FOURTH-ORDER OPERATORS ON MANIFOLDS WITH BOUNDARY

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Abstract. Recent work in the literature has studied fourth-order elliptic operators on manifolds with boundary. This paper proves that, in the case of the squared Laplace operator, the boundary conditions which require that the eigenfunctions and their normal derivative should vanish at the boundary lead to self-adjointness of the boundary-value problem. On studying, for simplicity, the squared Laplace operator in one dimension, on a closed interval of the real line, alternative conditions which also ensure self-adjointness set to zero at the boundary the eigenfunctions and their second derivatives, or their first and third derivatives, or their second and third derivatives, or require periodicity, i.e. a linear relation among the values of the eigenfunctions at the ends of the interval. For the first four choices of boundary conditions, the resulting one-loop divergence is evaluated for a real scalar field on the portion of flat Euclidean 4-space bounded by a 3-sphere, or by two concentric 3-spheres.
1. Introduction

The current attempts to develop quantum field theories of fundamental interactions have led to the consideration of fourth-order or even higher-order differential operators on closed Riemannian manifolds [1–5], or on manifolds with boundary [6, 7]. The analysis of the transformation properties under conformal rescalings of the background metric \( g \) leads, in particular, to the consideration of conformally covariant operators \( P \), which transform according to the law

\[
P_\omega = e^{-(m+4)\omega/2} P(\omega = 0) e^{(m-4)\omega/2}
\]

if \( g \) rescales as \( g_\omega = e^{2\omega} g \), \( m \) being the dimension of the Riemannian manifold which is studied. One of the physical motivations for this analysis lies in the possibility to use the Green functions of such operators to build the effective action in curved space-times [5].

Another enlightening example is provided by the ghost sector of Euclidean Maxwell theory in vacuum in four dimensions. The corresponding field equations are well known to be invariant under conformal rescalings of \( g \). On the other hand, the supplementary (or gauge) conditions usually considered in the literature are not invariant under conformal rescalings of \( g \). Even just in flat Euclidean 4-space, conformal invariance of the supplementary condition is only achieved on making the Eastwood–Singer choice [8]:

\[
\nabla_b \nabla^b \nabla^c A_c = 0
\] (1.2)

where \( A_c \) is the electromagnetic potential (a connection 1-form in geometric language). The preservation of Eq. (1.2) under gauge transformations of \( A_c \):

\[
{f} A_c \equiv A_c + \nabla_c f
\] (1.3)

is achieved provided that \( f \) obeys the fourth-order equation

\[
\Box^2 f = 0
\] (1.4)

where \( \Box^2 \) is the box operator composed with itself:

\[
\Box^2 \equiv \nabla_a \nabla^a \nabla_b \nabla^b.
\]
In the corresponding quantum theory via path integrals, one thus deals with two independent ghost fields (frequently referred to as the ghost and the anti-ghost), both ruled by $\square^2$, which is a fourth-order elliptic operator, and subject to the following boundary conditions (hereafter, $\nabla_N \equiv N^a \nabla_a$ denotes the covariant derivative along the inward-pointing normal $N^a$ to the boundary):

\[ [\varepsilon]_{\partial M} = 0 \quad (1.5) \]

\[ \left[ \nabla_N \varepsilon \right]_{\partial M} = 0. \quad (1.6) \]

Remarkably, since one now deals with a fourth-order elliptic operator, it is insufficient to impose just Dirichlet or Neumann (or Robin) boundary conditions. One needs instead both (1.5) and (1.6), which are obtained from the following requirements:

(i) Gauge invariance of the boundary conditions on $A_\mu$ [6, 7].

(ii) Conformal invariance of the whole set of boundary conditions [7].

(iii) Self-adjointness of the $\square^2$ operator (see section 2).

Although it remains extremely difficult to build a consistent quantization scheme via path-integral formalism for the full Maxwell field in the Eastwood–Singer gauge (the gauge-field operator on $A_\mu$ perturbations being, then, of sixth order [6, 7]), the investigation of the ghost sector remains of considerable interest in this case. There is in fact, on the one hand, the need to understand how to quantize a gauge theory in a way which preserves conformal invariance at all stages (as we just said), and on the other hand the attempt to extend the recent work on conformally covariant operators [1–5] to the more realistic case of manifolds with boundary.

Our paper begins, therefore, with a detailed derivation of the boundary conditions which ensure self-adjointness of the operator $\frac{d^4}{dx^4}$. For simplicity, the analysis is limited to one-dimensional problems, but the key properties are not affected by this sort of simplification. Section 3 proves the strong ellipticity and self-adjointness of the resulting boundary value problem. Section 4 studies the one-loop properties of a scalar field on a portion of flat Euclidean 4-space bounded by a 3-sphere, when the field is ruled by the $\square^2$ operator.
and is subject to the boundary conditions (1.5) and (1.6). Section 5 extends the analysis of section 4 to the part of flat Euclidean 4-space bounded by two concentric 3-spheres. Results and open problems are discussed in section 6, and relevant details are described in the appendix.

2. Self-adjointness of the operator $\frac{d^4}{dx^4}$

We are concerned with the squared Laplace operator acting on scalar fields on a flat Euclidean background, in the case when curvature effects result from the boundary only. Moreover, motivated by quantum cosmology and Euclidean quantum gravity, the boundary is assumed to be a 3-sphere of radius $a$, or a pair of concentric 3-spheres [6, 7]. The former case, in particular, may be viewed as the limiting case when the wave function of the universe is studied at small 3-geometries (i.e. as $a \to 0$), as shown in [9].

In our problem it is hence possible to expand the scalar field on a family of 3-spheres centred on the origin, according to the familiar relation [10]

$$\varepsilon(x, \tau) = \sum_{n=1}^{\infty} \varepsilon_n(\tau)Q(n)(x)$$

where $\tau \in [0, a]$, $Q(n)$ are the scalar harmonics on a unit 3-sphere, $S^3$, and $x$ are local coordinates on $S^3$. Thus, one is eventually led to study a one-dimensional differential operator of fourth order, and this makes it clear why all the essential information is obtained by the analysis of the operator $B \equiv \frac{d^4}{dx^4}$ on a closed interval of the real line, say $[0, 1]$. The operator $B$ is required to act on functions which are at least of class $C^4$ (see (2.22)), and the following definition of scalar product (anti-linear in the first argument) is considered:

$$\langle u, v \rangle \equiv \int_0^1 u^*(x)v(x)dx.$$  (2.2)

We now want to study under which conditions the operator $B$ is self-adjoint, which means that it should be symmetric, and its domain $D(B)$ should coincide with the domain of the adjoint $B^\dagger$. For this purpose, we first study the relation between the scalar products
\((Bu,v)\) and \((u,Bv)\). We have then to integrate repeatedly by parts, using the Leibniz rule to express
\[
\frac{d}{dx} \left( \frac{d^3 u^*}{dx^3 v} \right), \quad \frac{d}{dx} \left( \frac{d^2 u^*}{dx^2 d^2 v} \right), \quad \frac{d}{dx} \left( \frac{d u^*}{dx} \frac{d^2 v}{dx^2} \right), \quad \frac{d}{dx} \left( \frac{d^3 v}{dx^3} \right).
\]
This leads to
\[
(Bu,v) = \left[ \frac{d^3 u^*}{dx^3 v} \right]^1_0 - \left[ \frac{d^2 u^*}{dx^2 d^2 v} \right]^1_0 + \left[ \frac{du^*}{dx} \frac{d^2 v}{dx^2} \right]^1_0 - \left[ \frac{u^* d^2 v}{dx^3} \right]^1_0 + (u,Bv).
\]
(2.3)

Bearing in mind that the adjoint, \(B^\dagger\), of \(\frac{d^4}{dx^4}\) is again the operator \(\frac{d^4}{dx^4}\), it is thus clear that the condition \((Bu,v) = (u,B^\dagger v)\) is fulfilled provided that both \(u \in D(B)\) and \(v \in D(B^\dagger)\) obey the same boundary conditions, for which the four terms expressing the difference \((Bu,v) - (u,B^\dagger v)\) are found to vanish. Some ways to achieve this are as follows.

(i) First option:
\[
\begin{align*}
u(0) & = u(1) = 0 \quad u'(0) = u'(1) = 0 \\
v(0) & = v(1) = 0 \quad v'(0) = v'(1) = 0.
\end{align*}
\]
(2.4) (2.5)

(ii) Second option:
\[
\begin{align*}
u(0) & = u(1) = 0 \quad u''(0) = u''(1) = 0 \\
v(0) & = v(1) = 0 \quad v''(0) = v''(1) = 0.
\end{align*}
\]
(2.6) (2.7)

(iii) Third option:
\[
\begin{align*}u'(0) & = u'(1) = 0 \quad u'''(0) = u'''(1) = 0 \\
v'(0) & = v'(1) = 0 \quad v'''(0) = v'''(1) = 0.
\end{align*}
\]
(2.8) (2.9)

(iv) Fourth option:
\[
\begin{align*}u''(0) & = u''(1) = 0 \quad u'''(0) = u'''(1) = 0 \\
\end{align*}
\]
(2.10)
\begin{align*}
  v''(0) &= v''(1) = 0 \quad v'''(0) = v'''(1) = 0. 
  \end{align*}
  \tag{2.11}

(v) Periodic boundary conditions:
\begin{align*}
  \frac{u(1)}{u(0)} &= \beta \quad \tag{2.12} \\
  \frac{u'(1)}{u'(0)} &= \gamma \quad \tag{2.13} \\
  \frac{u''(1)}{u''(0)} &= \frac{1}{\gamma^*} \quad \tag{2.14} \\
  \frac{u'''(1)}{u'''(0)} &= \frac{1}{\beta^*}. \quad \tag{2.15}
\end{align*}

and the same for \( v \in D(B^\dagger) \), where \( \beta \) and \( \gamma \) are some constants, not necessarily equal. This is made possible by the fourth-order nature of our operator. By contrast, if we were studying the first-order operator \( i \frac{d}{dx} \) on the set of absolutely continuous functions on \([0, 1]\), periodic boundary conditions leading to self-adjointness would involve one and the same complex parameter \([11]\).

The solutions of the eigenvalue equation for the operator \( B \), i.e.
\begin{equation}
  Bu \equiv \frac{d^4u}{dx^4} = \lambda u 
  \tag{2.16}
\end{equation}

read
\begin{equation}
  u(x) = C_1 \cos \rho x + C_2 \sin \rho x + C_3 \cosh \rho x + C_4 \sinh \rho x \tag{2.17}
\end{equation}

where \( \rho \equiv \lambda^{1/4} \). In particular, the periodic boundary conditions \((2.12)-(2.15)\) lead to a linear algebraic system for the evaluation of the coefficients \( C_1, C_2, C_3 \) and \( C_4 \) which admits non-trivial solutions if and only if the determinant of the following matrix vanishes:
\begin{align*}
  \begin{pmatrix}
    (\cos \rho - \beta) & \sin \rho & (\cosh \rho - \beta) & \sinh \rho \\
    -\sin \rho & (\cos \rho - \gamma) & \sin \rho & (\cosh \rho - \gamma) \\
    (\sin \rho - \frac{1}{\gamma^*}) & -\sin \rho & (\cosh \rho - \frac{1}{\gamma^*}) & \sinh \rho \\
    \sin \rho & (\sin \rho - \frac{1}{\beta^*}) & (\cosh \rho - \frac{1}{\beta^*}) & \sinh \rho
  \end{pmatrix}.
\end{align*}
The above determinant, denoted by \( \delta \), turns out to have the form

\[
\delta = F_1 + F_2(\cos \rho + \cosh \rho) + F_3(\cos \rho)(\cosh \rho)
\]

(2.18)

where

\[
F_1 \equiv 4 + 2 \frac{\beta}{\beta^*} + 2 \frac{\gamma}{\gamma^*} + 2 \left( \frac{2\beta\gamma + 1}{\beta^*\gamma^*} \right) + 2\beta\gamma
\]

(2.19)

\[
F_2 \equiv - \left[ 2(\beta + \gamma) + 2(\beta\gamma + 1) \left( \frac{1}{\beta^*} + \frac{1}{\gamma^*} \right) + 2 \frac{(\beta + \gamma)}{\beta^*\gamma^*} \right]
\]

(2.20)

\[
F_3 \equiv 2 \left[ \frac{(\beta + 2\gamma)}{\beta^*} + \frac{(2\beta + \gamma)}{\gamma^*} + \beta\gamma + \frac{1}{\beta^*\gamma^*} \right].
\]

(2.21)

To sum up, if the conditions (2.4) and (2.5), or (2.6) and (2.7), or (2.8) and (2.9), or (2.10) and (2.11), or (2.12)-(2.15) are satisfied, the domains of \( B \) and of its adjoint turn out to coincide:

\[
D(B) = D(B^\dagger) \equiv \{ u : u \in AC^4[0,1], (2.4) \text{ or } (2.6) \}
\]

or (2.8) or (2.10) or (2.12) \(- (2.15) \) hold \}. (2.22)

With our notation, \( AC^4[0,1] \) is the set of functions in \( L^2[0,1] \) whose weak derivatives up to third order are absolutely continuous in \([0,1]\), which ensures that the weak derivatives, up to fourth order, are Lebesgue summable in \([0,1]\), and that all \( u \) in the domain are of class \( C^4 \) on \([0,1]\). Of course, symmetry of \( B \) is also obtained with the boundary conditions just described.

In other words, at least five sets of boundary conditions, (i) or (ii) or (iii) or (iv) or (v), can be chosen to ensure self-adjointness of the operator \( \frac{d^4}{dx^4} \). Hereafter, we first consider the option (i), since, as was stated in the introduction, it is the one which agrees with boundary conditions motivated by the request of gauge invariance and conformal invariance, if the scalar field is viewed as one of the two ghost fields of Euclidean Maxwell theory in the Eastwood–Singer gauge. We also stress again that nothing is lost on studying just the “prototype” operator \( \frac{d^4}{dx^4} \). The one-dimensional fourth-order operator may take a more complicated form in some set of local coordinates (see (4.1)), but is always reducible to the operator \( \frac{d^4}{dx^4} \) on the real line (more precisely, a closed interval of \( \Re \) in our problems).
3. Strong ellipticity and self-adjointness of the boundary value problem

A number of points discussed in the previous section need to be put on firmer ground. In particular, we are concerned with the issue of ellipticity of the boundary value problem. This is studied in terms of the leading symbol of our differential operator, which is a squared Laplacian on a Riemannian manifold with smooth boundary. It is indeed well known that the Fourier transform makes it possible to associate to a differential operator of order $k$ a polynomial of degree $k$, called characteristic polynomial or symbol. The leading symbol, $\sigma_L$, picks out the highest order part of this polynomial. For a squared Laplacian, denoted by $F$, it reads

$$\sigma_L(F; x, \xi) = |\xi|^4 I = g^{\mu\nu} g^{\rho\sigma} \xi_\mu \xi_\nu \xi_\rho \xi_\sigma I.$$  \hspace{1cm} (3.1)

With a standard notation, $x$ are local coordinates on $M$, and $\xi_\mu$ are cotangent vectors: $\xi_\mu \in T^*(M)$. The leading symbol of $F$ is trivially elliptic in the interior of $M$, since the right-hand side of (3.1) is positive-definite, and one has

$$\det(\sigma_L(F; x, \xi) - \lambda) = (|\xi|^4 - \lambda)^{\dim V} \neq 0$$  \hspace{1cm} (3.2)

for all $\lambda \in \mathbb{C} - \mathbb{R}_+$, where $V$ is the vector bundle over $M$ whose sections are the physical fields $\varphi$, acted upon by $F : C^\infty(V, M) \to C^\infty(V, M)$. In the presence of a boundary, however, one needs a more careful definition of ellipticity. First, for a manifold $M$ of dimension $m$, the $m$ coordinates $x$ are split into $m - 1$ local coordinates on $\partial M$, denoted by $\{\hat{x}^k\}$, and $r$, the geodesic distance to the boundary. Similarly, the $m$ coordinates $\xi_\mu$ are split into $m - 1$ cotangent vectors $\zeta_j \in T^*(\partial M)$, jointly with a real parameter $\omega \in T^*(\mathbb{R})$. The ellipticity we are interested in requires now that $\sigma_L$ should be elliptic in the interior of $M$, as specified before, and that strong ellipticity should hold. This means that a unique solution exists of the eigenvalue equation for the leading symbol:

$$\left[\sigma_L(F; \{\hat{x}^k\}, r = 0, \{\zeta_j\}, \omega \to -i \partial_r) - \lambda\right] \varphi(r) = 0$$  \hspace{1cm} (3.3)

subject to the boundary conditions and to a decay condition at infinity. Before defining these concepts, note that, in (3.3), $i\omega$ is eventually replaced by the operator of first derivative with respect to the geodesic distance to the boundary.
A complete formulation of boundary conditions needs some abstraction. For this purpose, one has to consider two vector bundles, \( W_F \) and \( W'_F \), over the boundary of \( M \), with a boundary operator \( B_F \), relating their sections, i.e.

\[
B_F : C^\infty(W_F, \partial M) \rightarrow C^\infty(W'_F, \partial M).
\]

All the information about normal derivatives of the fields is not encoded in \( B_F \) but in the boundary data \( \psi_F(\varphi) \in C^\infty(W_F, \partial M) \). For example, with boundary conditions involving \( \varphi \) and its first normal derivative, one has

\[
\psi_F(\varphi) = \begin{pmatrix} [\varphi]_{\partial M} \\ [\nabla N \varphi]_{\partial M} \end{pmatrix}
\]

(3.4)

\[
B_F = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}
\]

(3.5)

and the boundary conditions read \( B_F \psi_F(\varphi) = 0 \). The sections of the bundle \( W'_F \), which remained unspecified so far, are obtained by applying to the sections of \( W_F \) the operator whose main diagonal coincides with the main diagonal of \( B_F \). More precisely, if the boundary conditions are mixed, on writing \( B_F = P_F L \) for some projector \( P_F : W_F \rightarrow W'_F \) and some operator

\[
L : C^\infty(W_F, \partial M) \rightarrow C^\infty(W_F, \partial M)
\]

one has \( \psi'_F \in C^\infty(W'_F, \partial M) \) realized as \( \psi'_F = P_F \chi \), for some \( \chi \in C^\infty(W_F, \partial M) \). However, when the boundary operator (3.5) is considered, the projector \( P_F \) is turned into \( B_F \), and the strong ellipticity condition demands that a unique solution of Eq. (3.3) should exist, subject to the boundary condition

\[
\sigma_g(B_F)(\{\hat{x}^k\}, \{\zeta_j\}) \psi_F(\varphi) = \psi'_F(\varphi) \quad \forall \psi'_F(\varphi) \in C^\infty(W'_F, \partial M)
\]

(3.6)

and to the asymptotic condition

\[
\lim_{r \to \infty} \varphi(r) = 0.
\]

(3.7)
With a standard notation [7, 12], $\sigma_g(B_F)$ is the graded leading symbol of the boundary operator $B_F$ in the local coordinates $\{\hat{x}^k\}, \{\zeta_j\}$. When $B_F$ takes the form (3.5), $\sigma_g(B_F)$ may be defined by

$$\sigma_g(B_F) \equiv \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$  

Similarly to the case of the differential operator acting on physical fields, one is here mapping the boundary operator into its counterpart via Fourier transform. In the case of mixed boundary conditions for operators of Laplace type [13], $B_F$ has off-diagonal elements which are first-order tangential operators, whereas complementary projectors occur on the main diagonal. One then finds a more elaborated structure [13]:

$$\sigma_g(B_F) = \begin{pmatrix} \Pi & 0 \\ iT & I - \Pi \end{pmatrix}$$

where $T$ is an anti-self-adjoint matrix.

The asymptotic condition (3.7) picks out the solutions of the eigenvalue equation (3.3) which satisfy (3.6) with arbitrary boundary data $\psi'_F(\varphi)$ and vanish at infinite geodesic distance to the boundary. When all the above conditions are satisfied $\forall \zeta \in T^*(\partial M), \forall \lambda \in \mathbb{C} - \mathbb{R}_+, \forall (\zeta, \lambda) \neq (0, 0)$ and $\forall \psi'_F \in C^\infty(W'_F, \partial M)$, one says that the boundary value problem $(F, B_F)$ for the squared Laplacian is strongly elliptic with respect to the cone $\mathbb{C} - \mathbb{R}_+$. Following [14], we find the solution of (3.3), (3.6) and (3.7) in the form ($\chi_1$ and $\chi_2$ being some constants)

$$\varphi(r) = \chi_1 e^{-\rho_1 r} + \chi_2 e^{-\rho_2 r}$$  

where, on setting

$$|\zeta| \equiv \sqrt{g^{ij}(\hat{x})\zeta_i \zeta_j}$$  

$$\mu \equiv \left(|\zeta|^4 - \lambda\right)^{\frac{1}{4}}$$

we define

$$\rho_1 \equiv \frac{\mu}{\sqrt{2}}(1 + i)$$

10
\[ \rho_2 \equiv \frac{\mu}{\sqrt{2}} (1 - i). \quad (3.13) \]

In general, \( \mu \) has both a real part \( \mu_1 \) and an imaginary part \( \mu_2 \). The inequalities on \( \mu_1 \) and \( \mu_2 \) which are compatible with (3.9) and (3.12), (3.13) are \([14]\)

\[ (\mu_1 + \mu_2) > 0 \text{ and } (\mu_1 - \mu_2) > 0. \]

The boundary condition (3.6) leads to the equation \( A\chi = \psi \), where

\[ A \equiv \begin{pmatrix} 1 & 1 \\ -\rho_1 & -\rho_2 \end{pmatrix}. \quad (3.14) \]

This matrix is trivially invertible, and hence existence and uniqueness of the solution is guaranteed. On writing \( \psi' \equiv \begin{pmatrix} \psi_0' \\ \psi_1' \end{pmatrix} \), where, according to the rule described after (3.5), one has (for some constants \( \gamma_1 \) and \( \gamma_2 \))

\[ \psi_0' = \gamma_1 + \gamma_2 \quad (3.15) \]
\[ \psi_1' = -\rho_1 \gamma_1 - \rho_2 \gamma_2 \quad (3.16) \]

one finds

\[ \chi_1 = \gamma_1 \quad (3.17) \]
\[ \chi_2 = \gamma_2. \quad (3.18) \]

As stressed in [12], the condition of strong ellipticity is essential to ensure the existence of the asymptotic expansions normally assumed in the theory of heat-kernel asymptotics. In other words, if one cannot prove strong ellipticity for a given choice of boundary conditions, the local asymptotics of the fibre trace of the heat-kernel diagonal, and the corresponding, global asymptotics (resulting from integration over \( M \)) do not contain just the terms whose occurrence is ensured by invariance theory and by the Weyl theorem on the invariants of the orthogonal group [12]. There are, instead, highly singular contributions to the heat-kernel diagonal, so that their integral over \( M \) does not exist in any sense.
It is therefore reassuring to see that, precisely in the case more relevant for quantum field theory [6, 7], strong ellipticity holds, and hence the resulting one-loop theory is well defined. The following sections are devoted to a detailed evaluation of such one-loop approximation, but now we should clarify the self-adjointness issue for the squared Laplacian. For this purpose, let us recall that, given a symmetric operator $A$, with adjoint $A^\dagger$, its self-adjointness property can be studied by evaluating the dimension of the space of solutions of the equation $A^\dagger \varphi = i \varphi$, jointly with the corresponding dimension for the equation $A^\dagger \varphi = -i \varphi$. More precisely, one defines the deficiency sub-spaces [11]

$$\mathcal{H}_+ \equiv \text{Ker}(i - A^\dagger)$$

$$\mathcal{H}_- \equiv \text{Ker}(i + A^\dagger)$$

and the associated deficiency indices [11]

$$n_+(A) \equiv \text{dim}[\mathcal{H}_+]$$

$$n_-(A) \equiv \text{dim}[\mathcal{H}_-].$$

Two theorems are then very useful [11]:

**Theorem 3.1.** Given a closed symmetric operator $A$ with deficiency indices $n_+$ and $n_-$, $A$ is self-adjoint if and only if $n_+ = 0 = n_-$. Moreover, $A$ has self-adjoint extensions if and only if $n_+ = n_-$, and a one-one correspondence exists between self-adjoint extensions of $A$ and unitary maps from $\mathcal{H}_+$ onto $\mathcal{H}_-$.

**Theorem 3.2.** If $A$ is a symmetric operator with domain $D(A)$, and if a conjugation $C$ exists which maps $D(A)$ into $D(A)$ and commutes with $A$: $CA = AC$, then $n_+(A) = n_-(A)$, and hence $A$ has self-adjoint extensions.

In the case of the operator $B \equiv \frac{d^4}{dx^4}$ studied in section 2, since complex conjugation commutes with $B$, we immediately know from theorem 3.2 that the deficiency indices of $B$ are equal. The solutions of the equations $B^\dagger \varphi = \pm i \varphi$ on $L^2(0, \infty)$ (we shall later restrict to $[0,1]$) are weak solutions. However, by virtue of the elliptic regularity theorem,
these solutions are infinitely differentiable and hence strong solutions \cite{11}. Now the strong solutions of
\[
\frac{d^4}{dx^4} \varphi(x) = i\varphi(x)
\]  
(3.23)
can be written in the form \(\varphi(x) = e^{\alpha x}\), with \(\alpha\) a root of the equation \(\alpha^4 = i\). One then finds
\[
\alpha_1 = e^{i\frac{\pi}{8}} = \cos \frac{\pi}{8} + i\sin \frac{\pi}{8}
\]  
(3.24)
\[
\alpha_2 = ie^{i\frac{\pi}{8}} = -\sin \frac{\pi}{8} + i\cos \frac{\pi}{8}
\]  
(3.25)
\[
\alpha_3 = -e^{i\frac{\pi}{8}} = -\cos \frac{\pi}{8} - i\sin \frac{\pi}{8}
\]  
(3.26)
\[
\alpha_4 = -ie^{i\frac{\pi}{8}} = \sin \frac{\pi}{8} - i\cos \frac{\pi}{8}
\]  
(3.27)
Thus, only the strong solutions \(\varphi_2(x) \equiv e^{\alpha_2 x}\) and \(\varphi_3(x) \equiv e^{\alpha_3 x}\) are in \(L^2(0, \infty)\), and \(n_+(B) = 2\). Similarly, the strong solutions of the equation
\[
\frac{d^4}{dx^4} \varphi(x) = -i\varphi(x)
\]  
(3.28)
can be written in the form \(\varphi(x) = e^{\beta x}\), with \(\beta\) a root of the equation \(\beta^4 = -i\). One then finds
\[
\beta_1 = e^{-i\frac{\pi}{8}} = \cos \frac{\pi}{8} - i\sin \frac{\pi}{8}
\]  
(3.29)
\[
\beta_2 = ie^{-i\frac{\pi}{8}} = \sin \frac{\pi}{8} + i\cos \frac{\pi}{8}
\]  
(3.30)
\[
\beta_3 = -e^{-i\frac{\pi}{8}} = -\cos \frac{\pi}{8} + i\sin \frac{\pi}{8}
\]  
(3.31)
\[
\beta_4 = -ie^{-i\frac{\pi}{8}} = -\sin \frac{\pi}{8} - i\cos \frac{\pi}{8}
\]  
(3.32)
which implies that only the strong solutions \(\varphi_3(x) \equiv e^{\beta_3 x}\) and \(\varphi_4(x) \equiv e^{\beta_4 x}\) are in \(L^2(0, \infty)\), and hence \(n_-(B) = 2 = n_+(B)\). This property suggests also a non-trivial link with the strong ellipticity analysis, where we have seen that the asymptotic condition (3.7) selects only two of the original four contributions to the solution (3.9).
However, on $L^2[0,1]$, all strong solutions resulting from (3.24)–(3.27) and (3.29)–(3.32) are acceptable, and the deficiency indices of the operator $B$ are, therefore, $n_+(B) = n_-(B) = 4$. The domains for $B$ studied in section 2 and summarized in equation (2.22) are hence correctly interpreted as domains of self-adjoint extensions of $B$.

4. One-loop divergence on the Euclidean 4-ball

The definition and evaluation of functional determinants remains a topic of crucial importance in quantum field theory. Here the task is even more interesting, because we are studying a fourth-order elliptic operator on a manifold with boundary. As shown in [6, 7], the resulting eigenvalue equation for the modes occurring in the expansion (2.1) turns out to be, on the Euclidean 4-ball,

$$
\frac{d^4}{d\tau^4} + \frac{6}{\tau} \frac{d^3}{d\tau^3} - \frac{(2n^2 - 5)}{\tau^2} \frac{d^2}{d\tau^2} - \frac{(2n^2 + 1)}{\tau^3} \frac{d}{d\tau} + \frac{(n^2 - 1)^2}{\tau^4} \epsilon_n = \lambda_n \epsilon_n.
$$

(4.1)

Thus, on setting $M \equiv \lambda_n^{1/4}$, the solution of equation (4.1) is expressed by a linear combination of Bessel functions and modified Bessel functions [7], i.e.

$$
\epsilon_n(\tau) = A_{1,n} \frac{I_n(M\tau)}{\tau} + A_{2,n} \frac{K_n(M\tau)}{\tau} + A_{3,n} \frac{J_n(M\tau)}{\tau} + A_{4,n} \frac{N_n(M\tau)}{\tau}.
$$

(4.2)

Since the Euclidean 4-ball consists of a portion of flat Euclidean 4-space bounded by a 3-sphere, the coefficients $A_{2,n}$ and $A_{4,n}$ have to vanish $\forall n \geq 1$, to ensure regularity of $\epsilon_n$ at the origin. One is thus left with scalar modes of the form

$$
\epsilon_n(\tau) = A_{1,n} \frac{I_n(M\tau)}{\tau} + A_{3,n} \frac{J_n(M\tau)}{\tau}.
$$

(4.3)

We focus on a $\zeta(0)$ calculation for such a set of massless modes, subject to the boundary conditions (see (1.5), (1.6) and (2.4), (2.5))

$$
[\epsilon_n]_{\partial M} = 0
$$

(4.4)
because the resulting 1-loop analysis remains crucial in the course of studying quantum theory as a theory of small disturbances [15] of the underlying classical theory. Our calculation relies on the technique developed in [16] and applied several times by the present authors (see [17] and references therein). The starting point is the remark that, since $\zeta$-functions are $L^2$-traces of complex powers of elliptic operators, they admit an integral representation with the help of the Cauchy formula. For example, for a function $f$ analytic in the domain bounded by a curve $\gamma$, one has

$$\sum_{l=1}^{L} n_l f(z_l) = \frac{1}{2\pi i} \int_{\gamma} f(z) \frac{d}{dz} \log F(z) \, dz$$

(4.6)

where $F$ is a function having zeros at $z_1,\ldots,z_L$ with multiplicities $n_1,\ldots,n_L$, respectively. Thus, on choosing $f(z) \equiv z^{-s}$, one finds the desired integral representation of the $\zeta$-function in the form

$$\zeta(s) = \frac{1}{2\pi i} \int_{\gamma} z^{-s} \frac{d}{dz} \text{Tr} \log Q(z) \, dz$$

(4.7)

where $\gamma$ is the contour in the complex-$z$ plane which encircles all roots of the equation $Q(z) = 0$, with $Q$ the function expressing the equation obeyed by the eigenvalues by virtue of the boundary conditions. The contour $\gamma$ is then deformed into a new contour $\tilde{\gamma}$, which encircles the cut in the complex plane of the function $z^{-s}$, coinciding with the negative real axis. After some technical steps, one eventually finds

$$\zeta(s) = \frac{1}{2\pi i} \int_{\gamma} z^{-s} \frac{d}{dz} I(-z,s) \, dz$$

(4.8)

where $I(-z,s)$ is the regularized infinite sum defined by $I(-z,s) \equiv \sum_{n} n^{-2s} \log Q(z)$. More precisely, on denoting now by $f_n$ the function occurring in the equation obeyed by the eigenvalues by virtue of the boundary conditions, after taking out false roots (e.g.
$x = 0$ is a false root of the equation $J_\nu(x) = 0$, and writing $d(n)$ for the degeneracy of the eigenvalues parametrized by the integer $n$, one defines the function

$$I(M^2, s) \equiv \sum_{n=n_0}^{\infty} d(n)n^{-2s}\log f_n(M^2). \quad (4.9)$$

What is very useful is the analytic continuation “$I(M^2, s)$” to the complex-$s$ plane of the function $I(M^2, s)$, which is a meromorphic function with a simple pole at $s = 0$, i.e.

$$“I(M^2, s)" = \frac{I_{\text{pole}}(M^2)}{s} + I^R(M^2) + O(s). \quad (4.10)$$

The function $I_{\text{pole}}$ is the residue at $s = 0$, and makes it possible to obtain the $\zeta(0)$ value as

$$\zeta(0) = I_{\log} + I_{\text{pole}}(M^2 = \infty) - I_{\text{pole}}(M^2 = 0) \quad (4.11)$$

where $I_{\log}$ is the coefficient of the $\log(M)$ term in $I^R$ as $M \to \infty$. The contributions $I_{\log}$ and $I_{\text{pole}}(\infty)$ are obtained from the uniform asymptotic expansions of basis functions as $M \to \infty$ and their order $n \to \infty$, whilst $I_{\text{pole}}(0)$ is obtained by taking the $M \to 0$ limit of the eigenvalue condition, and then studying the asymptotics as $n \to \infty$. More precisely, $I_{\text{pole}}(\infty)$ coincides with the coefficient of $\frac{1}{n}$ in the asymptotic expansion as $n \to \infty$ of

$$\frac{1}{2}d(n)\log[\rho_\infty(n)]$$

where $\rho_\infty(n)$ is the $n$-dependent term in the eigenvalue condition as $M \to \infty$ and $n \to \infty$. The $I_{\text{pole}}(0)$ value is instead obtained as the coefficient of $\frac{1}{n}$ in the asymptotic expansion as $n \to \infty$ of

$$\frac{1}{2}d(n)\log[\rho_0(n)]$$

where $\rho_0(n)$ is the $n$-dependent term in the eigenvalue condition as $M \to 0$ and $n \to \infty$ [16, 17]. Although such a technique was originally developed for second-order elliptic operators, it can be easily generalized to study our fourth-order operators, provided that one bears in
mind that the eigenvalues have now dimension [length]$^{-4}$ (see notation after (4.1)). Hence one should replace $M^2$ by $M^4$ in (4.9)–(4.11), but the final results remain unaffected.

In our problem, the equations (4.3)–(4.5) lead to the eigenvalue condition (denoting by $a$ the radius of the 3-sphere)

$$\det \begin{pmatrix} I_n(Ma) & J_n(Ma) \\ -I_n(Ma) + MaI'_n(Ma) & -J_n(Ma) + MaJ'_n(Ma) \end{pmatrix} = 0 \tag{4.12}$$

which guarantees that non-trivial solutions exist for the coefficients $A_{1,n}$ and $A_{3,n}$ in (4.3).

At this stage, on using the limiting form of Bessel functions $I_n$ and $J_n$ when the argument tends to zero (see (A.1)–(A.4)), one finds that the left-hand side of (4.12) is proportional to $M^{2n}$ as $M \to 0$. Hence one has to multiply by $M^{-2n}$ to get rid of false roots. Moreover, in the uniform asymptotic expansion of Bessel functions as $M \to \infty$ and $n \to \infty$, both $I$ and $J$ functions contribute a $\frac{1}{\sqrt{M}}$ factor (see (A.5), (A.7) and (A.12)). These properties imply that $I_{\log}$ takes the value

$$I_{\log} = \frac{1}{2} \sum_{n=1}^{\infty} n^2 (-2n) = -\zeta_R(-3) = -\frac{1}{120}. \tag{4.13}$$

The calculation of $I_{\text{pole}}(\infty)$ relies on the asymptotic expansions (A.5), (A.7) and (A.12) of the appendix. One then finds that no $n$-dependent term occurs in the eigenvalue condition (4.12), which implies

$$I_{\text{pole}}(\infty) = 0. \tag{4.14}$$

Last, $I_{\text{pole}}(0)$ is obtained after working out $\rho_0(n)$ for (4.12). For this purpose, we remark that, as $M \to 0$ and $n \to \infty$, the first line of the matrix in (4.12) consists of two elements both equal to $\frac{1}{\Gamma(n+1)}$, whereas the second line consists of two elements both equal to $\frac{(n-1)}{\Gamma(n+1)}$ (bearing in mind that all powers of $(Ma/2)$ can be safely omitted, if one is interested in $\rho_0(n)$). Hence one finds exact cancellation of the contributions to $I_{\text{pole}}(0)$, i.e.

$$I_{\text{pole}}(0) = 0. \tag{4.15}$$
By virtue of (4.11) and (4.13)–(4.15) one finds, on the Euclidean 4-ball,

$$\zeta(0) = -\frac{1}{120}$$

(4.16)

for a real, massless scalar field.

It is interesting to notice that the effect of cancellation of contributions to $I_{\text{pole}}(0)$ arises because, whilst recurrence formulae for derivatives of Bessel functions:

$$2J_n'(z) = J_{n-1}(z) - J_{n+1}(z)$$

(4.17)

and those for modified Bessel functions:

$$2I_n'(z) = I_{n-1}(z) + I_{n+1}(z)$$

(4.18)

have different signs before $J_{n+1}(z)$ and $I_{n+1}(z)$, respectively, only $J_{n-1}(z)$ and $I_{n-1}(z)$ survive in the limit as $M \to 0$, and hence the determinant (4.12) is equal to zero in that limit. On differentiating the recurrence formulae (4.17) and (4.18) the appropriate number of times, one can easily check that the corresponding determinants for the boundary conditions (ii), (iii) and (iv) of section 2 are also equal to zero at $M = 0$. Thus, the contributions to $I_{\text{pole}}(0)$ with these boundary conditions vanish as well.

Moreover, the expression (4.13) for $I_{\text{log}}$ is trivially modified by adding to $(-2n)$ the integer numbers 1 for (ii), 3 for (iii) and 4 for (iv), respectively. Their contributions are proportional to

$$\sum_{n=1}^{\infty} n^2 = \zeta_R(-2) = 0.$$ 

Thus, the results (4.13) and (4.16) hold for the first four types of boundary conditions.

5. One-loop divergence in the two-boundary problem

In the two-boundary problem one studies a portion of flat Euclidean 4-space bounded by two concentric 3-spheres. This case is very interesting because it is more directly related to the familiar framework in quantum field theory, where one normally assigns boundary
data on two three-surfaces (it should be stressed, however, that unlike scattering problems
we are considering a path-integral representation of amplitudes in a finite region).

On denoting by \(a\) and \(b\), with \(a > b\), the radii of the two concentric 3-sphere boundaries,
we can consider the complete form (4.2) of our scalar modes, because no singularity at the
origin occurs in the two-boundary problem, and hence all linearly independent integrals
are regular, for all \(\tau \in [b, a]\). We now impose the boundary conditions (1.5) and (1.6),
which lead to the eigenvalue condition

\[
\begin{vmatrix}
I_n(Mb) & K_n(Mb) & J_n(Mb) & N_n(Mb) \\
F_{I_n}(Mb) & F_{K_n}(Mb) & F_{J_n}(Mb) & F_{N_n}(Mb) \\
I_n(Ma) & K_n(Ma) & J_n(Ma) & N_n(Ma) \\
F_{I_n}(Ma) & F_{K_n}(Ma) & F_{J_n}(Ma) & F_{N_n}(Ma)
\end{vmatrix} = 0 \quad (5.1)
\]

where, for \(Z = I, K, J\) or \(N\), we define

\[
F_{Z_n}(Mx) \equiv -Z_n'(Mx) + MxZ_n(Mx). \quad (5.2)
\]

On using the approximate forms (A.1)–(A.4) of Bessel functions, one finds that the left-
hand side of (5.1) is proportional to \(M^0\) as \(M \to 0\). Hence there are no false roots
of (5.1). As a next step we notice that, as \(M \to \infty\) and \(n \to \infty\), the \(I_n, K_n, J_n\)
and \(N_n\) functions contribute a \(\frac{1}{\sqrt{M}}\) factor, whereas \(F_{Z_n}\), defined in (5.2), contributes a
\(\sqrt{M}\) factor. Moreover, the dominant contribution to (5.1) as \(M \to \infty\) and \(n \to \infty\) is given by \(K_n(Mb), N_n(Mb), F_{K_n}(Mb), F_{N_n}(Mb)\) (in the first two rows), jointly with
\(I_n(Ma), J_n(Ma), F_{I_n}(Ma), F_{J_n}(Ma)\) (in the last two rows). One then finds that \(I_{\log}\) van-
ishes, because

\[
I_{\log} = \frac{1}{2} \sum_{n=1}^{\infty} n^2 \cdot 0 = 0. \quad (5.3)
\]

The value of \(I_{\text{pole}}(\infty)\) also vanishes, because no \(n\)-dependent term occurs in equation (5.1)
when \(n \to \infty\) and \(M \to \infty\). Moreover, \(I_{\text{pole}}(0)\) vanishes as well, since the determinant
leading to \(\rho_0(n)\) takes the form (by virtue of (A.1)–(A.4))

\[
D(n) = \det \begin{pmatrix}
0 & -\frac{1}{2} \Gamma(n) & 0 & -\frac{1}{2} \Gamma(n) \\
0 & -\frac{1}{2} \Gamma(n+1) & 0 & -\frac{1}{2} \Gamma(n) \\
\frac{1}{\Gamma(n+1)} & 0 & \frac{1}{\Gamma(n+1)} & 0 \\
\frac{1}{\Gamma(n+1)} & 0 & \frac{1}{\Gamma(n+1)} & 0
\end{pmatrix} \quad (5.4)
\]
and this vanishes exactly.

To sum up, we find that the $\zeta(0)$ value is zero in the two-boundary problem:

$$\zeta(0) = 0.$$  \hspace{1cm} (5.5)

With the same arguments presented in the end of section 4, the result (5.5) is found to hold for the first four boundary conditions studied in section 2.

6. Concluding remarks

The analysis of the squared Laplace operator in flat Euclidean backgrounds is motivated by the ghost sector of Euclidean Maxwell theory in a conformally invariant gauge, but has been here restricted to a real scalar field. Further motivations result from the theory of conformally covariant operators, which is an important branch of spectral geometry, and finds applications also in Euclidean quantum gravity [1–7]. The contributions of our investigation are as follows.

(i) The boundary conditions for which the squared Laplace operator is self-adjoint have been derived (cf [18]), taking as prototype the operator $\frac{d^4}{dx^4}$ on a closed interval of the real line. Interestingly, at least five sets of boundary conditions are then found to arise, and the option described by (2.4) and (2.5) coincides, if the field in (2.1) were a ghost field, with the boundary conditions obtained from the request of gauge invariance of the boundary conditions on $A_\theta$, when the Eastwood–Singer supplementary condition is imposed. The general reader, however, should be aware that the above boundary conditions have already been studied in the mathematical literature. For example, Eq. (1.5.51) of [18] studies the more involved boundary value problem

$$\left( \Delta^2 + \lambda \Delta \right) v = f \text{ in } \Omega$$  \hspace{1cm} (6.1)

$$v = 0 \text{ at } \partial \Omega$$  \hspace{1cm} (6.2)

$$\nabla_N v = \left(1 - \Delta_\Gamma \right)^{-\frac{3}{2}} \varphi \text{ at } \partial \Omega$$  \hspace{1cm} (6.3)
where $\Delta_\Gamma$ is chosen in such a way that $\left(1 - \Delta_\Gamma\right)^{\frac{1}{2}}$ is a suitable bijective operator \cite{18}. Here it is enough to remark that, since $\Delta^2 + \lambda \Delta \geq \Delta^2$ for all $\lambda \in \mathbb{R}_-$, the solution of (6.1)–(6.3) is unique for all smooth data. Hence strong ellipticity follows \cite{18}.

(ii) Given the fourth-order eigenvalue equation (4.1), the contribution of the corresponding eigenmodes to the one-loop divergence has been derived for the first time on the Euclidean 4-ball (see (4.16)), or on the portion of flat Euclidean 4-space bounded by two concentric 3-spheres (see (5.5)).

In our opinion, the property (i) is crucial because no complete prescription for the quantization is obtained unless suitable sets of boundary conditions are imposed, and sections 2 and 3 represent a non-trivial step in this direction. The result (ii) is instead relevant for the analysis of one-loop semiclassical effects in quantum field theory. In other words, if one has to come to terms with higher order differential operators in the quantization of gauge theories and gravitation, it appears necessary to develop techniques for a systematic investigation of one-loop ultraviolet divergences, as a first step towards a thorough understanding of their perturbative properties.

Some outstanding problems are now in sight. First, it appears interesting to extend our mode-by-mode analysis to curved backgrounds with boundary. In this case, the fourth-order conformally covariant differential operator is more complicated than the squared Laplace operator, and involves also the Ricci curvature and the scalar curvature of the background. Second, one should use Weyl’s theorem on the invariants of the orthogonal group to understand the general structure of heat-kernel asymptotics \cite{12} for fourth-order differential operators on manifolds with boundary. A naturally occurring question within that framework is, to what extent functorial methods \cite{7, 12} can then be used to compute all heat-kernel coefficients for a given form of the differential operator and of the boundary operator. Third, the recently considered effect of tangential derivatives in the boundary operator \cite{13, 19–21} might give rise to generalized boundary conditions for conformally covariant operators. The appropriate mathematical theory is still lacking in the literature,
but would be of much help for the current attempts to understand the formulation of quantum field theories on manifolds with boundary [7, 13, 17, 19–21].

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Appendix

In sections 4 and 5 we need the asymptotic expansions of Bessel functions when the argument tends to zero, or when both the argument and the order are very large [22]. In the former case, one finds, for all \( n \geq 1 \),

\[
I_n(x) \sim \frac{(x/2)^n}{\Gamma(n+1)} \quad (A.1)
\]

\[
J_n(x) \sim \frac{(x/2)^n}{\Gamma(n+1)} \quad (A.2)
\]

\[
K_n(x) \sim \frac{1}{2} \Gamma(n) (x/2)^{-n} \quad (A.3)
\]

\[
N_n(x) \sim -\frac{1}{\pi} \Gamma(n) (x/2)^{-n}. \quad (A.4)
\]

Moreover, when the argument is greater than the order, both being large, one finds, for modified Bessel functions [23],

\[
I_n(nz) \sim \frac{e^{ny}}{\sqrt{2\pi n(1+z^2)^{1/4}}} \sum_{s=0}^{\infty} \frac{U_s(y)}{n^s} \quad (A.5)
\]

\[
K_n(nz) \sim \sqrt{\frac{\pi}{2n}} \frac{e^{-ny}}{(1+z^2)^{1/4}} \sum_{s=0}^{\infty} (-1)^s \frac{U_s(y)}{n^s} \quad (A.6)
\]
\[ I_n'(nz) \sim \frac{(1 + z^2)^{1/4}}{z} \frac{e^{ny}}{\sqrt{2\pi n}} \sum_{s=0}^{\infty} \frac{V_s(y)}{n^s} \]  
(A.7)

\[ K_n'(nz) \sim -\sqrt{\frac{\pi}{2n}} \frac{(1 + z^2)^{1/4}}{z} e^{-ny} \sum_{s=0}^{\infty} (-1)^s \frac{V_s(y)}{n^s} \]  
(A.8)

where

\[ y \equiv \sqrt{1 + z^2 + \log \left( \frac{z}{1 + \sqrt{1 + z^2}} \right)} \]  
(A.9)

We do not need, in our calculations, the detailed form of the polynomials \( U_s \) and \( V_s \), which are generated by recurrence relations and can be found in [23]. For the Bessel functions \( J_n(nz) \) and \( N_n(nz) \), with both \( n \) and \( z \) very large, we need the asymptotic expansions in section 8.41 of [22], which are more conveniently expressed after setting \( z \equiv \sec \beta \):

\[ J_n(n \sec \beta) \sim \sqrt{\frac{2}{n\pi \tan \beta}} \left[ \cos \left( n \tan \beta - n\beta - \frac{\pi}{4} \right) \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma (2m + \frac{3}{2})}{\Gamma (\frac{1}{2})} \frac{A_{2m}}{(\frac{n}{2} \tan \beta)^{2m}} \right. \\
+ \sin \left( n \tan \beta - n\beta - \frac{\pi}{4} \right) \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma (2m + \frac{3}{2})}{\Gamma (\frac{1}{2})} \frac{A_{2m+1}}{(\frac{n}{2} \tan \beta)^{2m+1}} \right] \]  
(A.10)

\[ N_n(n \sec \beta) \sim \sqrt{\frac{2}{n\pi \tan \beta}} \left[ \sin \left( n \tan \beta - n\beta - \frac{\pi}{4} \right) \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma (2m + \frac{3}{2})}{\Gamma (\frac{1}{2})} \frac{A_{2m}}{(\frac{n}{2} \tan \beta)^{2m}} \right. \\
- \cos \left( n \tan \beta - n\beta - \frac{\pi}{4} \right) \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma (2m + \frac{3}{2})}{\Gamma (\frac{1}{2})} \frac{A_{2m+1}}{(\frac{n}{2} \tan \beta)^{2m+1}} \right] \]  
(A.11)

where \( A_{2m} \) and \( A_{2m+1} \) are numerical coefficients. In particular, it is crucial to pick out the dominant terms of the expansions (A.10) and (A.11), i.e.

\[ J_n(nz) \sim \sqrt{\frac{2}{n\pi}} (z^2 - 1)^{-1/4} \left[ \cos \left( n \left( \sqrt{z^2 - 1 - \arccos \frac{1}{z}} \right) - \frac{\pi}{4} \right) \right. \\
+ O(n^{-1}) \]  
(A.12)
\[
N_n(nz) \sim \sqrt{\frac{2}{n\pi}}(z^2 - 1)^{-\frac{1}{4}} \left[ \sin \left( n \left( \sqrt{z^2 - 1} - \arccos \frac{1}{z} \right) - \frac{\pi}{4} \right) + O(n^{-1}) \right].
\]

(A.13)

Thus, in (4.12), both \( J_n(Ma) \) and \( J'_n(Ma) \) contribute \( \frac{1}{\sqrt{M}} \) at large \( M \) (as well as \( I_n(Ma) \) and \( I'_n(Ma) \)). In section 5, all Bessel functions and their first derivatives contribute, for the same reason, a factor \( \frac{1}{\sqrt{M}} \) in the eigenvalue condition when both \( n \) and \( M \) tend to \( \infty \).

References


