}(\ell+1) & \ell P'_{\ell} \\ (\ell+1) P'_{\ell+1} & -P'_{\ell-1} \ell \end{bmatrix} \quad (6.14b)$$

*) A prime superscript denotes differentiation with respect to x . The P_{ℓ} are Legendre polynomials of the first kind.

Only G_ℓ occurs in photoproduction.

Eq. (6.12) may be inverted using the relations

$$P'_{\alpha\ell}(x) = \sum_{k=1,3,\dots}^{\alpha\ell-1} (\alpha k+1) P_k(x) \quad (6.15a)$$

$$P'_{\alpha\ell+1}(x) = \sum_{k=0,2,\dots}^{\alpha\ell} (\alpha k+1) P_k(x) \quad (6.15b)$$

where $\ell = 0, 1, 2, \dots$

The result is

$$\tilde{M}_\ell(s) = \int_{-1}^1 dx \begin{bmatrix} D_\ell(x) & 0 \\ 0 & E_\ell(x) \end{bmatrix} \tilde{\gamma}(s,t) \quad (6.16)$$

where

$$D_\ell = \begin{bmatrix} \frac{1}{\alpha(\ell+1)} \{ P_\ell, -P_{\ell+1}, \frac{\ell}{\alpha\ell+1} (P_{\ell-1} - P_{\ell+1}), \frac{\ell+1}{\alpha\ell+3} (P_\ell - P_{\ell+2}) \} \\ \frac{1}{\alpha\ell} \{ P_\ell, -P_{\ell-1}, \frac{\ell+1}{\alpha\ell+1} (P_{\ell+1} - P_{\ell-1}), \frac{\ell}{\alpha\ell-1} (P_\ell - P_{\ell-2}) \} \\ \frac{1}{\alpha(\ell+1)} \{ P_\ell, -P_{\ell+1}, \frac{1}{\alpha\ell+1} (P_{\ell+1} - P_{\ell-1}), 0 \} \\ \frac{1}{\alpha\ell} \{ -P_\ell, P_{\ell-1}, \frac{1}{\alpha\ell+1} (P_{\ell-1} - P_{\ell+1}), 0 \} \end{bmatrix} \quad (6.17a)$$

$$E_\ell = \frac{1}{\alpha} \begin{bmatrix} \frac{1}{\alpha+1} \{ P_{\ell+1}, P_\ell \} \\ \frac{1}{\alpha} \{ P_{\ell-1}, P_\ell \} \end{bmatrix} \quad (6.17b)$$

$D_\ell(x)$ is well known, having been given by CGLN⁴⁴).

In Appendix C the connection between the considerations of this Section and the helicity formalism of Jacob and Wick⁵¹⁾ is indicated. Photo- and electroproduction are usually analysed with multipole amplitudes and that approach is followed in this paper.

7. MULTIPOLE DISPERSION RELATIONS

In this Section the dispersion relations satisfied by the multipole amplitudes $\tilde{M}_\ell$ are derived from the considerations of Sections 5 and 6.

From the fixed- t dispersion relation Eq. (5.9) and Eqs. (6.9), (6.12) and (6.16) one finds for $\tilde{M}_\ell$

$$\begin{aligned} \text{Re } \tilde{M}_\ell(s) = & \int_{-1}^1 dx \begin{bmatrix} D_\ell(x) & 0 \\ 0 & E_\ell(x) \end{bmatrix} [C^{-1}(s)] [B(s,t)] \left\{ \frac{\tilde{f}(t)}{s-m_a^2} + [\tilde{f}] \frac{\tilde{f}(t)}{u-m_a^2} \right. \\ & \left. + \frac{(1-\tilde{f})}{2} \frac{\tilde{f}(t)}{t-u^2} \right\} + \frac{\rho}{\pi} \int_{(m+\mu)^2}^{\infty} ds' \int_{-1}^1 dx \begin{bmatrix} D_\ell(x) & 0 \\ 0 & E_\ell(x) \end{bmatrix} [C^{-1}(s)] [B(s,t)] \\ & \left\{ \frac{1}{s'-s} + [\tilde{f}] \frac{1}{s'-u} \right\} [B^{-1}(s',t)] [C(s')] \sum_{\ell'=0}^{\infty} \begin{bmatrix} G_\ell(x') & 0 \\ 0 & H_\ell(x') \end{bmatrix} \mathcal{I}_m \tilde{M}_{\ell'}(s') \end{aligned} \quad (7.1)$$

Here

$$x' = \frac{k' q}{k' q'} x + \frac{k_0' q_0' - k_0 q_0}{k' q'} \quad (7.2)$$

In this dispersion relation the expansion in terms of $\text{Im } \tilde{M}_{\ell'}(s')$ is assumed to converge even outside the physical region ($|x'| > 1$). Due to the kinematical singularities in A_2 and A_5 and the influence of the t -cut this is certainly not true everywhere. However, it is known from the pion-nucleon phase shift analysis⁴⁾ carried out using partial-wave dispersion relations that the s -channel partial-wave dispersion relations appear to be well satisfied up to pion lab. energies of ~ 1400 MeV. The analysis of Donnachie and Shaw^{8,24)} also indicate the validity of this approach up to 450 MeV in photoproduction. The t -cut comes mainly from ω and ρ exchange and this effect is known to be small²⁴⁾.

In practice it is more convenient to use as integration variable W' the c.m. energy in the multipole dispersion relations. For further discussion only the case of photoproduction will be considered. The development of the electroproduction dispersion relations is in no way different, only very much more lengthy. Equation (7.1) is rewritten as

$$\text{Re } \tilde{M}_\ell(W) = \tilde{\beta}_\ell(W) + \frac{Q}{\pi} \int_{m+\mu}^{\infty} dW' \frac{\text{Im } \tilde{M}_\ell(W')}{W' - W} + \tilde{C}_\ell(W) \quad (7.3)$$

where

$$\tilde{M}_\ell(W) = \frac{W}{K g_\ell} \begin{bmatrix} E_{\ell+} \\ (E_{\ell+} + m) E_{\ell-} \\ M_{\ell+} \\ (E_{\ell+} + m) M_{\ell-} \end{bmatrix} \quad (7.4)$$

with

$$J = [(W+m)^2 - \mu^2]^{1/2}$$

and

$$\tilde{\beta}_\ell(W) = \frac{W}{K g_\ell} \begin{bmatrix} E_{\ell+}^B \\ (E_{\ell+} + m) E_{\ell-}^B \\ M_{\ell+}^B \\ (E_{\ell+} + m) M_{\ell-}^B \end{bmatrix} \quad (7.5)$$

where the $(E_{\ell\pm}^B, M_{\ell\pm}^B)$ are the projected Born terms of Eq. (7.1).

The matrix $\tilde{C}_\ell$ is given by

$$\tilde{C}_\ell(w) = \frac{1}{\pi} \int_{m+\mu}^{\infty} dw' \sum_{\ell'} \tilde{K}_{\ell\ell'}(w, w') \text{Im} \tilde{m}_{\ell'}(w') \quad (7.6)$$

It contains the crossed channel contributions to $\text{Re} \tilde{m}_\ell$ as well as the contribution of s-channel multipoles, other than the direct term.

The kinematical factors $1/q^\ell$ give $\text{Im} \tilde{m}_\ell$ the proper threshold behaviour. They appear quite naturally as factors in the dispersion relations.

In practice only states with $\ell' = 0$ or 1 need be retained in Eq. (7.6), and of these the electric dipole E_{0+} transitions and the magnetic dipole $M_{1+}^{(3)}$ transition are by far the most important. The multipoles with higher ℓ' values can be neglected firstly because their maximum values are very much less than, for example, the maximum value of the $M_{1+}^{(3)}$ transition and secondly because their maxima occur at fairly high energies and in general the kernels $\tilde{K}_{\ell\ell'}(w, w')$ are rapidly decreasing functions of w' . This dominance in the crossed channel of multipoles leading to s- and p-wave states only is paralleled in pion-nucleon scattering, where the only important contributions to the crossed channel come from the resonant P_{33} and the low energy s-waves⁵²).

8. THE PROJECTED BORN TERMS FOR PHOTO- AND ELECTROPION PRODUCTION

From Eq. (7.1) it can be seen that the projected Born terms are obtained by a series of matrix multiplications, followed by integration. The integrals give rise to Legendre functions of the second kind:

$$Q_\ell(y) = \frac{1}{2} \int_{-1}^1 dx \frac{P_\ell(x)}{y-x} \quad (8.1)$$

These functions occur with arguments $\bar{q}_0$ and $\bar{E}_2$, defined by

$$t - \mu^2 = -2kq \left(\frac{2k_0 q_0 + k^2}{2kq} - x \right) = -2kq (\bar{q}_0 - x) \quad (8.2)$$

and

$$u - m^2 = -2kq \left(\frac{2k_0 E_2 + k^2}{2kq} + x \right) = -2kq (\bar{E}_2 + x) \quad (8.3)$$

Performing the matrix multiplications and integrating yields the following expressions:

$$\begin{aligned}
 E_{\ell+}^{(Born)} &= \frac{[W-m][(E_1+m)(E_2+m)]^{1/2}}{2m} \cdot \frac{1}{2(\ell+1)} \times \\
 &\left\{ \frac{2\delta_{\ell 0}}{W^2-m^2} \left[\Gamma_1 + (W-m)\Gamma_3 - \xi(W+m)\Gamma_3 \right] - \xi \left[\Gamma_1 - 2m\Gamma_3 \right] T_{\ell+} \right. \\
 &- \frac{(1-\xi)}{2} \left[4\Gamma_1 + K^2\Gamma_t \right] \left[\frac{\ell R_{\ell}^{\pi}}{(E_1+m)(W-m)} - \frac{q(\ell+1)R_{\ell+1}^{\pi}}{k(E_2+m)(W-m)} \right] \\
 &- 2\xi \left[\Gamma_1 - (W+m)\Gamma_3 \right] \frac{\ell R_{\ell}^N}{(E_1+m)(W-m)} + 2\xi \left[\Gamma_1 + (W-m)\Gamma_3 \right] \frac{q(\ell+1)R_{\ell+1}^N}{k(E_2+m)(W-m)} \left. \right\} \quad (8.4)
 \end{aligned}$$

$$\begin{aligned}
 E_{\ell-}^{(Born)} &= \frac{[W-m][(E_1+m)(E_2+m)]^{1/2}}{2m} \cdot \frac{1}{2\ell} \times \left\{ -\xi \left[\Gamma_1 - 2m\Gamma_3 \right] T_{\ell-} + \right. \\
 &\frac{(1-\xi)}{2} \left[4\Gamma_1 + K^2\Gamma_t \right] \left[\frac{(\ell+1)R_{\ell}^{\pi}}{(E_1+m)(W-m)} - \frac{q\ell R_{\ell-1}^{\pi}}{k(E_2+m)(W-m)} \right] \\
 &+ 2\xi \left[\Gamma_1 - (W+m)\Gamma_3 \right] \frac{(\ell+1)R_{\ell}^N}{(E_1+m)(W-m)} - 2\xi \left[\Gamma_1 + (W-m)\Gamma_3 \right] \frac{q\ell R_{\ell-1}^N}{k(E_2+m)(W-m)} \left. \right\} \quad (8.5)
 \end{aligned}$$

$$\begin{aligned}
 M_{\ell+}^{(Born)} &= \frac{[W-m][(E_1+m)(E_2+m)]^{1/2}}{2m} \cdot \frac{1}{2(\ell+1)} \left\{ -\xi \left[\Gamma_1 - 2m\Gamma_3 \right] T_{\ell+} + \right. \\
 &\frac{(1-\xi)}{2} \left[4\Gamma_1 + K^2\Gamma_t \right] \frac{R_{\ell}^{\pi}}{(E_1+m)(W-m)} + \\
 &2\xi \left[\Gamma_1 - (W+m)\Gamma_3 \right] \frac{R_{\ell}^N}{(E_1+m)(W-m)} \left. \right\} \quad (8.6)
 \end{aligned}$$

$$\begin{aligned}
 M_{\ell-}^{(\text{Born})} &= \frac{[W-m][(E_1+m)(E_2+m)]^{1/2}}{2m} \frac{1}{2\ell} \times \\
 &\left\{ \frac{2qk\delta_{\ell 1}}{(E_1+m)(E_2+m)(W-m)^2} \left[-\Gamma_1 + (W+m)\Gamma_3 - \xi(W-m)\Gamma_3 \right] + \right. \\
 &\xi[\Gamma_1 - 2m\Gamma_3] T_{\ell-} - \frac{(1-\xi)}{2} \left[4\Gamma_1 + \kappa^2\Gamma_t \right] \frac{R_e^\pi}{(E_1+m)(W-m)} \quad (8.7) \\
 &\left. - 2\xi \left[\Gamma_1 - (W+m)\Gamma_3 \right] \frac{R_e^N}{(E_1+m)(W-m)} \right\}
 \end{aligned}$$

$$\begin{aligned}
 S_{\ell+}^{(\text{Born})} &= \frac{[W-m][(E_1+m)(E_2+m)]^{1/2}}{2m} \frac{1}{2(\ell+1)} \times \\
 &\left\{ \frac{2k\delta_{\ell 0}}{(E_1+m)(W^2-m^2)} \left[-\Gamma_1 + (E_1+m)\Gamma_3 + \xi(W+m)\Gamma_3 + \frac{(1-\xi)}{4} k_0(W+m)\Gamma_t \right] \right. \\
 &+ \frac{(1-\xi)}{2} \frac{(2q_0-k_0)}{(W-m)} \left[\frac{\kappa^2\Gamma_t + 2\Gamma_1}{2} \right] \left[\frac{Q_\ell(\bar{q}_0)}{q(E_1+m)} - \frac{Q_{\ell+1}(\bar{q}_0)}{k(E_2+m)} \right] \quad (8.8) \\
 &+ \frac{(-1)^{\ell+1}}{W-m} \xi \left[-m(2q_0-k_0)\Gamma_3 + (2q_0-W)\Gamma_1 \right] \left[\frac{Q_\ell(\bar{E}_2)}{q(E_1+m)} + \frac{Q_{\ell+1}(\bar{E}_2)}{k(E_2+m)} \right] \\
 &+ \frac{(-1)^{\ell+1}}{W-m} \xi \left[m\Gamma_1 + (k_0W + \kappa^2 - \mu^2)\Gamma_3 \right] \left[\frac{Q_\ell(\bar{E}_2)}{q(E_1+m)} - \frac{Q_{\ell+1}(\bar{E}_2)}{k(E_2+m)} \right] \Big\}
 \end{aligned}$$

$$\begin{aligned}
 \int_{\ell-}^{(\text{Born})} &= \frac{[W-m][(E_1+m)(E_2+m)]^{1/2}}{2m} \frac{1}{2\ell} \times \\
 &\left\{ \frac{2q d_{\ell 1}}{(E_2+m)(W-m)^2} \left[\Gamma_1 + (E_1-m)\Gamma_3 + \left\{ (W-m)\Gamma_3 - \frac{(1-\xi)}{4} k_0(W-m)\Gamma_1 \right\} \right] \right. \\
 &+ \frac{(1-\xi)(2q_0-k_0)}{2(W-m)} \left[\frac{K^2}{2} \Gamma_1 + 2\Gamma_1 \right] \left[\frac{Q_{\ell}(\bar{q}_0)}{q(E_1+m)} - \frac{Q_{\ell-1}(\bar{q}_0)}{k(E_2+m)} \right] \\
 &+ \frac{(-1)^{\ell+1} \xi}{W-m} \left[-m(2q_0-k_0)\Gamma_3 + (2q_0-W)\Gamma_1 \right] \left[\frac{Q_{\ell}(\bar{E}_2)}{q(E_1+m)} + \frac{Q_{\ell-1}(\bar{E}_2)}{k(E_2+m)} \right] \\
 &\left. + \frac{(-1)^{\ell+1} \xi}{W-m} \left[m\Gamma_1 + (k_0 W + K^2 - \mu^2)\Gamma_3 \right] \left[\frac{Q_{\ell}(\bar{E}_2)}{q(E_1+m)} - \frac{Q_{\ell-1}(\bar{E}_2)}{k(E_2+m)} \right] \right\} \quad (8.9)
 \end{aligned}$$

In these equations, the following notation has been used:

$$R_{\ell}^N = \frac{(-1)^{\ell}}{(2\ell+1)} \left[Q_{\ell+1}(\bar{E}_2) - Q_{\ell-1}(\bar{E}_2) \right] \quad (8.10)$$

$$R_{\ell}^{\pi} = \frac{1}{(2\ell+1)} \left[Q_{\ell+1}(\bar{q}_0) - Q_{\ell-1}(\bar{q}_0) \right] \quad (8.11)$$

$$T_{\ell\pm} = (-1)^{\ell} \left[\frac{Q_{\ell}(\bar{E}_2)}{kq} - \frac{W+m}{(E_1+m)(E_2+m)(W-m)} Q_{\ell\pm 1}(\bar{E}_2) \right] \quad (8.12)$$

It should be kept in mind that Γ_1 and Γ_3 depend on $\pm, 0$ indices, which have been suppressed here for convenience.

The Born terms for photoproduction are obtained by putting $K^2 = 0$, which simplifies the kinematics and makes the functions $\Gamma_1(K^2)$ and $\Gamma_3(K^2)$ constant. Extracting a factor of $eg/(2m)$ from Eqs. (8.4) through (8.7) and multiplying by $m/4\pi W$ due to the different normalization [see Eq. (6.2)] the expressions of Schmidt and Guigay, as quoted in Ref. 24, are reproduced namely

$$E_{\ell+}^{(\text{Born})} = \frac{1}{\ell+1} \frac{Z}{\mu} \left\{ \delta_{\ell 0} \frac{(\xi-1)}{2} \left(\mu' - \frac{m}{W} \right) + \delta_{\ell 0} \frac{(\xi+1)}{2} \frac{m}{W} (\mu'+1) \right. \\ \left. - \xi (\mu'+1) \mathcal{T}_{\ell+} + (\xi-1) \left[\frac{2m}{W+m} \ell R_{\ell}^{\pi} - \frac{2m}{W-m} \frac{q(\ell+1)}{(E_z+m)} R_{\ell+1}^{\pi} \right] \right. \\ \left. - \xi \left[\mu' + \frac{2m}{W+m} \right] \ell R_{\ell}^N - \xi \left[\mu' - \frac{2m}{W-m} \right] \frac{q(\ell+1)}{(E_z+m)} R_{\ell+1}^N \right\} \quad (8.13)$$

$$E_{\ell-}^{(\text{Born})} = \frac{1}{\ell} \frac{Z}{\mu} \left\{ -\xi (\mu'+1) \mathcal{T}_{\ell-} + (1-\xi) \left[\frac{2m(\ell+1)}{W+m} R_{\ell}^{\pi} - \frac{2mq\ell}{(W-m)(E_z+m)} R_{\ell-1}^{\pi} \right] \right. \\ \left. + \xi \left[\mu' + \frac{2m}{W+m} \right] (\ell+1) R_{\ell}^N + \xi \left[\mu' - \frac{2m}{W-m} \right] \frac{q\ell}{(E_z+m)} R_{\ell-1}^N \right\} \quad (8.14)$$

$$M_{\ell+}^{(\text{Born})} = \frac{1}{\ell+1} \frac{Z}{\mu} \left\{ -\xi (1+\mu') \mathcal{T}_{\ell+} + \frac{(1-\xi)2m}{W+m} R_{\ell}^{\pi} + \xi \left[\mu' + \frac{2m}{W+m} \right] R_{\ell}^N \right\} \quad (8.15)$$

$$M_{\ell-}^{(\text{Born})} = \frac{1}{\ell} \frac{Z}{\mu} \left\{ \delta_{\ell 1} \frac{q}{(E_z+m)} \left[\frac{(\xi-1)}{2} \left(\mu' + \frac{m}{W} \right) - \frac{(\xi+1)}{2} \frac{m}{W} (\mu'+1) \right] \right. \\ \left. + \xi (\mu'+1) \mathcal{T}_{\ell-} + \frac{(\xi-1)2m}{W+m} R_{\ell}^{\pi} - \xi \left(\mu' + \frac{2m}{W+m} \right) R_{\ell}^N \right\} \quad (8.16)$$

Here we have defined

$$\mathcal{G}_{e_{\pm}} = m k T_{e_{\pm}} \quad (8.17)$$

$$\begin{aligned} \frac{Z}{m} &= \frac{[W-m][(E_1+m)(E_2+m)]^{1/2}}{16 \pi m k W} \frac{e q}{2} = \frac{\zeta}{16 \pi m W} \frac{e q}{2} \\ &= \left[\frac{E_2+m}{2W} \right]^{1/2} \frac{1}{2} \frac{e}{\sqrt{4\pi}} \frac{q}{\sqrt{4\pi}} \frac{1}{2m} \end{aligned} \quad (8.18)$$

ζ being defined by Eq. (7.4), and $\bar{E}_2, \bar{q}_0$ reducing to E_2/q and q_0/q respectively.

Again the $\pm, 0$ indices have been omitted. The appropriate anomalous magnetic moment combinations are

$$\mu^{\pm} = \mu'_P - \mu'_N, \quad \mu^0 = \mu'_P + \mu'_N \quad (8.19)$$

9. THE KERNELS FOR PHOTOPRODUCTION

In this section analytic expressions are given for the kernels defined by Eqs. (7.3) and (7.6), the kernels being given for any ℓ and for $\ell' = 0$ and 1. The kernels have not been simplified further, since in practical application the numerical evaluation will be done by computer. The analytic expressions obtained were checked for a series of W and W' values by direct numerical integration of Eq. (7.1).

In $\tilde{K}_{\ell\ell'}$, the elements of the first row are non-zero for $\ell \geq 0$, the elements of the second row for $\ell \geq 2$, and the elements of the third and fourth rows for $\ell \geq 1$, since the first allowed multipole transition amplitudes in each case are E_{0+} , E_{2-} , M_{1+} and M_{1-} respectively. Also since $\ell' \leq 1$, the second column, corresponding to contributions from $E_{\ell'-}$, is always zero and for $\ell' = 0$ only the first column will be non-zero.

Definitions used in the expressions for the kernels are

$$\alpha = \frac{k' W' + k E_2}{kq}$$

$$a = \frac{kq}{k'}$$

$$b = \frac{k' q'_0 - k q_0}{k'}$$

The notation for the matrix elements is that $K_{\ell\ell'}^{ij}(W, W')$ is the (i, j) -th matrix element of $\tilde{K}_{\ell\ell'}$. For convenience of writing, the expressions are given for $\xi K_{\ell\ell'}$, multiplied by some kinematical factor, so that on the right hand side terms in ξ correspond to the direct channel (i.e. s-channel) contributions and terms without ξ correspond to the crossed channel (i.e. u-channel) contributions. Since $\xi^2 = 1$ this causes no difficulty.

$$\xi k g^{l+1} K_{l_0}^{11}(w, w') = \frac{k g}{w-m} \delta_{l_0} - \frac{g l}{(l+1)(2l+1)} (-1)^{l-1} Q_{l-1}(\alpha)$$

$$+ \frac{(-1)^l}{2(l+1)} Q_l(\alpha) \left[w+w' - a \frac{(k'w' + kw)}{w-m} - \frac{2g^2}{E_a+m} \frac{l+1}{2l+3} \right]$$

$$+ \frac{g^2}{E_a+m} \frac{(-1)^{l+2}}{2l+3} Q_{l+2}(\alpha) + \frac{g}{(E_a+m)2(l+1)} Q_{l+1}(\alpha) [w'-w$$

$$+ a \frac{(k'w' + kw)}{w+m} + a(E_a+m) \frac{l}{2l+1}]$$

The above kernel connects Im ℓ_{o+} to Re ℓ_{l+}

$$\xi k g^{l+1} K_{l_0}^{21}(w, w') = \frac{g^2}{E_a+m} \frac{(-1)^l}{2l-1} Q_{l-2}(\alpha) - \frac{g(l+1)}{2(2l+1)} (-1)^{l+1} Q_{l+1}(\alpha)$$

$$+ \frac{(-1)^l}{2l} Q_l(\alpha) \left[w+w' - a \frac{(k'w' + kw)}{w-m} - \frac{g^2}{E_a+m} \frac{2l}{2l-1} \right]$$

$$+ \frac{g}{E_a+m} \frac{(-1)^{l-1}}{2l} Q_{l-1}(\alpha) \left[w'-w + a \frac{(k'w' + kw)}{w+m} + a(E_a+m) \frac{l+1}{2l+1} \right]$$

The above kernel connects Im ℓ_{o+} to Re ℓ_{l-}

$$\xi k g^{l+1} K_{l_0}^{31}(w, w') = \frac{(-1)^l}{2(l+1)} Q_l(\alpha) \left[w+w' - a \frac{(k'w' + kw)}{w-m} \right]$$

$$+ \frac{g}{E_a+m} \frac{(-1)^{l+1}}{2(l+1)} Q_{l+1}(\alpha) \left[w'-w + a \frac{(k'w' + kw)}{w+m} - a \frac{(E_a+m)}{2l+1} \right]$$

$$+ \frac{g}{(l+1)(2l+1)} (-1)^{l-1} Q_{l-1}(\alpha)$$

The above kernel connects Im ℓ_{o+} to Re η_{l+}

$$\begin{aligned}
 \frac{\xi k g^{2l+1}}{E_{2+m}} K_{2l}^{(4)}(w, w') &= \frac{(-1)^l}{2l} \phi_l(\alpha) \left[-w - w' + 2 \frac{(k'w' + kw)}{w - m} \right] \\
 + \frac{g}{E_{2+m}} \frac{(-1)^{l-1}}{2l} \phi_{l-1}(\alpha) &\left[w - w' - 2 \frac{(k'w' + kw)}{w + m} - 2 \frac{(E_{2+m})}{2l+1} \right] \\
 + \frac{g}{2(2l+1)} (-1)^{2l+1} \phi_{2l+1}(\alpha) &+ \delta_{2l} \left[\frac{k g^2}{(w+m)(E_{2+m})} - \frac{\xi k g^2}{(E_{2+m})(w'+w)} \right]
 \end{aligned}$$

The above kernel connects Im $\mathcal{E}_{0+}$ to Re $\mathcal{M}_{l-}$

$$2(l+1) \xi k g^{l+1} K_{2l}''(w, w') = (-1)^l \Phi_l(\alpha) [3b(w'+w) - 3a\alpha(w'+w)$$

$$- \frac{3g^2}{E_2+m} (w+m) \frac{l+1}{2l+3} - \frac{3k g_0}{(w'-m)} (w+m) - \frac{3\alpha k g}{w'-m} (w+m)$$

$$- \left(\frac{k'w'+kw}{w-m} + \frac{g^2}{E_2+m} \frac{l+1}{2l+3} \right) \left(-6\alpha \left(a + \frac{k g}{w'-m} \right) + 6b - \frac{6k g_0}{w'-m} \right.$$

$$\left. + 3(w'+m) \right) + (-1)^{l+1} \Phi_{l+1}(\alpha) \left[3b(w'-w) \frac{g}{E_2+m} \right.$$

$$\left. + \frac{3k g_0 g (w-m)}{(w'-m)(E_2+m)} + 3a\alpha(w-w') \frac{g}{E_2+m} + \frac{3\alpha k g^2 (w-m)}{(w'-m)(E_2+m)} \right.$$

$$\left. - 3g(w-m) \frac{l}{2l+1} + \left(\frac{g}{E_2+m} \frac{(k'w'+kw)}{w+m} + \frac{g l}{2l+1} \right) (6b + 3(w'+m)) \right.$$

$$\left. - \frac{6k g_0}{w'-m} - 6\alpha \left(a + \frac{k g}{w'-m} \right) \right] + \frac{g l}{2l+1} (-1)^{l-1} \Phi_{l-1}(\alpha) [3(w-m)$$

$$- 6b + \frac{6k g_0}{w'-m} - 3(w'+m) + 6a\alpha + \frac{6\alpha k g}{w'-m}]$$

$$+ \frac{g^2}{E_2+m} \frac{l+1}{2l+3} (-1)^{l+2} \Phi_{l+2}(\alpha) \left[3(w+m) + 6b - \frac{6k g_0}{w'-m} + 3(w'+m) \right.$$

$$\left. - 6a\alpha - \frac{6\alpha k g}{w'-m} \right] - \delta_{21} \left[\left(6a + \frac{6k g}{w'-m} \right) \left(\frac{k g}{3(w-m)} - \frac{g}{3} \right) (\Phi_1(\alpha) \right.$$

$$\left. - \alpha \Phi_0(\alpha) \right) + 4k g \xi \left(\frac{k g}{2(w'-m)(w-m)} + \frac{g(ww'+m^2)}{4k'w'w} \right) \right]$$

$$+ \delta_{20} \left[2k g \xi \left\{ \frac{3}{2k'w'w} \left((k+k')(w'w+m^2) + \frac{4a m^2}{2w'w} (w'+w) \right) \right. \right.$$

$$\left. + \frac{k g^2}{k'(E_2+m)(w'+w)} + \frac{g^2}{(E_2+m)(w'+w)} + \frac{k g^2}{(E_2+m)(w+m)(w'-m)} \right]$$

$$\begin{aligned}
 & + \frac{3kq_0}{(w-m)(w'-m)} \} + \frac{kq}{w-m} \left(6b - \frac{6kq_0}{w'-m} + 3(w'+m) \right) \\
 & + (\alpha Q_0(\alpha) - Q_1(\alpha)) \left(3a(w'+w) + \frac{3kq(w+m)}{w'-m} \right. \\
 & \left. - \left(6a + \frac{6kq}{w'-m} \right) \left(\frac{k'w' + kw}{w-m} + \frac{1}{3} \frac{kq^2}{(E_2+m)(w+m)} + \frac{1}{3} \frac{q^2}{E_2+m} \right) \right)
 \end{aligned}$$

The above kernel connects $\text{Im } \frac{1}{2}_{1+}$ to $\text{Re } \frac{1}{2}_{1+}$

$$\begin{aligned}
 \frac{2l}{E_2+m} kq^{l+1} K_{2l}^{2l}(w, w') &= (-1)^l Q_l(\alpha) \left[3b(w'+w) - \frac{3kq_0(w+m)}{w'-m} \right. \\
 & \left. - 3\alpha \left(a(w'+w) + \frac{kq(w+m)}{w'-m} \right) - \frac{3q^2(w+m)}{E_2+m} \frac{l}{2l-1} \right. \\
 & \left. - \left(\frac{k'w' + kw}{w-m} + \frac{q^2}{E_2+m} \frac{l}{2l-1} \right) \left(6b - \frac{6kq_0}{w'-m} + 3(w'+m) - 6a\alpha \right. \right. \\
 & \left. \left. - \frac{6\alpha kq}{w'-m} \right) \right] + \frac{q}{E_2+m} (-1)^{l-1} Q_{l-1}(\alpha) \left[3b(w'-w) + \frac{3kq_0(w-m)}{w'-m} \right. \\
 & + \alpha \left(3a(w-w') + \frac{3kq(w-m)}{w'-m} \right) - 3(w-m)(E_2+m) \frac{l+1}{2l+1} \\
 & \left. - \left(\frac{k'w' + kw}{w+m} + (E_2+m) \frac{l+1}{2l+1} \right) \left(-6b + \frac{6kq_0}{w'-m} - 3(w'+m) \right. \right. \\
 & \left. \left. + 6\alpha a + \frac{6\alpha kq}{w'-m} \right) \right] + \frac{q}{2l+1} (-1)^{l+1} Q_{l+1}(\alpha) \left[3(w-m) - 6b \right. \\
 & \left. + \frac{6kq_0}{w'-m} - 3(w'+m) + 6a\alpha + \frac{6\alpha kq}{w'-m} \right] \\
 & + \frac{q^2}{E_2+m} \frac{l}{2l-1} (-1)^{l-2} Q_{l-2}(\alpha) \left[3(w+m) + 6b - \frac{6kq_0}{w'-m} + 3(w'+m) \right.
 \end{aligned}$$

$$\begin{aligned}
 & -6a\alpha - \frac{6\alpha k_0}{w'-m} \Big] + \delta_{\ell 2} \Big[\frac{4g^2}{E_2+m} \left(a + \frac{k_0}{w'-m}\right) (\varphi_1(\alpha) \\
 & - \alpha \varphi_0(\alpha)) \left(\frac{k}{2(w+m)} - 1\right) + 4k_0 \xi \left(\frac{-g^2}{(E_2+m)(w'+w)} \right. \\
 & \left. + \frac{k_0^2}{2(E_2+m)(w'+w)k'} + \frac{k_0^2}{2(E_2+m)(w+m)(w'-m)} \right) \Big]
 \end{aligned}$$

The above kernel connects $\text{Im } \ell_{1+}$ to $\text{Re } \ell_{2-}$

$$\begin{aligned}
 & 2(\ell+1) \xi k_0^{\ell+1} K_{\ell 1}^{31}(w, w') = (-1)^\ell \varphi_\ell(\alpha) \Big[3b(w'+w) \\
 & - \frac{3k_0(w+m)}{w'-m} - 3a\alpha(w'+w) - \frac{3\alpha k_0(w+m)}{w'-m} \\
 & - \left(\frac{(k'w' + kw)}{w-m} \left(6b - \frac{6k_0}{w'-m} + 3(w'+m) - 6\alpha \left(a + \frac{k_0}{w'-m}\right) \right) \right. \\
 & \left. + \frac{g}{E_2+m} (-1)^{\ell+1} \varphi_{\ell+1}(\alpha) \left[3b(w'-w) + \frac{3k_0(w-m)}{w'-m} + 3a\alpha(w-w') \right. \right. \\
 & \left. \left. + \frac{3(w-m)(E_2+m)}{2\ell+1} + \frac{3\alpha k_0(w-m)}{w'-m} + \left(-\frac{(k'w' + kw)}{w+m} \right. \right. \right. \\
 & \left. \left. \left. + \frac{E_2+m}{2\ell+1} \right) \left(-6b + \frac{6k_0}{w'-m} - 3(w'+m) + 6a\alpha + \frac{6\alpha k_0}{w'-m} \right) \right] \right. \\
 & \left. + \frac{g}{2\ell+1} (-1)^{\ell-1} \varphi_{\ell-1}(\alpha) \left[-3(w-m) + 6b - \frac{6k_0}{w'-m} + 3(w'+m) - 6\alpha a \right. \right. \\
 & \left. \left. - \frac{6\alpha k_0}{w'-m} \right] + \delta_{\ell 1} \left[(\varphi_0(\alpha)\alpha - \varphi_1(\alpha)) \left(g + \frac{k_0}{w-m}\right) \left(2a + \frac{2k_0}{w'-m}\right) \right. \right. \\
 & \left. \left. - 4k_0 \xi \left(\frac{g(w'w + m^2)}{4k'w'w} + \frac{k_0}{2(w'-m)(w-m)} \right) \right] \right]
 \end{aligned}$$

The above kernel connects $\text{Im } \mathcal{E}_{1+}$ to $\text{Re } \mathcal{M}_{\ell+}$

$$\begin{aligned}
 & \frac{2\ell \xi k g}{E_2+m} \underset{\ell_1}{K}^{41}(w, w') = (-1)^\ell \mathcal{Q}_\ell(\alpha) [-3b(w'+w) \\
 & + \frac{3k g_0(w+m)}{w'-m} + 3\alpha(a(w'+w) + \frac{k g_0(w+m)}{w'-m}) \\
 & - \frac{(k'w'+kw)}{w-m} (-6b + \frac{6k g_0}{w'-m} - 3(w'+m) + 6a\alpha + \frac{6\alpha k g}{w'-m})] \\
 & + \frac{g}{E_2+m} (-1)^{\ell-1} \mathcal{Q}_{\ell-1}(\alpha) [3b(w-w') - \frac{3k g_0(w-m)}{w'-m} + \frac{3(w-m)(E_2+m)}{2\ell+1} \\
 & + \alpha(3a(w'-w) - \frac{3k g_0(w-m)}{w'-m}) - (6b - \frac{6k g_0}{w'-m} + 3(w'+m) \\
 & - 6\alpha a - \frac{6\alpha k g}{w'-m}) (\frac{k'w'+kw}{w+m} + \frac{E_2+m}{2\ell+1})] \\
 & - \frac{g}{2\ell+1} (-1)^{\ell+1} \mathcal{Q}_{\ell+1}(\alpha) [3(w-m) - 6b + \frac{6k g_0}{w'-m} - 3(w'+m) \\
 & + 6a\alpha + \frac{6\alpha k g}{w'-m}] + \delta_{\ell 1} [2k g \xi (\frac{-3b g}{(E_2+m)(w'+w)} \\
 & + \frac{g(w'w+m^2)}{2k'w'w} + \frac{k g}{(w-m)(w'-m)} + \frac{3k g g_0}{(E_2+m)(w+m)(w'-m)}) \\
 & + \frac{g}{E_2+m} (\mathcal{Q}_1(\alpha) - \alpha \mathcal{Q}_0(\alpha)) (3a(w'+m) + \frac{6k g m}{w'-m} + (6a \\
 & + \frac{6k g}{w'-m}) (\frac{k}{3} \frac{E_2+m}{w-m} + \frac{k'w'+kw}{w+m} + \frac{E_2+m}{3} - \frac{w+m}{3})) \\
 & - \frac{k g^2}{(E_2+m)(w+m)} (-6b + \frac{6k g_0}{w'-m} - 3(w'+m))] + \delta_{\ell 2} [(6a + \\
 & \frac{6k g}{w'-m}) (\alpha \mathcal{Q}_0(\alpha) - \mathcal{Q}_1(\alpha)) \frac{k g^2}{3(E_2+m)(w+m)} - 4k g \xi (\frac{k g^2}{2k'(E_2+m)(w'+w)} \\
 & + \frac{k g^2}{2(E_2+m)(w+m)(w'-m)})]
 \end{aligned}$$

The above kernel connects $\text{Im } \mathcal{E}_{1+}$ to $\text{Re } \mathcal{M}_{l-}$

$$\begin{aligned}
 2(l+1) \sum_{l_1} k g^{l+1} K_{(w, w')}^{l_1 3} &= (-1)^l Q_l(\alpha) \left[3b(w'+w) + \right. \\
 & 2(E'_2+m)(w-w') + 3 \frac{k g_0(w+m)}{w'-m} + 3 g^2 \frac{(w+m)}{E_2+m} \frac{l+1}{2l+3} \\
 & - 3\alpha \left(a(w'+w) - \frac{k g(w+m)}{w'-m} \right) - (6b + 4(E'_2+m) - 3(w'+m)) \\
 & + \frac{6k g_0}{w'-m} - 6a\alpha + \frac{6k g \alpha}{w'-m} \left(\frac{k'w'+kw}{w-m} + \frac{g^2}{E_2+m} \frac{l+1}{2l+3} \right) \Big] \\
 & + \frac{g}{E_2+m} (-1)^{l+1} Q_{l+1}(\alpha) \left[3b(w'-w) - 2(E'_2+m)(w'+w) - \frac{3k g_0(w-m)}{w'-m} \right. \\
 & - \alpha \left(3a(w'-w) + \frac{3k g(w-m)}{w'-m} \right) + 3(E_2+m)(w-m) \frac{l}{2l+1} \\
 & - \left(\frac{k'w'+kw}{w+m} + (E_2+m) \frac{l}{2l+1} \right) \left(6a\alpha - \frac{6\alpha k g}{w'-m} - 6b - 4(E'_2+m) \right. \\
 & \left. \left. - \frac{6k g_0}{w'-m} + 3(w'+m) \right) \right] + g \frac{l}{2l+1} (-1)^{l-1} Q_{l-1}(\alpha) \left[-3(w-m) - 6b \right. \\
 & \left. - 4(E'_2+m) - \frac{6k g_0}{w'-m} + 3(w'+m) + 6a\alpha - \frac{6\alpha k g}{w'-m} \right] \\
 & + \frac{g^2}{E_2+m} \frac{l+1}{2l+3} (-1)^{l+2} Q_{l+2}(\alpha) \left[-3(w+m) + 6b + 4(E'_2+m) \right. \\
 & \left. + \frac{6k g_0}{w'-m} - 3(w'+m) - 6a\alpha + \frac{6\alpha k g}{w'-m} \right] + \delta_{l1} \left[(6a - \right. \\
 & \left. \frac{6k g}{w'-m} \right) (Q_1(\alpha) - \alpha Q_0(\alpha)) \left(g \frac{l}{2l+1} - \frac{k g}{3(w-m)} \right)
 \end{aligned}$$

$$\begin{aligned}
 & + 4 \xi k g \left(\frac{k g}{2(W-m)(W'-m)} - \frac{g(W'W+m^2)}{4k'W'W} \right)] \\
 & + \delta_{20} \left[2 \xi k g \left(\frac{3}{2k'W'W} \left((k+k')(W'W+m^2) + \frac{m^2 m^2}{2W'W} (W'+W) \right) \right. \right. \\
 & + \frac{k g^2}{k'(E_2+m)(W'+W)} - \frac{g^2}{(E_2+m)(W'+W)} - \frac{2(E_2'+m)}{(W'+W)} - \frac{3k g_0}{(W'-m)(W-m)} \\
 & \left. \left. - \frac{k g^2}{(E_2+m)(W+m)(W'-m)} \right) + \frac{k g}{W-m} \left(6b + 4(E_2'+m) + \frac{6k g_0}{W'-m} + 3(W'+m) \right) \right. \\
 & + \left(\Psi_1(\alpha) - \alpha \Psi_0(\alpha) \right) \left(-3a(W'+W) + 3k g \frac{(W+m)}{W'-m} + \left(-\frac{6k g}{W'-m} + \right. \right. \\
 & \left. \left. 6a \right) \left(\frac{g^2}{E_2+m} \frac{l+1}{2l+3} + \frac{k g^2}{3(E_2+m)(W+m)} + \frac{k'W'+kW}{W-m} \right) \right)]
 \end{aligned}$$

The above kernel connects $\text{Im } \mathcal{M}_{1+}$ to $\text{Re } \mathcal{E}_{l+}$

$$\begin{aligned}
 & \frac{2 \xi k g^{l+1}}{E_2+m} K_{2l}^{23}(W, W') = (-1)^l \Psi_l(\alpha) \left[3b(W'+W) + 2(E_2'+m)(W-W') \right. \\
 & + \frac{3k g_0(W+m)}{W'-m} + 3g^2 \frac{(W+m)}{E_2+m} \frac{l}{2l-1} - \alpha \left(3a(W'+W) \right. \\
 & \left. - 3k g \frac{(W+m)}{W'-m} \right) + \left(\frac{k'W'+kW}{W-m} + \frac{g^2}{(E_2+m)} \frac{l}{2l-1} \right) \left(-6b - 4(E_2'+m) \right. \\
 & \left. \left. - \frac{6k g_0}{W'-m} + 3(W'+m) + 6a\alpha - \frac{6\alpha k g}{W'-m} \right) \right]
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{g}{E_2+m} (-1)^{\ell-1} \Phi_{\ell-1}(\alpha) \left[3b(w'-w) - 2(E_2'+m)(w'+w) - 3 \frac{k g_0 (W-m)}{W'-m} \right. \\
 & - \alpha \left(3a(w'-w) + 3 \frac{k g (W-m)}{W'-m} \right) + 3(E_2+m)(W-m) \frac{\ell+1}{2\ell+1} \\
 & + \left(\frac{k'w'+kw}{W+m} + (E_2+m) \frac{\ell+1}{2\ell+1} \right) \left(6b + 4(E_2'+m) + \frac{6k g_0}{W'-m} \right. \\
 & \left. \left. - 3(w'+m) - 6a\alpha + \frac{6\alpha k g}{W'-m} \right) \right] \\
 & - \frac{g(\ell+1)}{2\ell+1} (-1)^{\ell+1} \Phi_{\ell+1}(\alpha) \left[3(W-m) + 6b + 4(E_2'+m) + \frac{6k g_0}{W'-m} \right. \\
 & \left. - 3(w'+m) - 6a\alpha + \frac{6\alpha k g}{W'-m} \right] + \frac{g^2}{(E_2+m)2\ell-1} (-1)^{\ell-2} \Phi_{\ell-2}(\alpha) \left[6b + \right. \\
 & \left. 4(E_2'+m) - 3(W+m) + \frac{6k g_0}{W'-m} - 3(w'+m) - 6a\alpha + \frac{6\alpha k g}{W'-m} \right] \\
 & + \delta_{\ell 2} \left[(\Phi_1(\alpha) - \alpha \Phi_0(\alpha)) \left(-6a + \frac{6k g}{W'-m} \right) \left(\frac{2g^2}{3(E_2+m)} - \frac{k g^2}{3(E_2+m)(W+m)} \right) \right. \\
 & \left. + 4 \frac{k g^2}{2k'(E_2+m)(W'+W)} + \frac{g^2}{(E_2+m)(W'+W)} - \frac{k g^2}{2(E_2+m)(W'-m)(W+m)} \right]
 \end{aligned}$$

The above kernel connects $\text{Im } \mathcal{M}_{1+}$ to $\text{Re } \mathcal{E}_{\ell-}$

$$\begin{aligned}
 2(\ell+1) \frac{k g^{\ell+1}}{\ell!} K_{(w,w')}^{33} &= (-1)^{\ell} \Phi_{\ell}(\alpha) \left[3b(w'+w) + \right. \\
 & 2(E_2'+m)(W-w') + 3k g_0 \frac{(W+m)}{W'-m} - \alpha \left(3a(w'+w) - \right. \\
 & \left. \left. 3k g \frac{(W+m)}{W'-m} \right) - \frac{(k'w'+kw)}{W-m} \left(6b + 4(E_2'+m) + \frac{6k g_0}{W'-m} \right) \right]
 \end{aligned}$$

$$\begin{aligned}
 & -3(w'+m) - 6a\alpha + \frac{6\alpha kq}{w'-m} \Big] + \frac{q}{E_a+m} (-1)^{l+1} Q_{l+1}(\alpha) \left[3b(w'-w) - \right. \\
 & \left. 2(E_a'+m)(w'+w) - 3kq_0 \frac{(w-m)}{w'-m} + \alpha(3a(w-w') - 3kq \frac{(w-m)}{w'-m}) \right. \\
 & \left. - 3(E_a+m) \frac{(w-m)}{2l+1} + \left(\frac{E_a+m}{2l+1} - \frac{(k'w'+kw)}{w+m} \right) (-6b - 4(E_a'+m) \right. \\
 & \left. - \frac{6kq_0}{w'-m} + 3(w'+m) + 6a\alpha - \frac{6\alpha kq}{w'-m} \Big] \\
 & + \frac{q}{2l+1} (-1)^{l-1} Q_{l-1}(\alpha) \left[3(w-m) + 6b + 4(E_a'+m) + \frac{6kq_0}{w'-m} \right. \\
 & \left. - 3(w'+m) - 6a\alpha + \frac{6\alpha kq}{w'-m} \Big] + \delta_{l1} \left[(Q_1(\alpha) - \alpha Q_0(\alpha)) (-6a + \right. \right. \\
 & \left. \frac{6kq}{w'-m} \left(\frac{kq}{3(w-m)} + \frac{q}{3} \right) + 4 \frac{kq}{a(w-m)(w'-m)} \right. \\
 & \left. \left. - \frac{q(ww'+m^2)}{4k'w'w} \right) \right]
 \end{aligned}$$

The above kernel connects Im $\mathcal{M}_{l+}$ to Re $\mathcal{M}_{l+}$

$$\begin{aligned}
 \frac{2l \frac{kq}{E_a+m} (-1)^{l+1} K_{l+1}^{43}(w, w')}{E_a+m} &= (-1)^l Q_l(\alpha) \left[-3b(w'+w) + 2(E_a'+m)(w'-w) \right. \\
 & \left. - 3kq_0 \frac{(w+m)}{w'-m} + 3\alpha(a(w'+w) - kq \frac{(w+m)}{w'-m}) \right]
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{(k'w' + kw)}{w-m} \left(6b + 4(E_2' + m) + \frac{6kq_0}{w'-m} - 3(w' + m) - 6a\alpha \right. \\
 & \left. + \frac{6\alpha kq}{w'-m} \right) \Big] - \frac{q}{E_2 + m} (-1)^{\ell-1} \Phi_{\ell-1}(\alpha) \left[3b(w' - w) - 2(E_2' + m)(w' + w) \right. \\
 & \left. - 3kq_0 \frac{(w-m)}{w'-m} + 3\alpha \left(a(w-w') - \frac{kq(w-m)}{w'-m} \right) \right. \\
 & \left. + 3(E_2 + m) \frac{(w-m)}{2\ell+1} + \left(\frac{k'w' + kw}{w+m} + \frac{E_2 + m}{2\ell+1} \right) \left(6b + 4(E_2' + m) + \right. \right. \\
 & \left. \left. \frac{6kq_0}{w'-m} - 3(w' + m) - 6a\alpha + \frac{6\alpha kq}{w'-m} \right) \right] \\
 & + \frac{q}{2\ell+1} (-1)^{\ell+1} \Phi_{\ell+1}(\alpha) \left[3(w' - m) + 6b + 4(E_2' + m) + \frac{6kq_0}{w'-m} \right. \\
 & \left. - 3(w' + m) - 6a\alpha + \frac{6\alpha kq}{w'-m} \right] + \delta_{\ell 2} \left[\frac{1}{3} \frac{kq^2}{(E_2 + m)(w+m)} (\Phi_1(\alpha) - \right. \\
 & \left. \alpha \Phi_0(\alpha)) \left(-6a + \frac{6kq}{w'-m} \right) + 4\frac{kq}{2k'(E_2 + m)(w' + w)} + \right. \\
 & \left. \frac{kq^2}{2(E_2 + m)(w+m)(w'-m)} \right] + \delta_{\ell 1} \left[(\alpha \Phi_0(\alpha) - \Phi_1(\alpha)) (3a(w-w') - \right. \\
 & \left. 3kq \frac{(w-m)}{w'-m} + \left(-6a + \frac{6kq}{w'-m} \right) \left(\frac{k(E_2 + m)}{3(w-m)} + \frac{k'w' + kw}{w+m} + \frac{E_2 + m}{2\ell+1} \right) \frac{q}{E_2 + m} \right. \\
 & \left. + \frac{kq^2}{(E_2 + m)(w+m)} \left(6b + 4(E_2' + m) + \frac{6kq_0}{w'-m} - 3(w' + m) \right) - 2\frac{kq}{(w'-m)(w-m)} \right. \\
 & \left. - \frac{q}{2(E_2 + m)k'w'w} \left((E_2 + m)(w'w + m^2) + 2k'(w'w + m^2 - m^2) \right) \right. \\
 & \left. + \frac{3bq}{(E_2 + m)(w' + w)} + \frac{3kq_0q_0}{(E_2 + m)(w'-m)(w+m)} \right] \Big]
 \end{aligned}$$

The above kernel connects Im $\mathcal{M}_{1+}$ to Re $\mathcal{M}_{1-}$

$$\begin{aligned}
 2(l+1) \leq k q^{l+1} K_{l+1}^{14}(w, w') &= (-1)^l \varphi_l(\alpha) \left[w - w' - 2 \frac{(k'w' + kw)}{w - m} \right. \\
 &\quad \left. - \frac{2q^2}{(E_{a+m})(2l+3)} \right] - \frac{q}{E_{a+m}} (-1)^{l+1} \varphi_{l+1}(\alpha) \left[w + w' - 2 \frac{(k'w' + kw)}{w + m} \right. \\
 &\quad \left. - \frac{2l}{2l+1} (E_{a+m}) \right] - 2q \frac{l}{2l+1} (-1)^{l-1} \varphi_{l-1}(\alpha) \\
 &\quad + \frac{2q^2}{E_{a+m}} \frac{l+1}{2l+3} (-1)^{l+2} \varphi_{l+2}(\alpha) + \delta_{l0} \left[\frac{2kq}{w - m} - \frac{2 \leq kq}{w' + w} \right]
 \end{aligned}$$

The above kernel connects Im $\mathcal{M}_{1-}$ to Re $\mathcal{E}_{l+}$

$$\begin{aligned}
 \frac{2l \leq k q^{l+1}}{E_{a+m}} K_{l+1}^{24}(w, w') &= (-1)^l \varphi_l(\alpha) \left[w - w' - 2 \frac{(k'w' + kw)}{w - m} \right. \\
 &\quad \left. - \frac{q^2}{(E_{a+m})} \frac{2l}{2l-1} \right] - \frac{q}{E_{a+m}} (-1)^{l-1} \varphi_{l-1}(\alpha) \left[w' + w - 2 \frac{(k'w' + kw)}{w + m} \right. \\
 &\quad \left. - 2(E_{a+m}) \frac{(l+1)}{2l+1} \right] - 2q \frac{(l+1)}{2l+1} (-1)^{l+1} \varphi_{l+1}(\alpha) \\
 &\quad + 2 \frac{q^2}{E_{a+m}} \frac{l}{2l-1} (-1)^{l-2} \varphi_{l-2}(\alpha)
 \end{aligned}$$

The above kernel connects Im $\mathcal{M}_{1-}$ to Re $\mathcal{E}_{l-}$

$$\begin{aligned}
 2(l+1) \leq k g^{l+1} K_{l+1}^{34}(w, w') &= (-1)^l \phi_l(\alpha) \left[w - w' - 2 \frac{(k'w' + kw)}{w - m} \right] \\
 - \frac{g}{E_{a+m}} (-1)^{l+1} \phi_{l+1}(\alpha) &\left[w' + w - 2 \frac{(k'w' + kw)}{w + m} + 2 \frac{(E_{a+m})}{2l+1} \right] \\
 + \frac{2g}{2l+1} (-1)^{l-1} \phi_{l-1}(\alpha) &
 \end{aligned}$$

The above kernel connects Im $\mathcal{M}_{l-}$ to Re $\mathcal{M}_{l+}$

$$\begin{aligned}
 2l \leq k g^{l+1} K_{l+1}^{44}(w, w') &= (-1)^l \phi_l(\alpha) \left[w' - w + 2 \frac{(k'w' + kw)}{w - m} \right] \\
 + \frac{g}{E_{a+m}} (-1)^{l-1} \phi_{l-1}(\alpha) &\left[w' + w - 2 \frac{(k'w' + kw)}{w + m} - 2 \frac{(E_{a+m})}{2l+1} \right] \\
 + \frac{2g}{2l+1} (-1)^{l+1} \phi_{l+1}(\alpha) &+ \frac{2kg^2}{(E_{a+m})(w+m)} \delta_{l1}
 \end{aligned}$$

The above kernel connects Im $\mathcal{M}_{l-}$ to Re $\mathcal{M}_{l-}$

10. CONDITIONS FROM UNITARITY (AND T-VARIANCE)

If we define the reaction matrix R in terms of the S -matrix by

$$S = 1 + iR \quad (10.1)$$

then the unitarity of the S -matrix

$$S^\dagger S = 1 \quad (10.2)$$

together with time-reversal invariance when the final state differs from the initial state^{*)}, leads to the condition for the R -matrix

$$\text{Im} \langle \beta | R | \alpha \rangle = \frac{1}{2} \sum_{\nu} \langle \nu | R | \beta \rangle^* \langle \nu | R | \alpha \rangle \quad (10.3)$$

where $|\nu\rangle$ is a complete set of states. To first order in e , when intermediate states containing photons may be neglected, Eq. (10.3) imposes a phase condition on the individual multipole transition amplitudes in the region of partial-wave elasticity, namely that the phase of a multipole transition to a final pion-nucleon state is equal to the scattering phase shift of that pion-nucleon state. This result was first derived by Watson⁵⁶⁾ and is generally known as Watson's Theorem. The result can be derived very elegantly⁵⁷⁾ using the helicity formalism of Appendix C, and we outline that method here.

Firstly we introduce the helicity amplitudes for pion-nucleon scattering from an initial state of helicity μ , angles (ϑ, φ) to a final state of helicity μ' , angles (ϑ', φ') by

$$\begin{aligned} \langle \mu' | R | \mu \rangle &= g_{\mu'\mu}(\vartheta' \varphi'; \vartheta, \varphi) \\ &= \frac{1}{2\pi} \sum_J \sum_M (J + \frac{1}{2}) \langle \mu' | T^J | \mu \rangle D_{M, \mu'}^{J*}(\varphi', \vartheta', -\varphi') D_{M, \mu}^J(\varphi, \vartheta, -\varphi) \end{aligned} \quad (10.4)$$

*) Using time-reversal invariance here requires no additional assumptions in our case, since it is already implicit in writing down the dispersion relations in Section 7.

Retaining only intermediate states of one pion and one nucleon, then in terms of $f_{\mu,\lambda}(\vartheta,\varphi)$ of Eq. (C.2) and of $g_{\mu',\mu}(\vartheta',\varphi';\vartheta,\varphi)$, Eq. (10.3) becomes

$$I_m f_{\mu,\lambda}(\vartheta,\varphi) = \frac{1}{2} \sum_{\mu'} \int d\Omega' g_{\mu',\mu}^*(\vartheta',\varphi';\vartheta,\varphi) f_{\mu',\lambda}(\vartheta',\varphi') \quad (10.5)$$

Substituting Eqs. (C.2) and (10.4) in Eq. (10.5), and using the orthonormality relation

$$\int d\Omega' D_{M,N}^{J*}(\vartheta',\varphi',-\varphi') D_{M',N'}^J(\vartheta',\varphi',-\varphi') = \frac{4\pi}{(2J+1)} \delta_{JJ'} \delta_{MM'} \delta_{NN'} \quad (10.6)$$

we obtain immediately the unitarity relation for the helicity amplitudes

$$I_m \langle \mu | T^J | \lambda \rangle = \frac{1}{2} \sum_{\mu'} \langle \mu' | T^J | \mu \rangle^* \langle \mu' | T^J | \lambda \rangle \quad (10.7)$$

Using Eq. (C.10) - (C.13) and the parity relation of the scattering amplitudes

$$\langle \frac{1}{2} | T^J | \pm \frac{1}{2} \rangle = - \langle -\frac{1}{2} | T^J | \mp \frac{1}{2} \rangle \quad (10.8)$$

gives immediately

$$I_m \begin{pmatrix} M_{\ell\pm} \\ E_{\ell\pm} \end{pmatrix} = \begin{pmatrix} M_{\ell\pm} \\ E_{\ell\pm} \end{pmatrix} \frac{1}{2} \left\{ \langle \frac{1}{2} | T^J | \frac{1}{2} \rangle \pm \langle \frac{1}{2} | T^J | -\frac{1}{2} \rangle \right\}^* \quad (10.9)$$

Since the partial wave scattering amplitude is given by

$$f_{\ell\pm} = (\text{kinematical factor}) \left\{ \langle \frac{1}{2} | T^J | \frac{1}{2} \rangle \pm \langle \frac{1}{2} | T^J | -\frac{1}{2} \rangle \right\} \quad (10.10)$$

and recalling that

$$f_{\ell\pm} = \frac{1}{q} e^{i\delta_{\ell\pm}} \sin \delta_{\ell\pm} \quad (10.11)$$

Eq. (10.9) gives immediately that

$$\begin{pmatrix} M_{\ell\pm} \\ E_{\ell\pm} \end{pmatrix} = \begin{pmatrix} |M_{\ell\pm}| \\ |E_{\ell\pm}| \end{pmatrix} e^{i\delta_{\ell\pm} + i n \pi} \quad (10.12)$$

where n is an integer.

11. A MULTIPOLE SUM RULE

As we have already stated, the most important contributions to the photoproduction multipoles from the left-hand singularities are the Born terms and via crossing, the terms in $\text{Im } M_{1+}^{(3)}$ and $\text{Im } E_{0+}^{(0,1,3)}$. All of these contributions can be evaluated with considerable accuracy. The Born terms, being dependent only on the coupling constants, are known to within 3%, and the recent analysis of photoproduction data below 450 MeV by Donnachie and Shaw⁸⁾ shows that $\text{Im } M_{1+}^{(3)}$ is known nearly as well as this in the (dominant) energy region below 450 MeV. This analysis also shows that the E_{0+} transitions are known to within 10% below 450 MeV, and that all existing photoproduction data below 450 MeV are consistent with the fixed- t dispersion relations in the $(\text{Born} + M_{1+}^{(3)} + E_{0+})$ approximation. In other words, the contributions from transitions to higher angular momentum states in the u -channel and all contributions from the t -channel (except for one-pion exchange) can be ignored for photon energies below 450 MeV.

This statement is not necessarily true at high energies, however, since the energy dependence of one contribution is not necessarily the same as that of another, and a contribution which is relatively small at low energies need not be so at high energies. Qualitatively this is saying that the higher the energy the more important the "short-range" forces (corresponding to the more massive intermediate states in the u - and t -channels) should become, while at low energies the "long-range" forces (corresponding to the lowest-mass intermediate states in the u - and t -channels) should dominate. These "short-range" forces cannot be calculated a priori, but it is important to be able to make a realistic assessment of their effect. This can be done via a sum-rule, obtained on the assumption that the photoproduction cross-section does not diverge asymptotically (which is physically very reasonable). This is equivalent to saying that, at the worst, the real part of a given multipole goes asymptotically to a constant.

The sum-rule which can be derived is analogous to that for the scattering case⁵⁸), although in scattering no assumption about the asymptotic behaviour of the cross-section is required, unitarity being a sufficient condition.

The general form of a multipole dispersion relation may be written symbolically as

$$\begin{aligned} \operatorname{Re} M_{\ell\pm}(w) = B_{\ell\pm}(w) + \frac{P}{\pi} \int_{(m+\mu)}^{\infty} dw' \frac{\operatorname{Im} M_{\ell\pm}(w')}{(w'-w)} \\ + \frac{1}{\pi} \int_{-\infty}^{w_L} dw' \frac{\operatorname{Im} M_{\ell\pm}(w')}{(w'-w)} \end{aligned} \quad (11.1)$$

where $B_{\ell\pm}(w)$ is the contribution from the Born poles and the second integral represents the contribution from the rest of the left-hand singularities in the w -plane.

The $M_{\ell\pm}(w)$ are given by

$$M_{\ell\pm}(w) = C_{\ell\pm}(w) \begin{cases} M_{\ell\pm}(w) \\ E_{\ell\pm}(w) \end{cases} \quad (11.2)$$

with

$$C_{\ell+}(w) = \frac{w}{k \zeta q^{\ell}} \quad C_{\ell-}(w) = \frac{w(E_2+m)}{k \zeta q^{\ell}} \quad (11.3)$$

clearly

$$C_{l+} \sim W^{-l-1}, \quad C_{l-} \sim W^{-l} \quad \text{as } W \rightarrow \infty \quad (11.4)$$

and consequently, by the initial assumption on the asymptotic behaviour of $\text{Re } M_{l\pm}$, $\text{Re } B_{l\pm}$,

$$\text{Re } M_{l+} \sim W^{-l-1}, \quad \text{Re } M_{l-} \sim W^{-l} \quad \text{as } W \rightarrow \infty \quad (11.5)$$

and they may, in fact, vanish more rapidly than this, Eq. (11.5) merely being an upper limit.

We may write

$$\frac{1}{\pi} \int_{-\infty}^{W_L} dw' \frac{\text{Im } M_{l\pm}(w')}{(w' - w)} = - \frac{1}{\pi W} \int_{-\infty}^{W_L} dw' \text{Im } M_l(w') + O\left(\frac{1}{W^2}\right) \quad (11.6)$$

as $W \rightarrow \infty$, provided that the integral on the right-hand side of Eq. (11.6) exists

The asymptotic behaviour of $\text{Im } M_{\ell, \pm}$ (the asymptotic cross-section condition imposes the same limits on the behaviour of $\text{Im } M_{\ell, \pm}$ as on $\text{Re } M_{\ell, \pm}$) and the behaviour of kernels $K_{\ell\ell'}(W, W')$ ensure that this is true.

Sufficient conditions to enable us to write

$$\frac{P}{\pi} \int_{(m+\mu)}^{\infty} dW' \frac{\text{Im } M_{\ell, \pm}(W')}{(W' - W)} = - \frac{1}{\pi W} \int_{(m+\mu)}^{\infty} dW' \text{Im } M_{\ell, \pm}(W') \quad (11.7)$$

as $W \rightarrow \infty$ are⁵⁹⁾ firstly that the integral on the right-hand side of Eq. (11.7) exists, and secondly that $\text{Im } M_{\ell, \pm}(W)$ satisfy a Lipschitz condition. The first condition is certainly true for $\ell \geq 1$ (ℓ + multipoles) and for $\ell \geq 2$ (ℓ - multipoles) by the asymptotic cross-section assumption. The second condition can neither be disproved, nor proved, but it is not unreasonable. Certainly the limit must exist, and it is a reasonable conjecture that it is given by Eq. (11.7)

Finally the Born term may be written asymptotically as

$$B_{\ell, \pm}(W) = \frac{B'_{\ell, \pm}(W)}{\pi W} \quad \text{as } W \rightarrow \infty \quad (11.8)$$

$B'_{\ell, \pm}(W)$ is, at worst, a constant as $W \rightarrow \infty$, and in many cases decreases asymptotically to zero.

Substituting Eqs. (11.5), (11.6), (11.7) and (11.8) in Eq. (11.1), and taking the limit $W \rightarrow \infty$ leads immediately to the sum rule

$$B'_{l\pm}(w) - \int_{(m+\mu)}^{\infty} dw' I_m M_{l\pm}(w') - \int_{-\infty}^{w_L} dw' I_m M_{l\pm}(w') = 0 \quad (11.9)$$

for $l \geq 1$ ($l+$ multipoles) and for $l \geq 2$ ($l-$ multipoles).

The sum rule imposes strong restrictions on the magnitude of the contributions from the distant left-hand (and right-hand) singularities.

12. APPLICATION OF THE THEORY TO DATA ANALYSIS

The intention is to use the theory as a basis for a multipole analysis of photopion production up to photon energies of about 1100 MeV, i.e. to just above the old third resonance.

There are seven resonant amplitudes in pion nucleon scattering below 1 GeV, namely S_{31} , P_{33} , S_{11} , P_{11} , D_{13} , D_{15} , and F_{15} . With the exception of the P_{33} amplitude, each one of these has very strong inelastic effects associated with the resonance, most of the inelasticity being concentrated in these amplitudes. Some general qualitative features of the P_{33} amplitude and of the non-resonant amplitudes up to and including F-waves are given in Table 2. In the energy region being considered, all amplitudes above F-waves are essentially elastic and have very small phase shifts. The scattering data are taken from the paper by Donnachie⁶⁰).

TABLE 2

Amplitude	E_1 (MeV)	E_2 (MeV)	$P_{\max}$
P_{31}	700	870	25°
P_{33}	550	870	-
D_{33}	370	600	12°
D_{35}	700	870	3°
F_{35}	750	950	0.5°
F_{37}	750	1230	7°
P_{13}	530	700	9°
F_{17}	800	1230	3°

E_1 : lowest pion lab. energy at which inelasticity is observable

E_2 : lowest pion lab. energy at which the inelasticity parameter $\eta < 0.9$

$P_{\max}$: maximum value of the magnitude of the phase shift below 1 GeV.

From Table 2 it can be seen that the effective inelastic threshold is very much higher than the true inelastic threshold (171 MeV in pion-nucleon scattering, 322 MeV in photopion production), and consequently Watson's Theorem can be applied, without error, at least to the energy E_1 . Consequently, the imaginary part of the multipole transitions to these final states can be obtained up to this energy, and the contribution of the rescattering term evaluated. Thus up to the energy E_1 these multipole transition amplitudes can be taken as given and fixed, and need not be varied in a multipole fit to the experimental data. Even above this energy, if the scattering amplitude does not depart appreciably from being elastic (energies less than E_2) the departure from Watson's Theorem will be small, the phase of a multipole transition amplitude will not differ appreciably from the corresponding scattering phase shift, and a reasonable estimate of the rescattering can be made. The resultant multipoles can no longer be fixed, but they can be given reasonable errors and constrained, and used as "theoretical data" to supplement the experimental data.

The multipole transitions to the resonant amplitudes can be treated similarly if the energy is sufficiently low (see Table 3), i.e. the lower the energy the less one need vary to fit the data.

TABLE 3

Amplitude	E_3 (MeV)
S_{31}	410
S_{11}	310
P_{11}	250
D_{13}	370
D_{15}	580
F_{15}	580

E_3 : lowest pion lab. energy at which inelasticity is observable.

When analysing data in this way, there is no guarantee that a unique solution will be obtained -- in fact the reverse is usually the case. However, any solution obtained must satisfy the appropriate dispersion relation, and this imposes strong consistency conditions on the data, enabling decisions to be made among different sets of multipoles, i.e. the theory can be used to impose uniqueness on the solution.

The validity and power of this approach to data analysis has been demonstrated clearly in the phase-shift analysis of pion-nucleon scattering^{4, 60, 61}).

There is one further use to which the theory can be put. Almost all photoproduction data consist of photoproduction of π^+ and π^0 from protons, and photoproduction data from neutrons (via a deuteron target) or data on the inverse process, π^- radiative capture on protons, are few and have large errors. Consequently, an analysis will obtain the amplitudes $\mathcal{M}_{\ell\pm}^{(3)}$ and the combination $\mathcal{M}_{\ell\pm}^{(0)} + \frac{1}{3}\mathcal{M}_{\ell\pm}^{(1)}$, not the amplitudes $\mathcal{M}_{\ell\pm}^{(0)}$ and $\mathcal{M}_{\ell\pm}^{(1)}$ separately. However, once the phase of the $\mathcal{M}_{\ell\pm}^{(0)}$ and $\mathcal{M}_{\ell\pm}^{(1)}$ amplitudes is known (and this is found in the course of the fit) they can be separated by the theory.

The data is divided into two regions: the region of the first resonance (< 450 MeV photon lab. energy), and the region of the higher resonances (> 600 MeV photon lab. energy), with very little data in between. This division coincides conveniently with the division in the scattering data discussed above. In the low-energy region the theory, as developed here, describes the situation more or less completely [e.g. see Refs. 8, 24)]. In the high-energy region a detailed fit of the resonant amplitudes must be made. This work is now in progress, and will be described in a subsequent paper.

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APPENDIX A

CROSS-SECTION AND POLARIZATION FORMULAE

1. PHOTOPION PRODUCTION

The differential cross-section for the transition from an initial γ -N state i to a final pion-nucleon state f is given by

$$\frac{d\sigma}{d\Omega} = \frac{q}{k} |\langle \chi_f | \mathcal{Y} | \chi_i \rangle|^2 \quad (\text{A.1})$$

where

$$\begin{aligned} \mathcal{Y} = & i \sigma \cdot \underline{\epsilon} \mathcal{Y}_1 + \sigma \cdot \hat{q} \sigma \cdot (\hat{k} \times \underline{\epsilon}) \mathcal{Y}_2 \\ & + i \sigma \cdot \hat{k} \hat{q} \cdot \underline{\epsilon} \mathcal{Y}_3 + i \sigma \cdot \hat{q} \hat{q} \cdot \underline{\epsilon} \mathcal{Y}_4 \end{aligned} \quad (\text{A.2})$$

First of all we will evaluate those cross-sections in which the polarization of the final nucleon is unobserved. Summing over the final spin states yields

$$\sum_f \langle \chi_f | \mathcal{Y} | \chi_i \rangle^* \langle \chi_f | \mathcal{Y} | \chi_i \rangle = \langle \chi_i | \mathcal{Y}^\dagger \mathcal{Y} | \chi_i \rangle$$

where

$$\begin{aligned} \mathcal{Y}^\dagger \mathcal{Y} = & |\mathcal{Y}_1|^2 \{ \underline{\epsilon}^* \cdot \underline{\epsilon} + i \sigma \cdot (\underline{\epsilon}^* \times \underline{\epsilon}) \} \\ & + |\mathcal{Y}_2|^2 \{ \sigma \cdot (\hat{k} \times \underline{\epsilon}^*) \sigma \cdot (\hat{k} \times \underline{\epsilon}) \} \\ & + |\mathcal{Y}_3|^2 \{ \hat{q} \cdot \underline{\epsilon}^* \hat{q} \cdot \underline{\epsilon} \} \\ & + |\mathcal{Y}_4|^2 \{ \hat{q} \cdot \underline{\epsilon}^* \hat{q} \cdot \underline{\epsilon} \} \end{aligned}$$

$$\begin{aligned}
 & + \gamma_1^* \gamma_2 \left\{ -i \sigma \cdot (\hat{k} \times \underline{\epsilon}) \hat{q} \cdot \underline{\epsilon}^* + i \sigma \cdot \hat{q} \hat{k} \cdot (\underline{\epsilon} \times \underline{\epsilon}^*) \right. \\
 & \quad \left. - i \sigma \cdot \underline{\epsilon}^* \hat{q} \cdot (\hat{k} \times \underline{\epsilon}) + (\underline{\epsilon}^* \times \hat{q}) \cdot (\hat{k} \times \underline{\epsilon}) \right\} \\
 & + \gamma_1^* \gamma_3 \left\{ i \sigma \cdot (\underline{\epsilon}^* \times \hat{k}) \hat{q} \cdot \underline{\epsilon} \right\} \tag{A.3} \\
 & + \gamma_1^* \gamma_4 \left\{ \hat{q} \cdot \underline{\epsilon} \hat{q} \cdot \underline{\epsilon}^* + i \sigma \cdot (\underline{\epsilon}^* \times \hat{q}) \hat{q} \cdot \underline{\epsilon} \right\} \\
 & + \gamma_2^* \gamma_3 \left\{ \hat{q} \cdot \underline{\epsilon} \hat{q} \cdot \underline{\epsilon}^* + i \sigma \cdot \hat{k} \hat{q} \cdot (\hat{k} \times \underline{\epsilon}^*) \hat{q} \cdot \underline{\epsilon} \right. \\
 & \quad \left. + i \sigma \cdot (\hat{k} \times \underline{\epsilon}^*) \hat{q} \cdot \hat{k} \hat{q} \cdot \underline{\epsilon} \right\} \\
 & + \gamma_2^* \gamma_4 \left\{ \hat{q} \cdot \underline{\epsilon} i \sigma \cdot (\hat{k} \times \underline{\epsilon}^*) \right\} \\
 & + \gamma_3^* \gamma_4 \left\{ \hat{q} \cdot \underline{\epsilon}^* \hat{q} \cdot \underline{\epsilon} [\hat{q} \cdot \hat{k} + i \sigma \cdot (\hat{k} \times \hat{q})] \right\} \\
 & + \text{Hermitian Conjugate of the off-diagonal elements}
 \end{aligned}$$

Choosing a co-ordinate frame in which the production plane is the x,z plane and introducing the unit vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$ ($= \hat{k}$) in the x,y and z directions respectively, then for right and left circularly polarized photons (helicity ± 1),

$$\underline{\epsilon} \equiv \underline{\epsilon}_{\pm} = \mp \frac{1}{\sqrt{2}} (\hat{e}_1 \pm i \hat{e}_2) \tag{A.4}$$

and for linearly polarized photons

$$\begin{aligned}
 \underline{\epsilon} &= \cos \phi \hat{e}_1 + \sin \phi \hat{e}_2 \\
 &= \frac{1}{\sqrt{2}} \left\{ e^{i\phi} \underline{\epsilon}_- - e^{-i\phi} \underline{\epsilon}_+ \right\} \tag{A.5}
 \end{aligned}$$

A. Polarized nucleon, circularly polarized photon

Introducing the initial nucleon polarization by

$$P = \langle \chi_i | \sigma | \chi_i \rangle \quad (A.6)$$

then

$$\langle \chi_i | y_{\pm}^{\dagger} y_{\pm} | \chi_i \rangle = (1 \mp k \cdot P) \alpha + \beta \pm \sin \theta \hat{e}_1 \cdot P \gamma + \sin \theta \hat{e}_2 \cdot P \delta \quad (A.7)$$

where

$$\alpha = |y_1|^2 + |y_2|^2 - 2 \cos \theta \operatorname{Re} y_1^* y_2 + \sin^2 \theta \operatorname{Re} \{ y_1^* y_4 + y_2^* y_3 \} \quad (A.8)$$

$$\beta = \frac{1}{2} \sin^2 \theta \{ |y_3|^2 + |y_4|^2 + 2 \cos \theta \operatorname{Re} y_3^* y_4 \} \quad (A.9)$$

$$\gamma = \operatorname{Re} \{ y_1^* y_3 - y_2^* y_4 \} + \cos \theta \operatorname{Re} \{ y_1^* y_4 - y_2^* y_3 \} \quad (A.10)$$

$$\delta = \operatorname{Im} \{ y_1^* y_3 - y_2^* y_4 \} + \cos \theta \operatorname{Im} \{ y_1^* y_4 - y_2^* y_3 \} - \sin^2 \theta \operatorname{Im} \{ y_3^* y_4 \} \quad (A.11)$$

For the special case of the initial nucleon being polarized along, or apposite, to $\hat{k}$, one obtains respectively

$$\langle \chi_i | y_{\pm}^{\dagger} y_{\pm} | \chi_i \rangle = (1 \mp 1) \alpha + \beta \quad (A.12)$$

and

$$\langle \chi_i | y_{\pm}^{\dagger} y_{\pm} | \chi_i \rangle = (1 \pm 1) \alpha + \beta \quad (A.13)$$

Insertion of Eqs. (A.7), (A.12) or (A.13) in Eq. (A.1) gives the appropriate cross-sections.

B. Polarized nucleon, unpolarized photon

$$\begin{aligned} \frac{d\sigma(P)}{d\Omega} &= \frac{1}{2} \left\{ \frac{d\sigma_+(P)}{d\Omega} + \frac{d\sigma_-(P)}{d\Omega} \right\} \\ &= \frac{q}{k} \left\{ \alpha + \beta + \sin^2 \theta \hat{e}_2 \cdot P \delta \right\} \end{aligned} \quad (A.14)$$

C. Unpolarized nucleon, circularly polarized photon

$$\frac{d\sigma_+}{d\Omega} = \frac{d\sigma_-}{d\Omega} = \frac{d\sigma_0}{d\Omega} = \frac{q}{k} \left\{ \alpha + \beta \right\} \quad (A.15)$$

where $\frac{d\sigma_0}{d\Omega}$ is the cross-section for an unpolarized initial state.

[Note that in the helicity formalism,

$$\begin{aligned} \frac{d\sigma_+}{d\Omega} &= \frac{q}{2k} \left\{ |f_{\frac{1}{2}, -\frac{1}{2}}|^2 + |f_{\frac{1}{2}, -\frac{3}{2}}|^2 + |f_{-\frac{1}{2}, -\frac{1}{2}}|^2 + |f_{-\frac{1}{2}, -\frac{3}{2}}|^2 \right\} \\ &= \frac{q}{2k} \left\{ |H^+|^2 + |\Phi^+|^2 + |\Phi^-|^2 + |H^-|^2 \right\} = \frac{d\sigma_-}{d\Omega} \end{aligned}$$

which is in disagreement with the result obtained in the paper by G. Zweig⁵⁴.)]

D. Unpolarized nucleon, linearly polarized photon

From Eq. (A.5) it follows that

$$\overline{Y}^{\dagger}(\xi) Y(\xi) = \frac{1}{2} \left\{ \overline{Y}_+^{\dagger} Y_+ + \overline{Y}_-^{\dagger} Y_- - 2 \operatorname{Re} \left\{ e^{2i\phi} \overline{Y}_+^{\dagger} Y_- \right\} \right\}$$

and consequently

$$\begin{aligned} \frac{d\sigma(\xi)}{d\Omega} &= \frac{d\sigma_0}{d\Omega} + \frac{q}{k} \cos 2\phi \sin^2 \theta \left\{ \frac{1}{2} |\overline{Y}_3|^2 + \frac{1}{2} |\overline{Y}_4|^2 \right. \\ &\quad \left. + \operatorname{Re} \left\{ \overline{Y}_2^* \overline{Y}_3 + \overline{Y}_1^* \overline{Y}_4 \right\} + \cos \theta \operatorname{Re} \left\{ \overline{Y}_3^* \overline{Y}_4 \right\} \right\} \end{aligned} \quad (A.16)$$

Using Eqs. (A.8), (A.9) and (A.15), Eq. (A.16) can be recast easily into the form usually quoted²⁴⁾ viz.

$$\begin{aligned}
 \frac{k}{q} \frac{d\sigma}{d\Omega} = & \left\{ |y_1|^2 + |y_2|^2 + \frac{1}{2}|y_3|^2 + \frac{1}{2}|y_4|^2 + \operatorname{Re} y_1 y_4^* + \operatorname{Re} y_2 y_3^* \right\} \\
 & + \left\{ \operatorname{Re} y_3 y_4^* - 2 \operatorname{Re} y_1 y_2^* \right\} \cos \theta \\
 & - \left\{ \frac{1}{2}|y_3|^2 + \frac{1}{2}|y_4|^2 + \operatorname{Re} y_1 y_4^* + \operatorname{Re} y_2 y_3^* \right\} \cos^2 \theta \\
 & - \operatorname{Re} y_3 y_4^* \cos^3 \theta \\
 & + \sin^2 \theta \cos 2\theta \left\{ \frac{1}{2}|y_3|^2 + \frac{1}{2}|y_4|^2 + \operatorname{Re} y_2 y_3^* \right. \\
 & \left. + \operatorname{Re} y_1 y_4^* + \operatorname{Re} y_3 y_4^* \cos \theta \right\}
 \end{aligned} \tag{A.17}$$

Finally we obtain the recoil nucleon polarization, for a completely unpolarized initial state.

$$\begin{aligned}
 \tilde{P} \frac{d\sigma_0}{d\Omega} &= \frac{q}{4k} \sum_{\substack{\text{Initial} \\ \text{Spins}}} \langle \chi_i | y^+ \underline{\sigma} y | \chi_i \rangle \\
 &= \frac{q}{4k} \sum_{\substack{\text{Photon} \\ \text{Spin}}} \operatorname{Tr} [y^+ \underline{\sigma} y]
 \end{aligned} \tag{A.18}$$

Using Eq. (A.2) and

$$\sum_{\text{Photon Spin}} \varepsilon_i^T \varepsilon_j = \delta_{ij} - \hat{k}_i \hat{k}_j \quad (\text{A.19})$$

yields

$$P(\theta) \frac{d\sigma_0}{d\Omega} = \frac{q}{k} (\hat{k} \times \hat{q}) \operatorname{Im} \left\{ 2 y_1 y_2^* + y_1 y_3^* - y_2 y_4^* \right. \\ \left. + \cos\theta \{ y_1 y_4^* - y_2 y_3^* \} - \sin^2\theta y_3 y_4^* \right\} \quad (\text{A.20})$$

If the polarization perpendicular to the production plane is measured, Eq. (A20) gives the usual form²⁴⁾

$$P(\theta) \frac{k}{q} \frac{d\sigma_0}{d\Omega} = \sin\theta \operatorname{Im} \left\{ (2 y_1 y_2^* + y_1 y_3^* - y_2 y_4^* \right. \\ \left. - y_3 y_4^*) + (y_1 y_4^* - y_2 y_3^*) \cos\theta + y_3 y_4^* \cos^2\theta \right\} \quad (\text{A.21})$$

2. ELECTROPION PRODUCTION

The T-matrix, introduced in Section 4, is related to the differential cross-section by

$$d\sigma = \frac{1}{(2\pi)^5} \delta(K_2 + P_2 + Q - K_1 - P_1) \frac{m^2 m_e^2}{2} \times \\ \frac{|T|^2}{[(K_1 P_1)^2 - m^2 m_e^2]^{\frac{1}{2}}} \frac{dk_2}{E_2} \frac{dk_2}{k_{20}} \frac{dq}{q_0} \quad (\text{A.22})$$

which we will evaluate in the π -N centre-of-mass system.

Expressing T in terms of the amplitudes $y_1, \dots, y_6$ (Section 6) and summing over the electron and nucleon spins, requires us to evaluate

$$\frac{1}{4} \sum_{\text{Spins}} |T|^2 = \frac{1}{2} \sum_{\text{Electron Spins}} (a_i a_j^\dagger) \frac{1}{2} \text{Tr} [\mathcal{Y}_i \mathcal{Y}_j^\dagger] \quad (\text{A.23})$$

Because of the symmetry of $\frac{1}{2} \sum a_i a_j^\dagger$, $\frac{1}{2} \text{Tr} [\mathcal{Y}_i \mathcal{Y}_j^\dagger]$ may be symmetrized, since only this part will contribute to Eq. (A.23).

Then

$$\begin{aligned} \frac{1}{2} \text{Tr} [\mathcal{Y}_i \mathcal{Y}_j^\dagger]_{\text{symmetrized}} &= \delta_{ij} [|\mathcal{Y}_1|^2 + |\mathcal{Y}_2|^2 - 2 \cos \theta \text{Re} \mathcal{Y}_1 \mathcal{Y}_2^*] \\ &+ \hat{K}_i \hat{K}_j [|\mathcal{Y}_5|^2 + |\mathcal{Y}_6|^2 - |\mathcal{Y}_2|^2 + 2 \text{Re} (\mathcal{Y}_1 \mathcal{Y}_5^*) \\ &+ 2 \cos \theta \text{Re} (\mathcal{Y}_5 \mathcal{Y}_6^* - \mathcal{Y}_2 \mathcal{Y}_5^*)] \\ &+ \hat{q}_i \hat{q}_j [|\mathcal{Y}_3|^2 + |\mathcal{Y}_4|^2 + 2 \text{Re} (\mathcal{Y}_1^* \mathcal{Y}_4 + \mathcal{Y}_2^* \mathcal{Y}_3 + \cos \theta \mathcal{Y}_3^* \mathcal{Y}_4)] \\ &+ (k_i \hat{q}_j + \hat{q}_i k_j) \text{Re} [\mathcal{Y}_1^* \mathcal{Y}_2 + \mathcal{Y}_1^* \mathcal{Y}_3 + \mathcal{Y}_1^* \mathcal{Y}_6 \\ &+ \mathcal{Y}_2^* \mathcal{Y}_5 + \mathcal{Y}_3^* \mathcal{Y}_5 + \mathcal{Y}_4^* \mathcal{Y}_6 \\ &+ \cos \theta (\mathcal{Y}_4^* \mathcal{Y}_5 + \mathcal{Y}_3^* \mathcal{Y}_6 - \mathcal{Y}_2^* \mathcal{Y}_3)] \end{aligned} \quad (\text{A.24})$$

and

$$\begin{aligned} \frac{1}{2} \sum_{\text{Electron Spins}} a_i a_j^\dagger &= \frac{e^2}{2 m_e^2 K^4} \left\{ \frac{1}{2} K^2 \delta_{ij} + K_{2i} K_{1j} \right. \\ &+ K_{2j} K_{1i} - \frac{K_j}{K_0} (K_{10} K_{2i} + K_{20} K_{1i}) \\ &- \frac{K_i}{K_0} (K_{20} K_{1j} + K_{10} K_{2j}) \\ &\left. - \frac{K_i K_j}{K_0^2} \left(\frac{1}{2} K^2 - 2 K_{20} K_{10} \right) \right\} \end{aligned} \quad (\text{A.25})$$

Any differential cross-section in the π -N centre-of-mass system may be obtained by substituting Eqs. (A.24) and (A.25) into Eq. (A.23) and Eq. (A.22).

A P P E N D I X B

BORN TERMS FOR ELECTRO (PHOTO) PRODUCTION

The Born terms which appear in Eq. (5.9) are derived from the one-nucleon pole in the s-channel, the crossed one-nucleon exchange (in the u-channel), and the t-channel pion exchange. They are shown in Figs. 1a, 1c, 1e and according to the usual Feynman rules have the values

s-channel nucleon pole

$$g \bar{u}(p_2) \gamma_5 \frac{[i\gamma \cdot (K+P) - m]}{m^2 - s} [F_1(K^2) \gamma_\mu + F_2(K^2) \sigma_{\mu\nu} K_\nu] \epsilon_\mu u(p_1) \quad (B.1)$$

u-channel nucleon pole

$$g \bar{u}(p_2) [F_1(K^2) \gamma_\mu + F_2(K^2) \sigma_{\mu\nu} K_\nu] \epsilon_\mu \frac{[i\gamma \cdot (P_2 - K) - m]}{m^2 - u} \gamma_5 u(p_1) \quad (B.2)$$

t-channel pion pole

$$2ig \bar{u}(p_2) \gamma_5 u(p_1) \frac{F_\pi(K^2)}{u^2 - t} Q \cdot \epsilon \quad (B.3)$$

The isospin dependence has been suppressed. Explicitly it is

$$\text{s-channel: } \chi^T(\omega) \left[F_{\frac{1}{2}}^S \tau_\alpha + F_{\frac{1}{2}}^V (\delta_{\alpha 3} + \frac{1}{2} [\tau_\alpha, \tau_3]_-) \right] \chi(\omega)$$

$$\text{u-channel: } \chi^T(\omega) \left[F_{\frac{1}{2}}^S \tau_\alpha + F_{\frac{1}{2}}^V (\delta_{\alpha 3} - \frac{1}{2} [\tau_\alpha, \tau_3]_-) \right] \chi(\omega)$$

$$\text{t-channel: } \chi^T(\omega) F_\pi \frac{1}{2} [\tau_\alpha, \tau_3]_- \chi(\omega)$$

One finds the isospin dependence by associating a factor $(1 + \tau_3)/2$ with the nucleon electromagnetic current, a factor $\tau_3/2$ with the pion electromagnetic current. A factor $\tau_\alpha/2$ is associated with the strong $\pi - N$ vertex. Adding those terms associated with a charge form factor and expanding on the invariants M_i one obtains

$$\begin{aligned}
 \text{Charge Born terms} = & \bar{U}(P_2) M_1 U(P_1) \left\{ \frac{g}{2} (F_1^V \delta_{\alpha 3} + F_1^S \gamma_\alpha) \left[\frac{1}{s-m^2} + \frac{1}{u-m^2} \right] \right. \\
 & \left. + \frac{g}{2} F_1^V \frac{1}{2} [\gamma_\alpha, \gamma_3] - \left[\frac{1}{s-m^2} - \frac{1}{u-m^2} \right] \right\} \\
 & - \bar{U}(P_2) M_2 U(P_1) \left\{ g (F_1^V \delta_{\alpha 3} + F_1^S \gamma_\alpha) \frac{1}{t-\mu^2} \left[\frac{1}{s-m^2} + \frac{1}{u-m^2} \right] \right. \\
 & \left. + g F_1^V \frac{1}{2} [\gamma_\alpha, \gamma_3] - \frac{1}{t-\mu^2} \left[\frac{1}{s-m^2} - \frac{1}{u-m^2} \right] \right\} \\
 & - \bar{U}(P_2) M_5 U(P_1) \left\{ \frac{g}{2} (F_1^V \delta_{\alpha 3} + F_1^S \gamma_\alpha) \frac{1}{t-\mu^2} \left[\frac{1}{s-m^2} - \frac{1}{u-m^2} \right] \right. \\
 & \left. + \frac{g}{2} F_1^V \frac{1}{2} [\gamma_\alpha, \gamma_3] - \frac{1}{t-\mu^2} \left[\frac{1}{s-m^2} + \frac{1}{u-m^2} \right] \right\} \\
 & + \bar{U}(P_2) M_5 U(P_1) \frac{2g}{\kappa^2(t-\mu^2)} (F_\pi - F_1^V) \frac{1}{2} [\gamma_\alpha, \gamma_3] -
 \end{aligned} \tag{B.4}$$

Similarly adding together the anomalous magnetic moment Born terms one obtains

$$\begin{aligned}
 \text{Anomalous Moment Born terms} = & -\bar{U}(P_2) M_3 U(P_1) \left\{ \frac{g}{2} (F_2^V \delta_{\alpha 3} \right. \\
 & \left. + F_2^S \gamma_\alpha) \left[\frac{1}{s-m^2} - \frac{1}{u-m^2} \right] + \frac{g}{2} F_2^V \frac{1}{2} [\gamma_\alpha, \gamma_3] - \left[\frac{1}{s-m^2} + \frac{1}{u-m^2} \right] \right\} \\
 & - \bar{U}(P_2) M_4 U(P_1) \left\{ \frac{g}{2} (F_2^V \delta_{\alpha 3} + F_2^S \gamma_\alpha) \left[\frac{1}{s-m^2} + \frac{1}{u-m^2} \right] \right. \\
 & \left. + \frac{g}{2} F_2^V \frac{1}{2} [\gamma_\alpha, \gamma_3] - \left[\frac{1}{s-m^2} - \frac{1}{u-m^2} \right] \right\}
 \end{aligned} \tag{B.5}$$

As one would expect the crossing properties of the Born terms are the same as derived for the amplitudes A_i .

One can also expand the charge Born terms using the invariant $M_2' = 2i\gamma_5 F_\mu Q_\nu F_{\mu\nu}$. The result is the same for the coefficient of M_1 , but one finds that the coefficient of M_2' has a factor $1/(t - \mu^2 + k^2)$ instead of $1/(t - \mu^2)$ and the coefficient of M_5 is completely different, becoming

$$\frac{2g}{K^2} \left\{ \frac{F_\pi}{t - \mu^2} - \frac{F_1^\nu}{t - \mu^2 + k^2} \right\}$$

APPENDIX C

THE HELICITY FORMALISM

The connection between the traditional multipole description [Eq. (6.3)] and the helicity formalism⁵¹⁾ is given for photopion production.

These are eight helicity amplitudes $f_{\mu,\lambda}(\vartheta, \varphi)$, where μ is the helicity of the final nucleon and $\lambda = \lambda_N - \lambda_\gamma$ is the helicity of the initial state along the nucleon momentum. Parity reduces the number of independent amplitudes to four by the relation

$$f_{\mu,\lambda}(\vartheta, \varphi) = -f_{-\mu,-\lambda}(\vartheta, \pi - \varphi) \quad (C.1)$$

The angular momentum decomposition is given by⁵¹⁾

$$f_{\mu,\lambda}(\vartheta, \varphi) = \sum_{J=\frac{1}{2}}^{\infty} (J + \frac{1}{2}) \langle \mu | T^J | \lambda \rangle e^{i(\lambda - \mu)\varphi} d_{\lambda\mu}^J(\vartheta) \quad (C.2)$$

and the helicity amplitudes normalized by

$$\frac{d\sigma}{d\Omega} = \frac{1}{k} |f_{\mu,\lambda}(\vartheta, \varphi)|^2 \quad (C.3)$$

The connection between the helicity formalism and the multipole formalism is most easily obtained by specifying a co-ordinate frame with $\underline{p}_1$ as the z axis and the xz plane as the production plane, and then making a rotation within this frame to connect the final nucleon helicity state with the z-component nucleon spin state⁵³⁾.

Introducing a vector

$$\begin{aligned} \tilde{f} &= (f_{\frac{1}{2},\frac{3}{2}}, f_{\frac{1}{2},-\frac{3}{2}}, f_{\frac{1}{2},\frac{1}{2}}, f_{\frac{1}{2},-\frac{1}{2}}) \\ &= (H^-, \Phi^+, \Phi^-, H^+) \end{aligned} \quad (C.4)$$

(the latter notation is used in the paper by G. Zweig⁵⁴) one obtains

$$\tilde{f} = \tilde{R} \tilde{f} \quad (C.5)$$

where $\tilde{f}$ is given by Eq. (6.10) and

$$\tilde{R} = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & 0 & \cos \frac{\theta}{2} \sin \theta & \cos \frac{\theta}{2} \sin \theta \\ 0 & 0 & \sin \frac{\theta}{2} \sin \theta & -\sin \frac{\theta}{2} \sin \theta \\ 2 \cos \frac{\theta}{2} & -2 \cos \frac{\theta}{2} & -\sin \frac{\theta}{2} \sin \theta & \sin \frac{\theta}{2} \sin \theta \\ -2 \sin \frac{\theta}{2} & -2 \sin \frac{\theta}{2} & -\cos \frac{\theta}{2} \sin \theta & -\cos \frac{\theta}{2} \sin \theta \end{pmatrix} \quad (C.6)$$

The connection with the multipoles follows from Eq. (6.16), namely

$$\tilde{m}_\ell(s) = \int_{-1}^{+1} dx \mathcal{D}_\ell(x) \tilde{f}(s, t) \quad (C.7)$$

and

$$\tilde{f} = \tilde{R}^{-1} \tilde{f} = \tilde{R}^{-1} \sum_{\ell=0}^{\infty} \tilde{O}_\ell \tilde{T}_\ell \quad (C.8)$$

where Eq. (C.2) has been used in matrix form, with the diagonal matrix

$$\tilde{Q}_\ell = \begin{pmatrix} -\sin \frac{\theta}{2} \left[\sqrt{\frac{\ell}{\ell+2}} P'_{\ell+1} + \sqrt{\frac{\ell+2}{\ell}} P'_\ell \right] \\ \cos \frac{\theta}{2} \left[-\sqrt{\frac{\ell}{\ell+2}} P'_{\ell+1} + \sqrt{\frac{\ell+2}{\ell}} P'_\ell \right] \\ \cos \frac{\theta}{2} [P'_{\ell+1} - P'_\ell] \\ \sin \frac{\theta}{2} [P'_{\ell+1} + P'_\ell] \end{pmatrix} \quad (C.9)$$

and $\ell = j - \frac{1}{2}$.

Substituting Eq. (C.8) in Eq. (C.7) and performing the integration leads to

$$E_{\ell+} = \frac{\sqrt{2}}{i} \frac{1}{4(\ell+1)} \left\{ \sqrt{\frac{\ell}{\ell+2}} [-T_\ell^{(1)} + T_\ell^{(2)}] + [T_\ell^{(3)} - T_\ell^{(4)}] \right\} \quad (C.10)$$

$$E_{(\ell+1)-} = \frac{\sqrt{2}}{i} \frac{1}{4(\ell+1)} \left\{ \sqrt{\frac{\ell+2}{\ell}} [T_\ell^{(1)} + T_\ell^{(2)}] + [T_\ell^{(3)} + T_\ell^{(4)}] \right\} \quad (C.11)$$

$$M_{\ell+} = \frac{\sqrt{2}}{i} \frac{1}{4(\ell+1)} \left\{ \sqrt{\frac{\ell+2}{\ell}} [T_\ell^{(1)} - T_\ell^{(2)}] + [T_\ell^{(3)} - T_\ell^{(4)}] \right\} \quad (C.12)$$

$$M_{(\ell+1)-} = \frac{\sqrt{2}}{i} \frac{1}{4(\ell+1)} \left\{ \sqrt{\frac{\ell}{\ell+2}} [T_\ell^{(1)} + T_\ell^{(2)}] - [T_\ell^{(3)} + T_\ell^{(4)}] \right\} \quad (C.13)$$

and the inverse

$$T_{\ell}(1) = \langle \frac{1}{2} | T_{\ell} | \frac{3}{2} \rangle = \frac{\sqrt{\ell(\ell+2)}}{i\sqrt{2}} \left[E_{\ell+} - M_{\ell+} - E_{(\ell+)-} - M_{(\ell+)-} \right] \quad (C.14)$$

$$T_{\ell}(2) = \langle \frac{1}{2} | T_{\ell} | -\frac{3}{2} \rangle = \frac{\sqrt{\ell(\ell+2)}}{i\sqrt{2}} \left[M_{\ell+} - E_{\ell+} - E_{(\ell+)-} - M_{(\ell+)-} \right] \quad (C.15)$$

$$T_{\ell}(3) = \langle \frac{1}{2} | T_{\ell} | \frac{1}{2} \rangle = \frac{1}{i\sqrt{2}} \left[(\ell+2) (M_{(\ell+)-} - E_{\ell+}) - \ell (M_{\ell+} + E_{(\ell+)-}) \right] \quad (C.16)$$

$$T_{\ell}(4) = \langle \frac{1}{2} | T_{\ell} | -\frac{1}{2} \rangle = \frac{1}{i\sqrt{2}} \left[\ell (M_{\ell+} - E_{(\ell+)-}) + (\ell+2) (M_{(\ell+)-} + E_{\ell+}) \right] \quad (C.17)$$

REFERENCES

- 1) L.D. Roper, Phys.Rev.Letters 12, 340 (1964).
L.D. Roper, R.M. Wright and B.T. Feld, Phys.Rev. 138, B 140 (1965).
- 2) B.H. Bransden, P.J. O'Donnell and R.G. Moorhouse, Physics Letters 11, 339 (1964), Phys.Rev. 139, B 1566 (1965), Physics Letters 19, 420 (1965).
- 3) P. Bareyre, C. Bricman, A.V. Stirling and G. Villet, Physics Letters 18, 342 (1965).
- 4) P. Auvil, A. Donnachie, A.T. Lea and C. Lovelace, Physics Letters 12, 76 (1964).
A. Donnachie, A.T. Lea and C. Lovelace, Physics Letters 19, 146 (1965).
A. Donnachie, R.G. Kirsopp, A.T. Lea and C. Lovelace (in preparation).
- 5) R.H. Dalitz, Proc. of Oxford Intern. Conf. on Elementary Particles. (1965) p. 157.
- 6) R.G. Moorhouse, Phys.Rev.Letters 16, 772 (1966).
- 7) R.H. Dalitz and D.G. Sutherland, Phys.Rev. 146, 1180 (1966).
- 8) A. Donnachie and G. Shaw (to be published in Nuclear Physics).
- 9) M.A. Bég, B.W. Lee and A. Pais, Phys.Rev.Letters 13, 514 (1964).
- 10) H. Harari and H.J. Lipkin, Phys.Rev. 140, B 1617 (1965).
- 11) N. Cabibbo and L.A. Radicati, Physics Letters 19, 697 (1966).
S.L. Adler, Phys.Rev. 143, B 1144 (1966).
- 12) K. Kawarabayashi and M. Suzuki, CALT-68-81.
- 13) S.D. Drell and A.C. Hearn, Phys.Rev.Letters 16, 908 (1966).
- 14) S. Okubo (to be published in Ann.Phys. (N.Y.)).
- 15) M. Gell-Mann, Phys.Rev.Letters 8, 214 (1964).
G. Zweig, CERN pre-print (1964).
- 16) Z. Mati, Prog.Theor.Phys. 31, 331 (1964).
Y. Hara, Phys.Rev. 134, B 701 (1964).
- 17) S. Fubini, G. Furlan and C. Rossetti, Nuovo Cimento 40, 1171 (1965).

- 18) F.J. Gilman and H.J. Schnitzer (to be published in Phys.Rev.).
- 19) A.P. Contogouris, Phys.Rev. 124, 912 (1961).
A.C. Hearn and E. Leader, Phys.Rev. 126, 789 (1962).
- 20) A. Daudin et. al., CEA R-2525 (1964).
- 21) H.R. Crouch et. al., Phys.Rev.Letters 13, 640 (1964).
L.J. Lanzerotti, Phys.Rev.Letters 15, 210 (1965).
- 22) D. Atkinson, Phys.Rev. 128, 1908 (1962).
- 23) C. Lovelace, R.M. Heinz and A. Donnachie, Physics Letters 22, 332 (1966).
- 24) A. Donnachie and G. Shaw, Ann.Phys. (N.Y.) 37, 333 (1966).
- 25) N. Zagury, Phys.Rev. 145, 1112 (1966).
- 26) W. Korth, H. Rollnik, D. Schwela and R. Weizel, pre-print, Phys. Inst. der Universität Bonn (1965).
- 27) P. Finkler, UCRL-7953-T (1964).
- 28) W. Schmidt, Z.Phys. 182, 76 (1964).
- 29) G. Höhler and W. Schmidt, Ann.Phys. (N.Y.) 28, 34 (1964).
- 30) M. Bazin and J. Pine, Physics Letters 5, 168 (1963).
- 31) A.M. Baldin and A.I. Lebedev, Nuclear Physics 40, 44 (1963).
- 32) J. McKinley, Tech.Report No. 38, University of Illinois (1962).
- 33) J.S. Ball, Phys.Rev. 124, 2014 (1961).
- 34) Ph. Dennery, Phys.Rev. 124, 2000 (1961).
- 35) B. de Tollis and A. Verganelakis, Nuovo Cimento 22, 406 (1961).
- 36) A.E.A. Warburton and M. Gourdin, Nuovo Cimento 22, 362 (1961).
- 37) G. Höhler and K. Dietz, Z.Phys. 160, 453 (1960).
- 38) A.M. Baldin, Soviet Physics JETP 11, 416 (1960).
- 39) M. Gourdin, D. Lurié and A. Martin, Nuovo Cimento 18, 933 (1960).
- 40) B. de Tollis, E. Ferrari and H. Munczek, Nuovo Cimento 18, 198 (1960).

- 41) A.M. Baldin and B.B. Govorkov, Nuclear Physics 13, 193 (1959).
- 42) G. Höhler and A. Müllensiefen, Z.Phys. 157, 30 (1959).
- 43) K. Dietz and G. Höhler, Z.Naturforsch. 14a, 994 (1959).
- 44) G.F. Chew, M.L. Goldberger, F.E. Low and Y. Nambu, Phys.Rev. 106, 1345 (1957).
- 45) R.H. Dalitz and D.R. Yennie, Phys.Rev. 105, 1598 (1957).
- 46) J.D. Jackson, Brandeis Lectures, Vol. 1 (1962) (ed. Benjamin).
- 47) A.C. Hearn, Nuovo Cimento 21, 333 (1961).
- 48) S. Fubini, Y. Nambu and V. Wataghin, Phys.Rev. 111, 329 (1958).
- 49) K.M. Watson, Phys.Rev. 95, 228 (1954).
- 50) L.D. Pearlstein and A. Klein, Phys.Rev. 107, 836 (1957).
- 51) M. Jacob and G.C. Wick, Ann.Phys. (N.Y.) 7, 404 (1959).
- 52) A. Donnachie, J. Hamilton and A.T. Lea, Phys.Rev. 135, B 515 (1964).
- 53) J.S. Ball, UCRL-8858.
- 54) G. Zweig, Nuovo Cimento 32, 689 (1964).
- 55) G. Fidecaro et al., CERN pre-print.
- 56) K.M. Watson, Phys.Rev. 95, 228 (1954).
- 57) G. Shaw, Ph.D. Thesis, University of London (unpublished).
- 58) A. Donnachie and J. Hamilton, Phys.Rev. 138, B678 (1965).
- 59) J. Hamilton and W.S. Woolcock, Rev.Mod.Phys. 35, 737 (1963).
- 60) A. Donnachie, Scottish Universities Summer School Lecture Notes (to be published).
- 61) C. Lovelace, Proc. 1966 International Conference on High Energy Physics, Berkeley (to be published).

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FIG.1