Abelian-Projected Effective Gauge Theory of QCD with Asymptotic Freedom and Quark Confinement $^*$

Kei-Ichi KONDO$^{**}$

Department of Physics, Faculty of Science, Chiba University, Chiba 263-8522, Japan

Abstract

We give an outline of a recent proof that the low-energy effective gauge theory exhibiting quark confinement due to magnetic monopole condensation can be derived from QCD without any specific assumption. We emphasize that the low-energy effective abelian gauge theories obtained here give the dual description of the same physics in the low-energy region. They show that the QCD vacuum is nothing but the dual (type II) superconductor.
§1. Introduction

The ideal of dual superconductor vacuum of QCD as an origin of quark confinement was proposed by Nambu,1) Mandelstam2) and ’t Hooft in the mid-1970. Although there may be various scenarios for quark confinement, the picture of dual superconductor vacuum of QCD is the most intuitively appealing one, since this is based on the dual picture of the well-known ordinary superconductivity. In the ordinary superconductivity due to Cooper pair condensation, magnetic flux is squeezed into the tube-like region in the type II superconductor. Here the term ’dual’ implies that the role of electric field and magnetic field is interchanged. Hence it is expected that, if the magnetic monopole condensates, the color electric flux is squeezed into the string connecting the quark and anti-quark pair. This leads to the linear static potential. However, why the dual superconductor appears from QCD was a mystery before a resolution was proposed by ’t Hooft3) in 1981.

Abelian projection: The basic idea of abelian projection proposed by ’t Hooft3) is to remove as many non-Abelian degrees of freedom as possible, by partially fixing the gauge in such a way that maximal torus group $H$ of the gauge group $G$ remains unbroken. For $G = SU(N)$, $H = U(1)^{N-1}$. He claimed that under the abelian projection the $SU(N)$ gauge theory reduces to an $U(1)^{N-1}$ abelian gauge theory plus magnetic monopole.

Abelian dominance hypothesis: Soon after the proposal of abelian projection, the hypothesis of abelian dominance was proposed by Ezawa and Iwazaki.4) The abelian dominance claims that the non-Abelian components do not contribute to the physics in the low energy or at a long-distance scale. This was a hypothesis at that time. However, abelian dominance has been confirmed based on Monte Carlo simulation on the lattice by Suzuki and Yotsuyanagi5) in 1990.

Massive off-diagonal gluons Why is the abelian dominance realized in QCD? A basic observation is as follows. Abelian dominance will be achieved if the QCD dynamics makes the non-Abelian component heavier than the Abelian component, so that the non-Abelian components do not propagate at long-distance scale.

Wilsonian RG: How to prove abelian dominance? We need to derive the effective gauge theory of YM theory at resolution (length scale) $R$ where all the field variables with momentum $p \geq R^{-1}$ are integrated out in the sense of Wilsonian renormalization group (RG).

QCD at scale $R$ In the problem of quark confinement, $R$ could be the distance between two quarks. If $R$ is small, the perturbative picture is valid ($\Rightarrow$ Asymptotic freedom). As $R$ increases, the perturbative picture gradually becomes dubious. When $R$ reaches at a certain critical distance $R_c$, monopole condensation is expected to occur. For $R > R_c$, electric vortices emerge as stable topological excitations and confine quarks ($\Rightarrow$ confinement).
Starting from SU(2) Yang-Mills theory in 3+1 dimensions, we prove that the abelian-projected effective gauge theories are written in terms of the maximal abelian gauge field and the dual abelian gauge field interacting with monopole current. This is performed by integrating out all the remaining non-Abelian gauge field belonging to SU(2)/U(1). We show that the resulting abelian gauge theory recovers exactly the same one-loop beta function as the original Yang-Mills theory. Moreover, the dual abelian gauge field becomes massive if the monopole condensation occurs. This result supports the dual superconductor scenario for quark confinement in QCD. We give a criterion of dual superconductivity. Therefore there can exist the effective abelian gauge theory which shows both asymptotic freedom and quark confinement based on the dual Meissner mechanism.

In the following we give an outline of the proof. For the details and references, see the original paper with the same title. 6)


§2. Basic strategy

How to derive the low energy effective (maximal abelian) gauge theory (LEEGT) for a given non-Abelian gauge theory with a non-Abelian gauge group $G$. A strategy is as follows.

Step 1: Cartan decomposition,

$$G = H \otimes G/H, \quad A = \{a, A\} \in H \oplus G - H.$$  (2.1)

Step 2: Gauge fixing to fix off-diagonal parts (abelian gauge),

$$A_\mu = A^a_\mu T^a \in G - H.$$  (2.2)

Step 3: Integrate out off-diagonal non-Abelian parts.

Then we obtain the effective gauge theory written in terms of $a_\mu \in H$,

$$\int [da_\mu] \int [dA_\mu] e^{iS_{YM}[a, A]} \delta(F[A]) = \int [da_\mu] e^{iS_{LEEGT}[a]}.$$  (2.3)

However, this scenario has the following difficulties.

Q1: Where does the magnetic monopole come from?

Q2: How can we integrate out off-diagonal gluon fields which have quartic self-interaction?

Both difficulties are resolved by adopting an alternative strategy.
Step 1’: Introduce auxiliary (antisymmetric) abelian tensor field $B_{\mu\nu}$. This enables us to perform exact integration of the off-diagonal gluon fields, $A_\mu = A^a_\mu T^a \in \mathcal{G} - \mathcal{H}$. Then we obtain the abelian-projected effective gauge theory (APEGT) as a LEEGT of QCD,

$$\int [da_\mu] \int [dA_\mu] e^{iS_{YM}[a,A]} \delta(F[A]) = \int [da_\mu] \int [dB_{\mu\nu}] \int [dA_\mu] e^{iS_{APEGT}[a,B]} \delta(F[A]) = \int [da_\mu] \int [dB_{\mu\nu}] e^{iS_{APEGT}[a,B]}.$$  \hspace{1cm} (2.4)

§3. Derivation of APEGT

3.1. Step 1’: Cartan decomposition

Perform the Cartan decomposition,

$$A_\mu(x) = \sum_{A=1}^{3} A^A_\mu(x) T^A := a_\mu(x) T^3 + \sum_{a=1}^{2} A^a_\mu(x) T^a \in \mathcal{H} \oplus \mathcal{G} - \mathcal{H}. \hspace{1cm} (3.5)$$

Then the field strength

$$F_{\mu\nu}(x) := \sum_{A=1}^{3} F^A_{\mu\nu}(x) T^A := \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) - i[A_\mu(x), A_\nu(x)] \hspace{1cm} (3.6)$$

is decomposed as

$$F_{\mu\nu}(x) = [f_{\mu\nu}(x) + C_{\mu\nu}(x)] T^3 + S^a_{\mu\nu}(x) T^a,$$

$$f_{\mu\nu}(x) := \partial_\mu a_\nu(x) - \partial_\nu a_\mu(x),$$

$$S^a_{\mu\nu}(x) := D_{[\mu} [a]^{ab} A^b_{\nu]} - D_{\nu} [a]^{ab} A^b_{\mu},$$

$$C_{\mu\nu}(x) T^3 := -i[A_\mu(x), A_\nu(x)], \hspace{1cm} (3.7)$$

where the derivative $D_\mu [a]$ is defined by

$$D_\mu [a] = \partial_\mu + i[a_\mu T^3, \cdot], \hspace{0.5cm} D_\mu [a]^{ab} := \partial_\mu \delta^{ab} - \epsilon^{ab3} a_\mu. \hspace{1cm} (3.8)$$

Next, we rewrite the Yang-Mills (YM) action

$$S_{YM}[A] = -\frac{1}{2g^2} \int d^4x \text{ tr} (F_{\mu\nu} F^{\mu\nu}) = -\frac{1}{4g^2} \int d^4x \left[ (f_{\mu\nu} + C_{\mu\nu})^2 + (S^a_{\mu\nu})^2 \right]. \hspace{1cm} (3.9)$$

We wish to integrate out the off-diagonal gluons. Note that $C_{\mu\nu}$ is quadratic in $A$. Hence $C_{\mu\nu}^2$ is quartic in $A$. All the other terms are integrated out by Gaussian integration. In order to perform Gaussian integration for all the terms, we introduce auxiliary antisymmetric tensor
field $B_{\mu\nu}$. Note that $B_{\mu\nu}$ is an abelian tensor field. We can consider two equivalent theories depending on the way we introduce the auxiliary tensor field.

(I) If we introduce the auxiliary tensor field in such a way

$$-rac{1}{4g^2}(f_{\mu\nu} + C_{\mu\nu})^2 \to \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} B_{\rho\sigma}(f_{\mu\nu} + C_{\mu\nu}) - \frac{1}{4} g^2 B_{\mu\nu} B^{\mu\nu},$$

(3.10)

which corresponds to the tree-level duality, $B_{\mu\nu} \leftrightarrow \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} (f_{\rho\sigma} + C_{\rho\sigma})$, we obtain

$$S[A, B] = \int d^4x \left[ \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} B_{\rho\sigma}(f_{\mu\nu} + C_{\mu\nu}) - \frac{1}{4} g^2 B_{\mu\nu} B^{\mu\nu} - \frac{1}{4} g^2 (S_a^a)^2 \right].$$

(3.11)

This theory is equivalent to the BF-YM theory, i.e. a deformation of the topological BF theory, see Ref. 6.

(II) If we take

$$-rac{1}{4g^2}(C_{\mu\nu})^2 \to \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} B_{\rho\sigma} C_{\mu\nu} - \frac{1}{4} g^2 B_{\mu\nu} B^{\mu\nu},$$

(3.12)

which corresponds to $B_{\mu\nu} \leftrightarrow \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} C_{\rho\sigma}$, we are lead to the action,

$$S[A, B] = \int d^4x \left[ -\frac{1}{4g^2} (f_{\mu\nu} f_{\mu\nu} + 2f_{\mu\nu} C_{\mu\nu}) + \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} B_{\rho\sigma} C_{\mu\nu} - \frac{1}{4} g^2 B_{\mu\nu} B^{\mu\nu} \right.$$

$$\left. - \frac{1}{4} g^2 (S_a^a)^2 \right].$$

(3.13)

We can show that two theories are equivalent to each other, see Ref. 6. In what follows, we focus on the action (3.13).

3.2. Step 2: Maximal abelian gauge fixing

We consider the gauge fixing,

$$F^\pm[A, a] := (\partial^\mu \pm i\xi a^\mu) A_\mu^\pm = 0,$$

(3.14)

$$F^3[a] := \partial^\mu a_\mu = 0,$$

(3.15)

where we have used the $(\pm, 3)$ basis. *) The gauge fixing with $\xi = 0$ is the Lorentz gauge, $\partial^\mu A^\mu = 0$. In particular, $\xi = 1$ corresponds to the differential form of the maximal abelian gauge (MAG) which is expressed as the minimization of the functional

$$\mathcal{R}[A] := \frac{1}{2} \int d^4x [(A_\mu^1(x))^2 + (A_\mu^2(x))^2] = \int d^4x A_\mu^+(x) A_\mu^-(x).$$

(3.16)

*) $O^\pm := (O^1 \pm iO^2)/\sqrt{2}$. In this basis, $\sum_\pm P^\pm Q^\mp = P^+ Q^- + P^- Q^+ = P^a Q^a$, and $\sum_\pm (\pm) P^\mp Q^\pm = -P^+ Q^- + P^- Q^+ = \epsilon^{a35} P^a Q^b(a, b = 1, 2)$. 

5
The differential MAG condition (3.14) corresponds to a local minimum of the gauge fixing functional $R[A]$, while the MAG condition (3.16) requires the global (absolute) minimum. The differential MAG condition (3.14) fixes gauge degrees of freedom in SU(2)/U(1) and is invariant under the residual U(1) gauge transformation. An additional condition (3.15) fixes the residual U(1) invariance. Both conditions (3.14) and (3.15) then completely fix the gauge except possibly for the Gribov problem. It is known that the differential MAG (3.14) does not spoil renormalizability of YM theory.

From physical point of view, we expect that MAG introduces the non-zero mass $m_\alpha$ for the off-diagonal gluons, $A_\mu^1, A_\mu^2$. This is suggested from the form (3.16) which is equal to the mass term for $A_\mu^1, A_\mu^2$, although we need an independent proof of this statement. This motivates us to integrate out the off-diagonal gluons in the sense of Wilsonian renormalization group and allows us to regard the resulting theory as the low-energy effective gauge theory written in terms of massless fields alone which describes the physics in the length scale $R > m_A^{-1}$. The abelian dominance will be realized in the physical phenomena occurring in the scale $R > m_A^{-1}$.

We introduce the Lagrange multiplier field $\phi^\pm$ and $\phi^3$ for the gauge-fixing function $F^\pm[A]$ and $F^3[A]$, respectively. It is well known that the gauge fixing term and the Faddeev-Popov ghost term are obtained using the BRST transformation $\delta_B$ as

$$L_{GF} = -i\delta_B G, \quad (3.17)$$

where $G$ carries the ghost number $-1$ and is a hermitian function of Lagrange multiplier field $\phi^\pm, \phi^3$, ghost field $c^a$, antighost field $\bar{c}^a$, and the remaining field variables of the original lagrangian. The simplest choice is given by

$$G = \sum_{\pm} \bar{C}^\pm (F^\pm[A, a] + \frac{\alpha}{2} \phi^\pm) + \bar{C}^3 (F^3[a] + \frac{\beta}{2} \phi^3). \quad (3.18)$$

Under the local U(1) gauge transformation,

$$a_\mu \rightarrow a_\mu + \partial_\mu \omega, \quad O^\pm \rightarrow e^{\mp i\omega} O^\pm \quad O^3 \rightarrow O^3. \quad (3.19)$$

Hence $a_\mu$ transforms as a U(1) gauge field, while $A_\mu^\pm$ and $B_{\mu\nu}^\pm$ behave as charged matter fields under the U(1) gauge transformation. It turns out that $B_{\mu\nu}^3$ and $C_{\mu\nu} = i \sum_{\pm} (\pm) A_\mu^\pm A_\nu^\mp$ are U(1) gauge invariant as expected.

For the gauge fixing function (3.18), straightforward calculation leads to the gauge fixing lagrangian (3.17),

$$L_{GF} = \phi^a F^a[A, a] + \frac{\alpha}{2} (\phi^a)^2 + i \bar{C}^a D^{\mu ab}[a] \xi D^{bc}_{\mu}[a] C^c$$

---

*) This statement has been proved soon after this talk, see Ref. 7.
\[-i\xi \tilde{C}^a [A^a_\mu A^{\mu b} - A^c_\mu A^{\mu c} \delta^{ab}] C^b + \phi^3 F^3[a] + \frac{\beta}{2} (\phi^3)^2 + i \tilde{C}^3 \partial^\mu \partial_\mu C^3 - i \tilde{C}^3 \partial^\mu (\epsilon^{ab3} A^a_\mu C^b) + i \tilde{C}^a \epsilon^{ab3} [(1 - \xi) A^b_\mu \partial^\mu + F^b[a, A]] C^3. \]  

(3.20)

This reduces to the usual form in the Lorentz gauge, $\xi = 0$.

3.3. **Step 3: Integration over $SU(2)/U(1)$**

We perform integration over the off-diagonal fields $\phi^a, A^a_\mu, C^a, \tilde{C}^a$ and $B^a_{\mu\nu}$ for BF-YM case) belonging to the Lie algebra of $SU(2)/U(1)$ and obtain the effective abelian gauge theory written in terms of the diagonal fields $a_\mu, B_{\mu\nu}$.

Under the MAG, the total action is given by

\[
S_{YM} = S_{YM[a, A, B, C, \tilde{C}]} = S_1[a, B] + S_2[a, C, \tilde{C}] + S_0[a, B, C, \tilde{C}],
\]

(3.21)

\[
S_1 = \int d^4x \left[ -\frac{1}{4g^2} f_{\mu\nu} f^{\mu\nu} - \frac{1}{4} g^2 B_{\mu\nu} B^{\mu\nu} \right],
\]

(3.22)

\[
S_2 = \int d^4x \left[ i \tilde{C}^a D^{abc} [a] D^{cb}_\mu [a] C^b + i \tilde{C}^3 \partial^\mu \partial_\mu C^3 + \phi^3 (\partial^\mu a_\mu) + \frac{\beta}{2} (\phi^3)^2 \right],
\]

(3.23)

\[
S_0 = -i \ln \int [dA^a_\mu] \exp \left\{ i \int d^4x \left[ \frac{1}{2g^2} A^a_\mu Q^{ab}_\mu A^b_\mu + A^a_\mu G^{a}_\mu \right] \right\}
\]

\[
= -\frac{1}{2} \ln \det (Q^{ab}_\mu) + \frac{g^2}{2} \alpha G^{a}_\mu (Q^{-1})^{ab}_\mu G^{b}_\mu,
\]

(3.24)

\[
Q^{ab}_\mu := (D_\mu[a] D_\mu[a])^{ab} \delta^{\mu\nu} - 2\epsilon^{ab3} f_{\mu\nu} + \frac{1}{2} g^2 \epsilon^{ab3} \epsilon_{\mu\nu\rho\sigma} B^{\rho\sigma}
\]

\[
= -2g^2 (\tilde{C}^a C^b - \tilde{C}^c C^d \delta^{ab}) \delta_{\mu\nu} - D_\mu[a]^a_{\nu} D^a_{\nu} + \frac{1}{\alpha} D_\mu[a]^{ac} D^c_{\nu} [a]^{cb}, \quad (3.25)
\]

\[
G^c_\mu := i (\partial_\mu \tilde{C}^3) \epsilon^{cb3} C^b - i \tilde{C}^a \epsilon^{ab3} \epsilon^{ca3} a_\mu C^3 - i \partial_\mu (\tilde{C}^a \epsilon^{ab3} C^b), \quad (3.26)
\]

where we have rescaled the parameter $\alpha$ to absorb the $g$ dependence.

The residual U(1) invariant theory is obtained by putting $\phi^3 = 0$ and $\tilde{C}^3 = C^3 = 0$ (hence $G^a_\mu = 0$). Therefore, the resulting APEGT is greatly simplified,

\[
S_{YM} = \int d^4x \left[ -\frac{1}{4g^2} f_{\mu\nu} f^{\mu\nu} - \frac{1}{4} g^2 B_{\mu\nu} B^{\mu\nu} + i \tilde{C}^a D^{abc} [a] D^{cb}_\mu [a] C^b \right]
\]

\[
- \frac{1}{2} \ln \det (Q^{ab}_\mu).
\]

(3.27)

The logarithmic determinant is calculated using the $\zeta$-function regularization.

\[
\ln \det Q = -\lim_{s \to 0} \frac{d}{ds} \frac{\mu^{2s}}{\Gamma(s)} \int_0^\infty dt \ t^{s-1} \text{Tr}(e^{-tQ}).
\]

(3.28)
This is a gauge-invariant regularization. Hence the result respects the residual U(1) gauge invariance. The result is

\[
\frac{1}{2} \ln \det Q^{ab}_{\mu\nu} = \int d^4x \left[ \frac{1}{4g^2} z_a f_{\mu\nu} f^{\mu\nu} + \frac{1}{4} z_b g^2 B_{\mu\nu} B^{\mu\nu} + \frac{1}{2} z_c B_{\mu\nu} \tilde{f}_{\mu\nu} \right. \\
\left. + \text{ghost self-interaction terms} \right. \\
\left. + \text{higher derivative terms} \right],
\]  

(3.29)

where

\[
z_a = -\frac{20}{3} \frac{g^2}{16\pi^2} \ln \mu, \quad z_b = +2\kappa \frac{g^2}{16\pi^2} \ln \mu, \quad z_c = +4\kappa \frac{g^2}{16\pi^2} \ln \mu.
\]  

(3.30)

If we neglect the ghost self-interaction terms and higher derivative terms, off-diagonal ghost and anti-ghost fields can be integrated out,

\[
S_c = \ln \int [d\bar{C}] [dC] \exp \left\{ -\int d^4x \bar{C}^a D^{ac}_\mu [a] D_{\mu}^{cb} [a] C^b \right\} \\
= \ln \det (D^{ac}_\mu [a] D_{\mu}^{cb} [a]) \\
= \int d^4x \left[ \frac{1}{4g^2} z'_a f_{\mu\nu} f^{\mu\nu} + \cdots, \quad z'_a := \frac{2}{3} \kappa \frac{g^2}{16\pi^2} \ln \mu. \right.
\]  

(3.31)

Finally we obtain

\[
S_{APEGT}[a, B] = \int d^4x \left[ -\frac{Z_a}{4g^2} f_{\mu\nu} f^{\mu\nu} - \frac{1}{4} z_b g^2 B_{\mu\nu} B^{\mu\nu} + \frac{1}{2} z_c B_{\mu\nu} \tilde{f}_{\mu\nu} \right],
\]  

(3.32)

where

\[
Z_a := 1 - z_a + z'_a = 1 + \frac{22}{3} \kappa \frac{g^2}{16\pi^2} \ln \mu.
\]  

(3.33)

§4. Asymptotic freedom

Defining the running coupling constant,

\[
g(\mu) = Z_a^{1/2} g,
\]  

(4.34)

the \(\beta\)-function is easily calculated:

\[
\beta(g) := \mu \frac{dg(\mu)}{d\mu} = -\frac{b_0}{16\pi^2} g(\mu)^3, \quad b_0 = \frac{11C_2(G)}{3} > 0.
\]  

(4.35)

Thus the APEGT exhibits asymptotic freedom in the sense that APEGT has exactly the same beta function as the original YM theory (at one-loop). In other words, the APEGT is
the abelian gauge theory with QCD-like running coupling constant \( g(\mu) \),

\[
S_{\text{APEGT}}[a, B] = \int d^4x \left[ -\frac{1}{4g(\mu)^2} f_{\mu\nu} f^{\mu\nu} + (B_{\mu\nu} - \text{dependent terms}) \right],
\]

\[
\frac{1}{g(\mu)^2} = \frac{1}{g(\mu_0)^2} + \frac{b_0}{8\pi^2} \ln \frac{\mu}{\mu_0},
\]

apart from the \( B_{\mu\nu} \)-dependent terms.

If \( B_{\mu\nu} \)-dependent terms are absent, the kinetic term for the field \( a_\mu \) does not change due to higher order expansion and hence the beta function is unchanged, namely, the same as the one-loop result. Therefore, \( B_{\mu\nu} \)-dependent terms can give non-perturbative contribution.

§5. APEGT with magnetic monopole current

We show that APEGT contains magnetic monopole. The two-form \( B_{\mu\nu} \) has the Hodge decomposition in four dimensions,

\[
B^{(2)} = db^{(1)} + \delta C^{(3)} = db^{(1)} + \delta \star \chi^{(1)} = db^{(1)} + \star d\chi^{(1)}.
\]

This is equivalent to

\[
B_{\mu\nu} = b_{\mu\nu} + \tilde{\chi}_{\mu\nu}, \quad b_{\mu\nu} := \partial_\mu b_\nu - \partial_\nu b_\mu, \quad \tilde{\chi}_{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta}(\partial^\alpha \chi^\beta - \partial^\beta \chi^\alpha).
\]

The tensor \( B_{\mu\nu} \) has six degrees of freedom, while the fields \( b_\mu \) and \( \chi_\mu \) have eight. This mismatch is not a problem, since two degrees are redundant; the gauge transformation \( b_\mu(x) \rightarrow b'_\mu(x) = b_\mu(x) - \partial_\mu \theta, \chi_\mu(x) \rightarrow \chi'_\mu(x) = \chi_\mu(x) - \partial_\mu \phi \), leave \( B_{\mu\nu} \) invariant. In the function integral, the integration over \( B_{\mu\nu} \) is replaced by an integration over \( b_\mu \) and \( \chi_\mu \), provided that the gauge degrees of freedom are fixed,

\[
[dB_{\mu\nu}] = [db_\mu][d\chi_\mu]\delta(F[b])\delta(F[\chi]).
\]

At one-loop level, integration over the redundant variable \( \chi \) leads to

\[
S_{\text{APEGT}}[a, b, k] = \int d^4x \left[ -\frac{Z_a}{4g^2} f_{\mu\nu} f^{\mu\nu} - \frac{1 + z_b}{4} g^2 b_\mu b^{\mu\nu} - z_c b_\mu k^\mu \right],
\]

where we have defined the magnetic current,

\[
k^\mu := \partial^\nu \tilde{f}_{\mu\nu}, \quad \tilde{f}_{\mu\nu} := \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} f^{\rho\sigma}.
\]

The magnetic current is non zero under the singular gauge transformation such that the gauge transformed vector potential satisfies the MAG. The resulting APEGT is written
in terms of 1) abelian gauge field $a_\mu$, 2) the dual abelian gauge field $b_\mu$ and 3) magnetic monopole current $k_\mu$ which couples to $b_\mu$. This theory has $U(1)_e \times U(1)_m$ symmetry

$$a_\mu \rightarrow a_\mu + \partial_\mu \omega \quad (U(1)_e),$$

$$b_\mu \rightarrow b_\mu + \partial_\mu \theta \quad (U(1)_m),$$

where $U(1)_m$ symmetry is guaranteed by the topological conservation $\partial_\mu k^\mu = 0$. The APEGT is considered as a quantum field theoretical realization of 't Hooft idea.

§6. Dual effective Abelian gauge theory

In order to obtain the dual theory with monopole condensation which leads to the dual Meissner effect and quark confinement, we consider the theory written in terms of $b_\mu$ alone. This can be done as follows. It was pointed out that the magnetic monopole in APEGT can be calculated from the original YM theory using the off-diagonal gluons. In fact, we can show that the U(1)-invariant current $K^\mu := \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \partial_\nu (\epsilon^{ab3} A^a_\rho A^b_\sigma)$ gives the magnetic monopole part of the abelian magnetic current $k_\mu$ which contains both the magnetic monopole part and the Dirac string part. It turns out that the magnetic monopole charge calculated from this definition satisfies the Dirac quantization condition. The existence of magnetic monopole is a consequence of mathematical identity, $\Pi_2(SU(2)/U(1)) = \Pi_1(U(1)) = \mathbb{Z}$.

By extracting the $b_\mu$ dependent pieces from the action (3.21),

$$S_{YM} = S_{YM}|_{b_\mu=0} + \int d^4 x \left[ -\frac{1}{4} g^2 b_{\mu\nu} b^{\mu\nu} + b^\mu K_\mu \right],$$

and inserting the identity $1 = \int [dK^\mu] \delta (K^\mu - \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \partial_\nu (\epsilon^{ab3} A^a_\rho A^b_\sigma))$, the partition function $Z_{YM}$ is written as

$$Z_{YM}[J] := \int [db_\mu] \exp \left\{ -\int d^4 x \left[ -\frac{1}{4} g^2 b_{\mu\nu} b^{\mu\nu} \right] \right\}
\times \int d\mu \int [dK^\mu] \delta (K^\mu - \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \partial_\nu (\epsilon^{ab3} A^a_\rho A^b_\sigma))
\times \exp \left\{ -S_{YM}|_{b_\mu=0} - \int d^4 x b^\mu K_\mu \right\},$$

where $d\mu$ denotes the integration measure over all the fields.

The dual effective theory with an action $S[b]$ is obtained by integrating out all the fields except for $b_\mu$,

$$Z_{YM}[J] := \int [db_\mu] \exp \left\{ -S[b] \right\},$$

$$S[b] = -\frac{1}{4} g^2 \int d^4 x b_{\mu\nu} b^{\mu\nu} + \ln \langle \exp \left[ \int d^4 x b_\mu (x) K_\mu (x) \right] \rangle_0,$$
where the expectation value for a function $f$ of the field is defined by

$$
\langle f(A) \rangle_0 := \int d\tilde{\mu} \int [dK^\mu] \delta(K^\mu - \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \partial_\nu (\epsilon^{ab}_3 A^a_\rho A^b_\sigma)) \times \exp \left\{ - S_{YM} |_{b_\mu=0} \right\} f(A), \tag{6.49}
$$

where $d\tilde{\mu}$ denotes the normalized measure without $[db_\mu]$ so that $\langle 1 \rangle_0 = 1$. It turns out that the cumulant expansion leads to

$$
S[b] = -\frac{1}{4} g^2 \int d^4 x b_\mu(x) b^{\mu}(x) + \int d^4 x \langle K_\mu(x) \rangle_0 b^{\mu}(x) + \frac{1}{2} \int d^4 x \int d^4 y \langle K_\mu(x) K_\nu(y) \rangle_c b^{\mu}(x) b^{\nu}(y) + O(b^3), \tag{6.50}
$$

where $\langle K_\mu(x) K_\nu(y) \rangle_c$ is the connected correlation function obtained from the normalized expectation value.

We can obtain a similar expression for the APEGT using the action (5.41). Hence the argument in the next subsection can be extended also to the APEGT.

§7. Dual Meissner effect due to monopole condensation

Recall that the effective dual abelian theory $S[b]$ has $U(1)_m$ symmetry. The magnetic current satisfies the conservation $\partial_\mu K^\mu = 0$, consistently with the $U(1)_m$ symmetry.

When $U(1)_m$ is unbroken, the correlation function of the magnetic monopole current is transverse,

$$
\langle K_\mu(x) K_\nu(y) \rangle_c = \left( \delta_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu \right) M(x - y). \tag{7.51}
$$

As long as the magnetic $U(1)_m$ symmetry is not broken, the dual gauge field $b_\mu$ is always massless as can be seen from (6.50) and (7.51). Therefore non-zero mass for the dual gauge field implies breakdown of the $U(1)_m$ symmetry.

If $U(1)_m$ symmetry is broken in such a way that

$$
\langle K_\mu(x) K_\nu(y) \rangle_c = g^2 \delta^4(x - y) f(x) + \cdots, \tag{7.52}
$$

the mass term is generated,

$$
S[b] = \int d^4 x \left[ -\frac{1}{4} g^2 b_\mu(x) b^{\mu}(x) + \frac{1}{2} g^2 m_b^2 b_\mu(x) b_\mu(x) + \cdots \right], \tag{7.53}
$$

if we write $f(x) = m_b^2$. This can be called the dual Meissner effect; the dual gauge field acquires a mass given by

$$
m_b^2 = \frac{1}{4g^2} \Phi(0), \tag{7.54}
$$
if the monopole loop condensation occurs in the sense that,
\[ \Phi(x) := \lim_{y \to x} \frac{(K_\mu(x)K_\mu(y))_c}{\delta^{(4)}(x-y)} \neq 0. \] (7.55)

This is a criterion of dual superconductivity of QCD. In the translation invariant theory, \( \Phi(x) \) is an \( x \)-independent constant which depends only on the gauge coupling constant \( g \).

It should be remarked that \( \Phi \) is not the local order parameter in the usual sense. In order to find the non-zero value of \( m_b \), we must extract, from the magnetic monopole current correlation function \( (K_\mu(x)K_\nu(y))_c \), a piece which is proportional to the Dirac delta function \( \delta^{(4)}(x-y) \) diverging as \( y \to x \). Therefore, if such type of strong short-range correlation between two magnetic monopole loops does not exist, \( \Phi \) is obviously zero. This observation seems to be consistent with the result of lattice simulations. The monopole loops exist both in the confinement and the deconfinement phases. However, in the deconfinement phase the monopole currents are dilute and the vacuum contains only short monopole loops with some non-zero density. In the confinement phase, on the other hand, the monopole trajectories form the infinite long loops and the monopole currents form a dense cluster, although there is a number of small mutually disjoint clusters.

Is this dual theory the same as the dual GL theory? It should be remarked that APEGT doesn’t need any scalar field. In this sense, the mechanism in which the dual gauge field acquires a mass is different from the dual Higgs mechanism. Nevertheless, we can always introduce the scalar field into APEGT so as to recover the spontaneously broken \( U(1)_m \) symmetry,
\[ \frac{1}{2} m_b^2 b_\mu(x)b_\mu(x) \to \frac{1}{2} m_b^2 (b_\mu(x) - \partial_\mu \theta(x))^2 = |(\partial_\mu - ib_\mu(x))\phi(x)|^2, \] (7.56)
where we identify
\[ \phi(x) = \frac{m_b}{\sqrt{2}} e^{i\theta(x)}. \] (7.57)

Indeed, the result is invariant under \( b_\mu \to b_\mu + \partial_\mu \alpha \) and \( \theta \to \theta + \alpha \) (\( \phi \to e^{i\alpha} \phi \)). Such a scalar field is called the St"uckelberg field or Batalin-Fradkin field. The case (7.57) is obtained as an extreme type II limit (London limit),
\[ \lim_{\lambda \to \infty} V(\phi), \quad V(\phi) := \lambda(|\phi(x)|^2 - m_b^2/2)^2, \] (7.58)

or non-linear \( \sigma \) model with a constraint in the integration measure,
\[ \delta(|\phi(x)|^2 - m_b^2/2). \] (7.59)
The value $\phi_0$ at which the potential $V(\phi)$ has a minimum is proportional to the mass $m_b$ of dual gauge field,

$$m_b = \sqrt{2\phi_0} = \sqrt{\Phi}/(2g).$$

(7.60)

In the deconfinement phase, the minimum is given by $\phi_0 = 0$ ($m_b = 0$), while in the confinement phase the minimum is shifted from zero $\phi_0 \neq 0$ ($m_b \neq 0$) which corresponds to monopole condensation. Thus the dual abelian gauge theory with an action $S[b]$ is equivalent to (the London limit of) the dual GL theory (or the dual Abelian Higgs model with radial part of the Higgs field being frozen),

$$S[b] = \int d^4x \left[ -\frac{1}{4} b_{\mu\nu} b^{\mu\nu} + |(\partial_\mu - ig^{-1} b_\mu)\phi|^2 + \lambda(|\phi|^2 - \phi_0^2)^2 + \cdots \right],$$

where we have rescaled the field $b_\mu \rightarrow b_\mu/g$. Note that the inverse coupling $g^{-1}$ has appeared as a coupling constant. This implies that the dual theory is suitable for describing the strong coupling region. Thus the dual GL theory is derived from YM theory without any specific assumption.

§8. Dual description of low-energy physics

Regarding the APEGT with an action $S[a,b,k]$ as an interpolating theory, we can obtain the dual description of the same physics, say quark confinement, based on $S[a], S[b]$. The respective effective theory is obtained by integrating out the remaining field variables. It is valid as an low-energy effective (abelian) gauge theory of QCD in the length scale $R > R_c := m_A^{-1}$, since the off-diagonal parts with mass of order $m_A$ are integrated out.
(a) Dual GL theory $S_{DGL}[b]$ (non asymptotic-free abelian gauge theory)

$$S_{DGL}[b] = \int d^4x \left[ \frac{-1}{4} b_{\mu\nu} b^{\mu\nu} + |(\partial_\mu - ig^{-1}b_\mu)\phi|^2 + \lambda(|\phi|^2 - \phi_0^2)^2 + \cdots \right]. \tag{8.62}$$

This dual GL theory exhibits dual superconductivity, since it has Nielsen-Olesen vortex solution corresponding to the Abrikosov vortex in the GL theory describing the usual superconductivity. When the monopole condensation occurs, $S[b]$ shows dual Meissner effect $m_b = \sqrt{2}\phi_0 \neq 0$. This leads to the linear static potential. Therefore we obtain the dual superconductor vacuum of QCD.

(b) Zwanziger-Suzuki (ZS) theory $S[a]$ (asymptotic-free abelian gauge theory) The effective abelian theory written in terms of $a_\mu$ is obtained (Using the Zwanziger formalism, this theory was first derived by Suzuki\(^8\)) assuming the abelian dominance, i.e. neglecting the off-diagonal gluons from the beginning,

$$S_{ZS}[a] = \int d^4x \left[ \frac{-1}{4g(\mu)^2} f_{\mu\nu}(x)f^{\mu\nu}(x) + \frac{1}{2} a_\mu(x) \frac{n^2m_b^2 X_{\mu\nu}(\partial)a^\nu(x)}{(n \cdot \partial)^2 + n^2m_b^2} \right], \tag{8.63}$$

where $n$ is an arbitrary fixed four-vector appearing in the Zwanziger formalism. The coupling constant $g(\mu)$ is the running coupling constant obeying the same $\beta$ function as the YM theory. Note that the local $U(1)_e$ symmetry is not broken and $a_\mu$ is massless, since $\partial^\mu X_{\mu\nu} \equiv 0 \equiv \partial^\nu X_{\mu\nu}$.

Both effective theories (8.62), (8.63) leads to the linear static potential with the string tension $\sigma$,

$$V(r) = \sigma r, \quad \sigma = \frac{Q^2}{4\pi} m_b^2 f(\kappa_{GL}), \tag{8.64}$$

where $f(x)$ is a dimensionless function. The essential part $m_b^2$ in the string tension follows simply due to the dimensional analysis, irrespective of the details of the calculation.

(c) Monopole action $S[k]$ (Lattice version was proposed by Smit and van der Sijs\(^9\))

$$S_m[k] = \int d^4x \left[ -\frac{1}{4g^2(\mu)} f_{\mu\nu}(x)f^{\mu\nu}(x) + \frac{1}{g^2} k^\mu(x)D_{\mu\nu}(x-y)k^\nu(y) \right], \tag{8.65}$$

where $D_{\mu\nu}(x-y)$ is the massless vector propagator. By using the monopole action and energy(action)-entropy argument, we can show that in the strong coupling region long monopole loops give dominant contribution to the path integral and that the non-zero monopole condensation is obtained.
§9. Addenda

The above strategy for obtaining APEGT can be extended to $SU(N)$ YM theory, see Ref. 10.

In this report, we have not discussed the low-energy effective theory from a viewpoint of topological field theory. By pushing ahead the idea of topological field theory, quark confinement has been proven recently, see Ref. 7 where massiveness of the off-diagonal gluons is also proved. The mass $m_A$ is essentially the same as the Haldane gap in one-dimensional quantum Heisenberg antiferromagnets.

Acknowledgements

I would like to thank organizers of YKIS’97 for inviting me to give a talk.

References

6) K.-I. Kondo, hep-th/9709109 (revised), to be published in Phys. Rev. D.
7) K.-I. Kondo, hep-th/9801024.
10) K.-I. Kondo, in preparation.