An Algorithm to Simplify Tensor Expressions

R. Portugal
Department of Applied Mathematics, University of Waterloo, Waterloo, Ontario, N2L 3G1 - Canada.

Abstract

The problem of simplifying tensor expressions is addressed in two parts. The first part presents an algorithm designed to put tensor expressions into a canonical form, taking into account the symmetries with respect to index permutations and the renaming of dummy indices. The tensor indices are split into classes and a natural place for them is defined. The canonical form is the closest configuration to the natural configuration. In the second part, the Gröbner basis method is used to simplify tensor expressions which obey the linear identities that come from cyclic symmetries (or more general tensor identities, including non-linear identities). The algorithm is suitable for implementation in general purpose computer algebra systems. Some timings of an experimental implementation over the Riemann package are shown.

1 Introduction

Recently, some attempts to describe algorithms for simplifying tensor expressions have appeared in the literature.[1][2][3] The tensor symmetry properties and the presence of dummy indices make the problem complex. Fulling et al.[1] described an algorithm to enumerate the independent monomials built from the Riemann tensor and its covariant derivative. They presented explicit tables for monomials of order 12 or less in derivatives of the metric. Although explicit bases are presented for monomial of order 8 or less, they have not described neither a systematic algorithm to build the basis nor a method to simplify generic tensor expressions built using the Riemann tensor. Using the same algorithm, Wybourne and Meller[4] enumerated the order-14 invariants. On the other hand, Ilyin and Kryukov[2] did present an algorithm to simplify general tensor expressions based on the group algebra of the permutation group. Though the method is elegant from the mathematical point of view, it is inefficient. Quoting the authors: “hardware development is very fast now, and it will be possible to solve problems with 11 indices with the help of our program”. Dresse presented in the second part of his PhD dissertation[3] a new algorithm to simplify tensor expression based on the backtrack algorithms for combinatorial objects.[5] Most of his effort has been directed to solve the “dummy index problem” (described bellow).

In this work, the problem is addressed in two parts. In the first, the tensor expressions are put into a canonical form taking into account the symmetries with respect to index permutations...
and renaming of dummy indices. In the second part, the Gröbner basis method (see Geddes at al.[6] and refs. therein) is used to address the linear identities that come from cyclic symmetries (or more general tensor identities). The algorithm is suitable for implementation in general purpose computer algebra systems, since many functionalities of these systems (besides the Gröbner basis implementation) are required.

It is known that the names of the dummy indices have no intrinsic meaning, and the presence of two or more of them in a tensor expression creates symmetries with respect to renaming. This leads to algorithms of complexity $O(n!)$ where $n$ is the number of dummy indices. Dummy indices difficult the definition of tensor rules and the use of pattern matching. One way to solve the dummy index problem is to rename them in terms of the index positions and the name of the tensors (see step 9 of the algorithm). However, the tensor symmetries may change the index positions, invalidating the process. This kind of renaming method is invariant if there is a prescribed method to put the indices into a canonical position.

In section 2, an algorithm to put tensor expressions into a canonical form is described. The canonical position of the indices is based on definitions 3 to 6. Some parts of the definitions of this section involve conventions that can be changed without losing the “canonicalization” character of the algorithm. The ideal format for the canonical tensor is

$$T F^+ S_{II} A^+ B^+ C^+ S_I C^- B^- A^- F^-$$

where $F^+$, $S_{II}$, $A^+$, $B^+$, $C^+$, $S_I$, $C^-$, $B^-$, $A^-$ and $F^-$ represent sequences of indices whose meanings are defined in step 7 of the algorithm. The indices of class $S_I$ can have contravariant or covariant character. They are placed in the midst of the classes that have the character fixed. This ideal format is only achieved in special cases, as when the tensor $T$ is totally symmetric or antisymmetric. The canonical position of the indices is the one closest to this format, where the notion of “closest” is precisely defined.

Product of equal tensors is addressed by reduction to one tensor of rank equal to the sum of the rank of the factors. This reduction process is used in other parts of the algorithm in order to simplify the implementation.

In section 3, some timings of an experimental implementation of the algorithm over the Riemann package[7] are presented. Polynomials built from the Riemann tensor are challenges for any algorithm to simplify tensor expressions. I believe that special techniques can improve the timings for the kind of symmetries of the Riemann tensor, but none has been implemented for the demonstration of this section. The implementation in the Maple system[8] can easily handle products of three Riemann tensors.

In section 4, the problem of the simplification of tensor expressions under the presence of side tensor identities, which come generally from cyclic symmetries, is addressed. The Gröbner basis implementation of the Maple system is used to accomplish the full simplification.

# 2 The algorithm

Consider the definition of the normal and canonical functions given in Geddes et al.[6] Let $\mathcal{A}$ be a set of algebraic expressions which admits a canonical function. Consider the operations of multiplication, addition and contraction of tensors as defined in the tensor algebra.[9][10][11] If a coordinate system has been selected, the tensor algebra can be performed through the tensor components. In this work,
a tensor expression is any expression written in terms of non-assigned tensor components obeying the rules of the tensor algebra whose coefficient factors are members of the set \( \mathcal{A} \). Consider Riemannian spaces, in which exists a fundamental metric tensor which establish a relation between the covariant and contravariant tensor indices.

2.1 Definitions

**Hypothesis 1:** Tensors do not obey any side tensor identities except symmetries with respect to index permutations or symmetries with respect to renaming or the inversion of character of dummy indices.

“Symmetries with respect to index permutations” means that the tensors obey one or more equations of the kind

\[
T_{i_1 \cdots i_n} = \epsilon T_{\pi(i_1 \cdots i_n)},
\]

where \( \epsilon = 1 \) or \( \epsilon = -1 \) and \( \pi(i_1 \cdots i_n) \) is any permutation of \( i_1 \cdots i_n \).

**Definition 1:** *Induced symmetry* of a sub-set of indices of a tensor.

The induced symmetries of a sub-set of indices of a tensor are the symmetries that the sub-set inherits from the symmetries of the whole set of indices. Pairs of dummy indices are treated as independent free indices, hence not permutable.

For example, the induced symmetry of the first two indices of the Riemann tensor is the skew symmetry. The second and third indices have no induced symmetry regardless any contraction between the first and fourth indices.

**Definition 2:** *Equivalent* index configurations.

Two index configurations\(^2\) of a tensor \( T \) extracted from a tensor product are said to be equivalent if one configuration can be put into the other by the use of any of the following properties:

1. Character inversion of the dummy indices,
2. Renaming the dummy indices,
3. Index permutation allowed by the symmetries of the tensors of the tensor product.

**Definition 3:** Suppose the tensor \( T \) has \( n \) indices and let \( (\lambda_1, \cdots, \lambda_p) \) be a partition of \( n \) where \( p \) is a positive integer less or equal than \( n \). The indices of \( T \) can be grouped in disjoint classes \( C_1, \cdots, C_p \) where a generic class \( C_i \) has \( \lambda_i \) indices. The indices of each class can be substituted with numbers in such way that the indices of class \( C_i \) run from \( \sum_{j=1}^{i-1} \lambda_j + 1 \) to \( \sum_{j=1}^{i} \lambda_j \). Consider all index configurations of the tensor \( T \) and let the indices be substituted with the corresponding numbers. The configurations are in one-to-one correspondence with the elements of the symmetric group \( S_n \).[12][13] The following criteria establish an order of decreasing configurations with respect to classes \( C_1 \) to \( C_p \):

a. Smaller value of the position of the first index of class \( C_1 \) in the tensor \( T \). If the positions are equal, consider the position of the next index of class \( C_1 \). If the positions of all indices of class \( C_1 \) are equal, consider the positions of the indices of the next classes until \( C_p \).

\(^2\)The index configuration is the list of indices of the tensor, taking into account the character of each index.
b. Smaller value of the first index member of class $C_1$ that appears in the tensor $T$. If the first indices of class $C_1$ that appear in all configurations are equal, consider the next index member of class $C_1$. If the indices of class $C_1$ appear in the same order, consider the order of the indices of the next classes until $C_p$.

Definition 3a alone compares the position of the classes, while definition 3b alone compares the order of the indices. Given a set of equivalent configurations of a tensor $T$, definition 3 allows one to select the smallest configuration of the set with respect to a given partition of the number of indices. The smallest configuration is unique. If definition 3b is not applied, or if it is applied for some but not all classes, instead of having one smallest configuration, one may have a sub-set of smallest configurations.

**Definition 4:** Character normal configurations.

Let $C^+$ and $C^-$ be the classes of the contravariant and covariant indices respectively of a tensor $T$ extracted from a tensor product. Consider the set of all equivalent index configurations of $T$. The character normal configurations consist of the sub-set of smallest index configurations according to definition 3a with respect to classes $C^+$ and $C^-$.}

**Definition 5:** Index normal configurations.

Consider the definition of group I and II given in step 3 and the definition of classes $F^+, S_{II}, A^+, B^+, C^+, S_I, C^-, B^-, A^-$ and $F^-$ given in step 7a. The present criteria applies when a tensor ($T$) of group I is extracted from a tensor product. Consider the set of all equivalent index configurations of $T$. Let the indices be relabelled as described in step 7c. The index normal configurations of the tensor $T$ consist of the sub-set of smallest configurations that are character normal configurations, and satisfy definition 3a for classes $F^+, S_{II}, A^+, B^+, C^+, S_I, C^-, B^-, A^-$ and $F^-$ and definition 3b for classes $F^+, S_{II}, B^+, C^+, S_I, C^-, B^-, A^-$ and $F^-$.}

**Definition 6:** Index canonical configuration.

The index canonical configuration is the only element of the set of index normal configurations which fully satisfies definition 3 with respect to classes $F^+, S_{II}, A^+, B^+, C^+, S_I, C^-, B^-, A^-$ and $F^-$ in this order.

The order of the indices of classes $A^+$ and $A^-$ and the order of the indices of the sub-classes of $S_I$ (not including $S_I$) are not considered in the definition of the index normal configurations. In fact, if classes $A$ and $S_I$ have less than two elements each, the set of index normal configurations has one element, which is the index canonical configuration.

An algorithm to simplify tensor expressions must recognize when an expression vanishes. The following three lemmas guarantee that the algorithm presented here can recognize null tensor products.

**Lemma 1:** Let $P$ be a product of non-null tensors and suppose that no factor vanishes. Let $A$ be a factor and define the tensor $S$ as

$$S = \frac{P}{A}.$$
Suppose that $A$ and $S$ share $n$ contracted indices. Let $s$ be the rank of $S$, and $a$ be $n$ plus the number of free indices of $A$.

The product $P$ is zero if and only if there exists a factor $A$ having the symmetry

$$A^{i_1 \cdots i_n} = -A^{\pi(i_1 \cdots i_n)}$$

(1)

where the permutation $\pi$ acts only on the indices contracted with the indices of $S$, and $P$ is invariant under the application of the permutation $\pi$ on the corresponding indices of $S$, that is

$$P = A^{i_1 \cdots i_n} S_{\pi(j_1 \cdots j_s)},$$

(2)

where $n$'s are equal to $n$ i's and the permutation $\pi$ acts on names, not on positions.\(^4\) $A$ may have indices contracted internally which have not been represented in (1) and (2). Symmetry (1) takes account of permutations and character inversions of the dummy indices within $A$.

**Proof:** ($\Rightarrow$)(by *reductio ad absurdum*) There are two cases:

1. If $P$ is not invariant for any factor $A$ that has the symmetry (1), then

$$P \neq A^{i_1 \cdots i_n} S_{\pi(j_1 \cdots j_s)}.$$  

Using (1) and renaming the dummy indices, it follows that $P \neq -P$, therefore $P \neq 0$.

2. If no factor admits a symmetry $\pi$ of the form (1), from the supposition that no factor of $P$ vanishes, it follows that $P \neq 0$.

So far only products of non-vanishing factors have been considered. What are the conditions that cancel a single generic tensor with some or all indices contracted? The answer can be obtained from lemma 1. Suppose that $T$ is a tensor of rank $m$ with $2n$ indices contracted. This tensor can be written as

$$g_{i_1 j_1} \cdots g_{i_n j_n} T^{\sigma(i_1 j_1 \cdots i_n j_n f_1 \cdots f_{m-2n})},$$

(3)

where $\sigma$ is the permutation of $i_1 j_1 \cdots i_n j_n f_1 \cdots f_{m-2n}$ that specifies the actual order of the indices of $T$, and $g_{i j}$ is the metric supposedly symmetric. The free indices are $f_1 \cdots f_{m-2n}$. For this case, the tensor $S$ of lemma 1 is

$$S_{i_1 j_1} \cdots i_n j_n = g_{i_1 j_1} \cdots g_{i_n j_n},$$

and is symmetric under the interchange of $i_p j_p$ into $j_p i_p$ for $p \leq n$, and is totally symmetric under the pair interchange of $i_p j_p$ into $i_q j_q$ for $p, q \leq n$. Tensor $S$ does not vanish (again from lemma 1).

If the second factor of (3) does not vanish, there are two cases to consider regarding lemma 1. In the first case, tensor $T$ has antisymmetry (1) while $S$ is symmetric under the same permutation acting on the corresponding indices independently of $S$ being multiplied by $T$. In the second case, the indices of $S$ has the symmetry (2) only if one considers the contractions of indices between $S$ and $T$. These cases reflect items (i) and (ii) respectively of the following lemma.

\(^4\)The character of the indices of $A$ and $S$ need not be contravariant and covariant respectively. The only restriction is that the character of the dummy indices are opposite.
Lemma 2: Let $T$ be a non-null tensor of rank $m$ with $2n$ indices contracted ($2n \leq m$). Let $\sigma$ be a permutation such that

$$T^{\sigma(i_1 j_1 \cdots i_n j_n f_1 \cdots f_{m-2n})}$$

(4)

describes an index configuration with $m$ free indices, such that after $j_p$ ($p \leq n$) indices are lowered by the metric terms as described in (3), one obtains the actual index positions of $T$. Suppose that (4) does not vanish. $T$ is zero if and only if at least one of the following items is satisfied.

(i) Consider (4). Independently of any contraction, there is a permutation $\rho$ acting on $i_1 j_1 \cdots i_n j_n$ such that

$$T^{\rho \sigma}(i_1 j_1 \cdots i_n j_n f_1 \cdots f_{m-2n}) = \epsilon T^{\sigma}(i_1 j_1 \cdots i_n j_n f_1 \cdots f_{m-2n})$$

where $\epsilon = 1$ or $\epsilon = -1$ and at least one of the following items is satisfied.

(i.1) $T^{\rho \sigma}(i_1 j_1 \cdots i_n j_n f_1 \cdots f_{m-2n})$ is antisymmetric under one or more interchanges of $i_p j_p$ into $j_p i_p$ for $p \leq n$

(i.2) $T^{\rho \sigma}(i_1 j_1 \cdots i_n j_n f_1 \cdots f_{m-2n})$ is antisymmetric under one or more pair interchange of $i_p j_p$ into $i_q j_q$ for $p, q \leq n$.

(ii) There is an index character configuration such that $T$ is antisymmetric under a permutation $\pi$ acting on the contravariant indices (like (1)) and $T$ is invariant under the same permutation acting on the corresponding covariant indices (like (2) with all indices in the same tensor).

Proof: [14]

For example, suppose that $T$ is a tensor of rank 6 with the symmetries

$$T^{ijklmn} = -T^{klijmn},$$

$$T^{ijklmn} = T^{jiklmn},$$

$$T^{ijklmn} = T^{ijlkmn},$$

$$T^{ijklmn} = T^{ijklmn}.$$

Consider the index configuration

$$T^{ikl}_{\ ikl}$$

(5)

which is equivalent to zero. Due to the contraction of the first and fourth indices, (5) is antisymmetric under the interchange of $k^+$ and $l^+$, and is symmetric under the interchange of $k^-$ and $l^-$. This is an example of item (ii) of lemma 2.

There is one case not analyzed yet. A tensor $T$ may vanish due to symmetries regardless of any index contraction. This case has no practical applications but the algorithm must recognize what are the combinations of symmetries that cancel the tensor. That recognition can be done at the moment of defining the tensor, before the algorithm is executed. From now on, suppose that tensors are not zero if there is no index contraction.

Let $\mathcal{M}$ be the set of all mathematically equivalent tensor products generated by all possible equivalent configurations of the indices of the tensors of $\mathcal{P}$ ($\mathcal{P}$ is a product of non-null tensors when
all indices are considered free). Definitions 4-6 can be extended to products of tensors by application on each factor.

**Lemma 3:** Two elements of $\mathcal{M}$ have the *index canonical configuration* with opposite signs if and only if $\mathcal{P}$ is zero.

**Proof:** ($\Rightarrow$) If $\mathcal{P}$ is mathematically equivalent to $\mathcal{P}'$ and $-\mathcal{P}'$ simultaneously then $\mathcal{P} = 0$.

($\Leftarrow$) If $\mathcal{P}$ is zero, either one of the factors vanishes or, using lemma 1, there is a factor $A$ antisymmetric on some of its indices (like (1)) such that $\mathcal{P}$ satisfies (2). If one of the factors vanishes, item (i) or item (ii) of lemma 2 are obeyed. For all possible cases, two equivalent opposite terms with the *index canonical configurations* are generated due to the presence of contracted antisymmetric indices.

In worst cases, the algorithm recognizes that a factor is zero in step 7f, that a product is zero in step 7h, and that a sum is zero in step 10.

**Rule 1:** Consider a tensor product. The character of the first (from left to right) index of a pair of summed indices (if some exists) is contravariant, and the character of the second is covariant. The same rule applies for the summed indices within a single tensor.

### 2.2 The algorithm

The algorithm is divided into steps grouped by types of action that must be followed in increasing order unless otherwise stated. The main goal is to put the indices into a canonical position. The canonical position involves the relative position of the indices inside the tensor as well as their character. Definition 6 is a precise specification of the canonical position. All dummy indices are renamed and the original names are thrown away. The renaming of the dummy indices takes into account their position inside the tensors and the order of the tensors. Therefore the renaming process leads to canonical names only after the dummy indices have been put into a canonical position and an ordering for the tensors has been established.

**Step 1.** (Expanding) Expand the tensor expression modulo the coefficients. Select the tensor factors of the first term.\(^5\) From now on, consider how to put a tensor product into a canonical form. Steps 1 to 9 are applied to all terms of the expanded expression, one at a time.

**Step 2.** (Raising or lowering indices) If there are any metric tensor with indices contracted with other tensors, the corresponding indices are raised or lowered according to the contraction character. The same goes for other special tensors built from the metric.

**Step 3.** (Splitting in group I and II) Split the product in two groups. Group I consists of tensors with symmetries. Group II consists of tensors with no symmetries. Tensors of group I are placed in the left positions and tensors of group II in the right positions using the commutativity property of the product.

\(^5\)For expressions consisting of one tensor, only steps 6 and 9 are performed if the tensor has no symmetries; steps 7 and 9 are performed if the tensor has symmetries. Classes $S_I$ and $S_{II}$ are empty is this case.
Step 4a. (Merging equal tensors) Tensors of group II with the same name and same number of indices and with no free indices merge into a new tensor. Suppose that each original tensor has \( n \) indices and that there are \( N \) equal tensors, then the new tensor has \( nN \) indices and is totally symmetric under the interchange of the groups of the \( n \) indices. After the merging, the resulting tensors are incorporated into group I. The information about the relation with original tensors is stored, since it will be used at the end of the algorithm to substitute the new tensor with the original tensor names.

Step 4b. Tensors of group I with same names, same number of indices and same number of free indices merge to form a new tensor. The new tensor has the same symmetry under the interchange of group of indices as described in step 4a, and each group of indices inherits the symmetries of the original tensors. In step 4a, only tensors of group II with no free indices are merged. In this step, the tensors may have free indices.

Step 5a. (Sorting the tensor names) The tensors are lexicographically sorted inside each group. Tensors with same name but with different number of indices are sorted according to the number of indices.

Step 5b. Tensors with the same name and same number of indices are sorted according to the number of free indices.

Step 5c. Tensors of group II with the same name, same number of indices, and same number of free indices are sorted according to the name of the first free index. Notice that the free indices have fixed positions for tensors of group II.

Step 6. (Fixing the index character of group II) Consider the tensors of group II. The summed indices that are contracted within the tensors or those that are contracted with other tensors of group II may have their character changed in order to obey rule 1. All other summed indices (the ones contracted with tensors of group I) are put covariant. The free indices remain untouched.

Step 7a. (Splitting in classes) Step 7 is performed for all tensors of group I. Consider the first tensor of group I (the current tensor). It has symmetries that can involve all indices or only some of them. The discussion that follows applies only for the indices involved in the symmetries or contracted with indices involved in the symmetries. The pairs of summed indices not involved in the symmetries must obey rule 1 and the free indices not involved in the symmetries remain untouched throughout the whole algorithm. Consider the indices that can have their positions affected by the symmetry. They are split into classes:

- Class \( F^+ \): contravariant free indices
- Class \( S_{II} \): summed indices contracted with tensors of group II.
- Classes \( A^+, B^+, C^+ \), \( A^-, B^-, C^- \): summed indices inside the tensor
- Class \( S_I \): summed indices contracted with tensors of group I.
- Class \( F^- \): covariant free indices

\(^6\)This step can be displaced and performed together with step 9
Let $f^+, s_{II}, a, b^+, c^+, s_I, F^+, A, B^+, C^+, S_I, C^-, B^-$ and $F^-$ respectively. Classes $A^+$ and $A^-$ have the same size.

The summed indices inside the tensor are members of classes $A^+, B^+, C^+, A^-, B^-$ or $C^-$. When both indices of a pair of summed indices are involved in the symmetries, the contravariant index is a member of class $A^+$, and the covariant is a member of class $A^-$. If a index of a pair of summed indices is involved with the symmetries while the corresponding one is not, there are four cases. Let us call $i^+$ the index involved in the symmetries and $i^-$ the corresponding index not involved in the symmetries. These indices are equal but have different characters (here the signs + and - do not describe the character). Suppose that the relative position of $i^+$ with respect to $i^-$ cannot be inverted due to the symmetries. If $i^+$ is at the right of $i^-$, then $i^+$ is a member of class $B^+$; if $i^+$ is at the left, then $i^+$ is a member of class $B^-$. If the relative positions of $i^+$ and $i^-$ can change, then $i^+$ is a member of class $C^+$ or $C^-$ depending on whether $i^+$ is contravariant or covariant.

As soon as class $A$ is determined, it is verified whether there are antisymmetric contracted indices. If so, the tensor is null and the algorithm returns to step 2 for the next term.\(^7\)

The indices of class $S_I$ are contracted with the indices of the tensors of group I. Some of the latter indices may not be involved in the symmetries. They form sub-class $S_I^0$, which is further split into $S_I^{0^+}$ and $S_I^{0^-}$, corresponding to the indices of $S_I^0$ contracted with the tensor to the right or to the left of the current tensor respectively. The remaining indices of $S_I$ are split into sub-classes. The indices of the current tensor contracted with the next (to the right) tensor of group I form the first sub-class ($S_I^{1^+}$); the indices contracted with the next tensor form the second sub-class ($S_I^{2^+}$), and so on until the last tensor of group I. If the current tensor is the first of group I, the sub-division in classes is complete; otherwise the sub-division continues and the next sub-class consists of the indices the current tensor contracted with the first tensor of group I. The following sub-class consists of the indices contracted with the second tensor of group I and so on until the last tensor of group I that has not been considered yet.

Step 7b.(Fixing the index character of group I) In order to obey rule 1, the indices of class $S_{II}$ are put contravariant. The corresponding indices of tensors of group II have already been put contravariant in step 6. The indices of class $S_I$ are put contravariant if they are contracted with the tensor at the right, or covariant if they are contracted with the tensor at the left of the current tensor. The indices of class $B^+$ are put contravariant and the corresponding ones are put covariant. The indices of class $B^-$ are put covariant and the corresponding ones are put contravariant. The character of the indices of classes $F^-$ is maintained throughout the whole algorithm. The character of classes $A^+, C^+, A^-$ and $C^-$ may still change.

Step 7c.(Ordering and the numbering of the indices) This step establishes the order of the indices of classes $F^+, S_{II}, B^+, C^+, S_I, C^-, B^-$ and $F^-$ to be used in step 7e. Also, it specifies the numbers each index receives in order for definition 3 to be applied. To begin with, let us discuss the order of the indices of classes $F^+$ and $F^-$. First, sort the free indices of class $F^+$ using their original names. The first sorted free index is substituted with the number 1, the second by 2, and so on until the $f^+$ th index, which is substituted with $f^+$. The same process is performed for the indices of class $F^-$.\(^7\)The vanishing of the tensor due to the presence of antisymmetric contracted indices can be a natural consequence of the application of steps 7d to 7f together with lemma 3. For the sake of efficiency, it is better to verify the presence of antisymmetric contracted indices as soon as class $A$ is determined.
which receive the numbers $f^+ \ldots a^- + 1, \ldots, f^+ \ldots f^-$.\(^8\)

Class $S_H$ consists of summed indices contracted with the tensors of group II. At this point, the tensors of group II are ordered. The positions of the indices of these tensors are used as a reference to order the indices of class $S_H$. The first summed index in group II (from left to right) that is a member of class $S_H$ is the first index of $S_H$. It receives the number $f^+ + 1$. The second receives the number $f^+ + 2$ and so on until $f^+ + s_H$.

The indices of classes $B^+, B^-, C^+$ and $C^-$ are contracted with indices that have no symmetries; therefore, they have an ordering reference. They follow the same method used for the indices of class $S_H$.

Class $S_I$ can have its sub-classes ordered. The first sub-class is $S_I^{0+}$, followed by sub-classes $S_I^{1}$, $S_I^{2}$ and so on, and the last sub-class is $S_I^{0-}$. The indices of the classes $S_I^{0+}$ and $S_I^{0-}$ can be ordered since they are contracted with indices not involved in the symmetries. They follow the same method used for the indices of class $S_H$. The indices of the other sub-classes cannot be ordered at this point. Also, the indices of class $A$ cannot be ordered at this point.

Here follows, explicitly, the numbers reserved for each class:

- Class $F^+$: 1, 2, \ldots, $f^+$
- Class $S_H$: $f^+ + 1, f^+ + 2, \ldots, f^+ + s_H$
- Class $A^+$: $f^+ + s_H + 1, \ldots, f^+ \ldots a^+$
- Class $B^+$: $f^+ \ldots a^+ + 1, \ldots, f^+ \ldots b^+$
- Class $C^+$: $f^+ \ldots b^+ + 1, \ldots, f^+ \ldots c^+$
- Class $S_I$: $f^+ \ldots c^+ + 1, \ldots, f^+ \ldots s_I$
- Class $C^-$: $f^+ \ldots s_I + 1, \ldots, f^+ \ldots c^-$
- Class $B^-$: $f^+ \ldots c^- + 1, \ldots, f^+ \ldots b^-$
- Class $A^-$: $f^+ \ldots b^- + 1, \ldots, f^+ \ldots a^-$
- Class $F^-$: $f^+ \ldots a^- + 1, \ldots, f^+ \ldots f^-$

Step 7d. (Generating the character configurations) Both indices of a pair from class $A$ can change their positions due to the symmetries, but the relative position of some pairs may be fixed. The characters of the pairs, that cannot invert the relative position, can be chosen such that they obey rule 1. The characters of the remainder indices of class $A$ (let $a_0$ be the number of pairs of class $A$ that can invert their relative position) and the characters of indices of class $C$ cannot be chosen a priori. Each pair has two states which are contravariant-covariant and covariant-contravariant, corresponding to the characters of the indices. The algorithm generates $2^{(a_0+c)}$ possible character configurations by changing the states of pairs that can invert their relative position.

If the current tensor is totally symmetric, steps 7d and 7f need not be performed. The canonical form can be obtained straightforwardly, avoiding the slowest steps.

Step 7e. (Applying the symmetries) At this point, more than one equivalent configuration may have been generated. The symmetries are applied to all configurations in order to perform the following tasks.\(^9\) The contravariant indices are pushed to the left positions as far as possible and the covariant

\(^8\)The notation $f^+ \ldots f^-$ means $f^+ + s_H + a^+ + b^+ + c^+ + s_I + c^- + b^- + a^- + f^-$. The variables $a^+$ and $a^-$ are equal to $a$.

\(^9\)The application of the symmetries is a straightforward procedure that can be implemented for each kind of symmetry. In the case of the Maple system, tensors can be represented by tables and the symmetries by indexing functions, which can perform the tasks described in step 7e.
indices to right positions as far as possible. A sign change may be generated if there are antisymmetric indices. The state configurations that are members of the set of character normal configurations of the current tensor are selected. The symmetries are applied to the selected configurations again in order to put classes $F^+, S_{II}^+, A^+, B^+, C^+, S_I^+, C^-, B^-, A^-, F^-$ as closely as possible in the smallest position as prescribed in definition 3a, and after that, the indices of classes $F^+, S_{II}^+, B^+, C^+, S^0, C^-, B^-, A^-, F^-$ are put as closely as possible in their order as prescribed in definition 3b (see step 7c for relabelling). The sub-set of index normal configurations is selected. At this point, the character and the positions of all classes have been determined. Only the order of the indices of class $A$ and of the indices of the sub-classes of $S_I$ (not including $S^0_I$) has not been determined yet.

**Step 7f.** (Generating equivalent configurations of class $A$) In general, the ordering of the indices of the sub-classes of $S_I$ depends on the ordering of the indices of class $A$ and vice-versa. These classes must be ordered together. For now, suppose that all classes $S_I^i$ $(i > 0)$ are empty. In this case, steps 7a to 7g can be performed for each factor of the product since they are independent of each other.

The aim of this step is the generation of permutations of class $A$ that preserve the character arrangement of the indices. The selected configurations of step 7e are submitted to all possible re-orderings of class $A$ (class $A^+$ plus $A^-$) allowed by induced symmetry of the indices of class $A^+$ together with the indices of class $A^-$. The character normal configurations are selected.

If the induced symmetry of class $A$ is totally symmetric, the generation of permutation are not necessary since the canonical positions can be obtained straightforwardly.

In many cases, it is sufficient to generate the re-orderings allowed by the induced symmetries of class $A^+$ and class $A^-$ independently. This kind of re-ordering automatically maintain the character configuration.

**Step 7g.** (Selecting the index canonical configuration) The indices of all configurations are substituted by their correspondent numbers (step 7c), and the index canonical configuration with respect to classes $F^+, S_{II}^+, A^+, B^+, C^+, S_I, C^-, B^-, A^-, F^-$ is selected.

If two elements with the index canonical configuration are selected and they have opposite signs then the current product is zero (lemma 3). In this case, the algorithm returns to step 2 for the next term.

**Step 7h.** (Ordering indices of the sub-classes of $S_I$) The order of the sub-classes of $S_I$ and the order of the indices of $S^0_I$ have already been determined. The present step finds out the order of the indices of sub-classes $S_I^i$ $(i > 0)$ of all tensors of the current product. Consider class $A$ and all sub-classes $S_I^i$ $(i > 0)$ of the first tensor. These classes together have a induced symmetry. The same can be stated about the other factors of the product. All indices of all classes $A$ and all sub-classes $S_I^i$ $(i > 0)$ of all factors are put together in order to form a new tensor totally contracted. The order of the indices in the new tensor follows the order of the factors and the order of appearance in each factor. The symmetries of the new tensor is composed by all induced symmetries acting in the corresponding indices (see example 2).

By a recursive call of the algorithm, the indices of the new tensor are in class $A$ and are ordered by the method described for this class. The dummy indices that come from sub-classes $S_I^i$ cannot invert their relative positions (within a pair) hence step 7d need not generate character configurations for these indices.

If the induced symmetry of the indices of the sub-classes $S_I^i$ of a factor coincides with the actual
symmetry of these indices (taking into account the contractions of class \( A \)) then the indices of class \( A \) of this factor need not be included in the new tensor.

**Step 8.** (Recovering merged tensors) The symmetric (by group of indices) tensors that have been formed by merging tensors with equal names, equal number of indices and equal number of free indices in step 4a and 4b are converted back by the inverse process to a product of tensors with the original names but with the new positions of indices generated by the previous steps.

**Step 9.** (Renaming the dummy indices) At this stage of the algorithm, the indices are in their final position. The dummy indices are renamed following the rules: There are two cases. The first occurs when the whole pair of dummy indices resides inside a tensor. If the tensor name is \( A \) and the number of indices is \( m \), then the dummy index name will be \( A_{m,i,j} \), where \( i \) is the position of the contravariant index and \( j \) is the position of the covariant index and \( _{\text{,}} \) is some separator.\(^{10}\) If the same name appears in other tensors of the product, no name conflict is generated. The second case occurs when the dummy indices involve two tensors. Suppose that the first tensor has the name \( A \) with \( m \) indices and the second has the name \( B \) with \( n \) indices, then the dummy index will be renamed \( A_{m,i,B_{n,j}} \), where \( i \) is the position of the contravariant index inside the tensor \( A \) and \( j \) is the position of the covariant index inside the tensor \( B \). This renaming process proceeds from left to right. If a second dummy index receives the same name, then the number 1 is appended in its name \( A_{m,i,B_{n,j},1} \), for example. If a third summed index receives the same name, then the number 2 is appended in its name: \( A_{m,i,B_{n,j},2} \), and so on.\(^{11}\)

**Step 10.** (Collecting equal tensor terms) After performing steps 1 to 9 for all terms, collect equal tensor products and put the coefficient factors into the canonical form.

**Step 11.** (Sorting the addition)\(^{12}\) Each tensor product can be converted to a string by concatenating with separators the names of the tensors and the indices in the order they appear in the product. These strings are sorted. The order of the terms in the sum is rearranged to be in the same order as the sorted concatenated strings.

### 2.3 Proof that the algorithm is a canonical function

Let \( \mathcal{T} \) be the set of all tensor expressions which obey hypothesis 1. The algorithm described in section 2.2 is a function \( \mathcal{F} : \mathcal{T} \rightarrow \mathcal{T} \).

**Theorem:** \( \mathcal{F} \) is a canonical function.

**Demonstration:** The proof has three parts.

1. All operations performed by the algorithm obey the rules of the tensor algebra, hence preserve the mathematical equivalence of tensor expressions.

\(^{10}\)The separator is a symbol not present in tensor expressions.

\(^{11}\)The method of appending a number to the repeated dummy index names can be fully avoided if step 8 is performed after step 9.

\(^{12}\)In general, this step is not necessary when the algorithm is implemented over multiple purpose computer algebra systems.
2. For all \( E_1, E_2 \in \mathcal{T} \) such that \( E_1 = E_2 \), \( \mathcal{F}(E_1) \equiv \mathcal{F}(E_2) \). After step 1, \( E_1 \) and \( E_2 \) are sums of tensor products. Consider a generic tensor product. It is clear from the sorting uniqueness that the position of the tensors is unique after the application of the algorithm. The indices of tensors of group II and the indices of classes \( F^+, S_{II}, B^+, S_I, B^- \) and \( F^- \) of tensors of group I go to a unique character configuration, since this is a matter of convention.

The proof that the indices of tensors of group I (including the merged tensors of step 4) come to a unique configuration is consequence of the fact that the algorithm runs over all index character configurations of the indices of classes \( A \) and \( C \) and over all allowed index position configurations of classes \( A \) and \( S_I \). Only one configuration is selected as the canonical configuration unless the product is zero (lemma 3). After step 9, the dummy indices have canonical names, completing the canonicalization of a tensor product. Steps 10 and 11 put a sum of tensor products into the canonical form.

3. In item 2 is missing the details about the special case when one tensor expression is zero, that is, if \( E = 0 \) then \( \mathcal{F}(E) \equiv 0 \). From lemmas 1 and 2 follow that the indices responsible for the cancelation of a factor are in class \( A \) and for a product are in class \( S_I \). Since the algorithm generates the set of all mathematically equivalent tensor products by permuting the indices of classes \( A \) and \( S_I \), lemma 3 garantee that any null product inside the tensor expression is recognized. Other non-null tensor products that may still exist are put into a canonical form (item 2) guaranteeing the cancellation of a sum of products in step 10.

In the algorithm, there are shortcuts for special kinds of symmetries avoiding the generation of character configurations of step 7d or permutations of step 7f. These shortcuts increase the efficiency but they must satisfy items 1-3 of the proof.

2.4 Examples

Example 1: Consider the tensor expression \( R^i_{\ b a} \) (contraction of the Riemann tensor). Step 7a establishes that classes \( A \) and \( F^- \) are the only non empty classes in this case. Class \( A^+ \) is \([i^+]\), class \( A^- \) is \([i^-]\) and class \( F^- \) is \([a, b]\). Step 7c establishes that the contravariant index \( i \) receives the number 1, the covariant index \( i \) receives the number 2, and the indices \( a \) and \( b \) receive the numbers 3 and 4 in this order, since \( a \) precedes \( b \). Step 7d establishes that there are two character configurations to be considered: \( R^i_{\ b a} \) and \( R^i_{\ b a} \). The symmetries are applied (step 7e) in order to put these configurations into the equivalent ones \( R^i_{\ b i a} \) and \( R^i_{\ a i b} \). Both are selected, since both are members of the set of character normal configurations. No further changes are generated by the symmetries, so the configurations that are members of the set of index normal configurations must be selected. The indices are substituted with their respective numbers, yielding \([1,4,2,3]\) and \([1,3,2,4]\) respectively. Definition 3a with respect to the partition \((1,1,2)\) – corresponding to classes \([1]\), \([2]\) and \([3,4]\) – selects both configurations. Definition 3b with respect to class \([3,4]\) selects \([1,3,2,4]\) as the only member of the set of index normal configurations. Since there are no induced symmetries, no more index configurations are generated. The index canonical configuration is \([1,3,2,4]\). Therefore, the canonical form is \( R^i_{\ a i b} \), where \( i \) is \( R_{\ 4 .1 .3} \), as prescribed in step 9.

Example 2: Consider the tensor expression \( T^j S^{k l} T^i R_{i l j k} \), where \( S^{k l} \) is a totally symmetric tensor and \( R_{i l j k} \) is the Riemann tensor. Step 3 splits the expression into group I: \([S^{k l}, R_{i l j k}]\)
and group II: \([T^j, T^i]\). Step 4b merges \(T^j\) and \(T^i\) into one totally symmetric tensor (let be called \(T_{T^j}T^j\)) and adds to group I. Group II is empty now. Step 5a establishes the order for group I: \([T_{i,l j k}, S^k l, T_{T^j}T^j]\). Step 7a establishes that the only non-empty class in this case is \(S_I\). For the first tensor one has \(S_I^1 = [l, k]\) and \(S_I^2 = [j, i]\). The order of these indices has not been established yet. These are the only classes since all indices have been covered. Step 7b fixes the character as: \([R^{l i j k}, S_{k l}, T_{T^j}T^j]\). Step 7e converts \(R^{l i j k}\) into \(R^{l i k j}\) and maintains \(S^k l\) and \(T_{T^j}T^j\), invariant.

Step 7h generates a new tensor. Let us call \(N^{l i k j}_{kl j i}\). It has the symmetries of the Riemann tensor (excluding the cyclic symmetry) in the first four indices \((l^+, i^+, k^+, j^+);\) is symmetric under the interchange of the fifth and sixth indices \((k^-, l^-)\) and is symmetric under the interchange of the seventh and eighth indices \((j^-, i^-)\). The algorithm is called recursively, and \(N^{l i k j}_{kl j i}\) is converted to \(N^{N^{l i k j}_{lkl j}}\), determining the order of the indices. After step 8 one has: \([R^{l i k j}, S_{l k}, T_i, T_j]\). Step 9 establishes that canonical form for the expression is \(R^{l i k j}S_{l k}T_iT_j\) where \(l = R^\_4\_1S^\_2\_1\), \(i = R^\_4\_2T^\_1\_1\), \(k = R^\_4\_3S^\_2\_2\) and \(j = R^\_4\_4T^\_1\_1\).

### 3 An experimental implementation

Algorithms to simplify tensor expressions have been implemented in some computer algebra systems.[15][16][17][18][19] Some implementations use pattern matching which requires a big database of tensor rules, and even worse, sometimes the user must enter the rules. The underlying method used by the Ricci package[15] seems similar in some aspects to the method presented here. All these implementations have not solved the dummy index problem, therefore the main simplifier spends a long time or cannot simplify tensor expressions with many dummy indices.

In this section, I present an experimental implementation of the algorithm described in section 2 over the Riemann package.[7] The new package can be obtained from websites,\(^{13}\) and my purpose is to supersede the Riemann package in the near future with the new functionalities introduced to deal with tensor components abstractly.

The function \texttt{normalform} uses the algorithm to put tensor expressions into normal forms. No attempt has been made to put the output into the canonical form, that is, ordered with respect to the sum and to the product of terms, since this last step is unnecessary for the purpose of any general computer algebra system.

In the examples below, contravariant indices have positive signs and covariant indices have negative signs. Tensors are indexed variables and their symmetries are declared by the command \texttt{definetensor}. Some tensors are pre-defined: Christoffel symbols, Riemann tensor, Ricci tensor and Ricci scalar are pre-defined with the names \(\Gamma\) (Christoffel symbols) and \(R\) (Riemann, Ricci and Ricci scalar). More details can be found in the new help pages for the commands \texttt{normalform}, \texttt{definetensor} and \texttt{symmetrize}, and in the help pages of the Riemann package.

Expressions that are zero due to tensor symmetries are readily simplified:\(^{14}\)

\[
\texttt{> readlib(showtime)();}
\]
\[
\texttt{> definetensor(T[i,j,k,l], sym[1,2] and asym[3,4]);}
\]

\(^{13}\)See the addresses: http://www.cbpf.br/~portugal/Riegeom.html or http://www.astro.queensu.ca/~portugal/Riegeom.html

\(^{14}\)All calculations have been performed in a Pentium 120 MHz with 32 Mb of RAM, running Maple V release 5 over Windows 95.
\[ T^{ijkl} \]

time = 0.01, bytes = 13478
> expr1 := printtensor(R[i,j,k,l]*T[-i,-k,-j,-l]);

\[ R^{ijkl} T_{ikjl} \]

time = 0.05, bytes = 8124
> normalform(expr1);

0

time = 0.12, bytes = 90619
> expr2 := printtensor(T[i,j,k,l]*V[-i]*V[-j]+V[b]*V[a]*T[-a,-b,l,k]);

\[ T^{ijkl} V_i V_j + V^b V^a T_{ablk} \]

time = 0.01, bytes = 8042
> normalform(expr2);

0

time = 0.30, bytes = 338050
> off;

Polynomials constructed with the Riemann tensor exemplify the algorithm’s worst performance. No special techniques have been implemented for the kind of symmetries of the Riemann tensor. In the next example, all possible ways to write the scalars formed by the product of two Riemann tensors are generated. The 40.320 expressions reduce to 4 independent non-null forms if the cyclic identity of the Riemann tensor is not considered. In the next section, one can verify that the cyclic identity reduces to 3 independent scalars (cf. ref. [1]).

> S := \{ op(combinat[permute])([a,b,c,d,-a,-b,-c,-d]) \}:;
> nops(S);

40320

> readlib(showtime)();
> S1 := map(x->abs(normalform(R[op(1..4,x)]*R[op(5..8,x)])), S);

\{ 0, \left| R^{RI} R^{R2 R2 R3 R4 R4} R_{R1 R1 R2 R2 R3 R3 R4 R4} \right|, \left| R^{RI} R^{R2 R3 R3 R2 R4 R4} R_{R1 R1 R3 R2 R2 R3 R4 R4} \right|, \left| R_{R1 R1 R2 R2} R^{RI} R^{R2 R2} \right|, \left| (R) \right|^2 \}

\(^{15}\)Simplified names for the dummy indices are used for this demonstration. This choice does not provide truly canonical names.
The program spends less than one second per expression on average. Next, one example involving
the product of three Riemann tensors is provided:

\[
-R^{R1R3 R2R4 R3R1 R4R2} R^{R1R3 \cdot I} R2R4 \cdot I R1R3 R2R4 R3R1 R4R2 R1R3 \cdot I R2R4 \cdot I
\]

time = 26.41, bytes = 6303158
> off;

\section{Simplification of tensor expressions}

The algorithm of section 2 achieves a full simplified form of the tensor expressions if the tensors do
not obey side identities. To accomplish the simplification of tensor expressions obeying side relations,
the Gröbner basis method is used. The general strategy is to put the tensor expression and the side
relations into a canonical form of the algorithm of section 2, and simplify the new tensor expression
with respect to the new side relation.

Here follows an example of how the Gröbner basis method is used to simplify the expression
\[
R^{abcd} R^{acbd} - \frac{1}{2} R^{abcd} R^{abcd}.
\]

\[
\text{expr} := \text{printtensor}(R[a,b,c,d]*R[-a,-c,-b,-d]-1/2*R[a,b,c,d]*R[-a,-b,-c,-d]);
\]

\[
\text{EXPR} := \text{normalform} (\text{expr});
\]

\[
\text{side rel} := \text{printtensor}(R[a,b,c,d]*\text{symmetrize}(R[-a,-b,-c,-d],\text{cyclic}[b,c,d]));
\]

\[
\text{SR} := \text{normalform} (\text{side rel});
\]

\[
\text{simplify} (\text{EXPR}, \{\text{SR}=0\});
\]

0

From this example one notices that the scalars \(R^{abcd} R_{abcd}\) and \(R^{abcd} R_{abcd}\) are not inde-
pendent.
5 Conclusion

An algorithm to simplify tensor expressions based on computable definitions have been described. It has two parts. In the first part, the expression is put into a canonical form, taking into account symmetries with respect to index permutations and the renaming of dummy indices. The definition of the canonical form involves some conventions that can be changed without disqualifying the definition. The conventions are based on the implementation simplicity and on a readable display for tensor expression. In the second part, cyclic identities or more general kinds of tensor identities are addressed through the Gröbner basis method. The expression and the side relations are both put into a canonical form for the Gröbner method to work successfully.

In this work, a precise definition of the canonical form for tensor expressions is provided. No restriction is imposed on the kind of symmetries that the tensors can obey. For most of the symmetries that occur in practise, the algorithm is very fast. The symmetries of the Riemann tensor reveal the algorithm’s worst performance, but even in this case it is useful for practical applications.

The invariant renaming of the dummy indices plays an important role in the efficiency of the algorithm, since it neutralizes the symmetries that come from dummy index renaming. This is a solution for the dummy indices problem mentioned in the introduction.

An experimental implementation of the algorithm over the Riemann package[7] is available free from web sites. All calculations and timings presented in this work can be reproduced in the Maple system.

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References


See footnote 12.


