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HIGH-ENERGY BEHAVIOUR OF THE BARDAKÇI-RUEGG AMPLITUDEA. Białas ^{*)} and S. Pokorski ^{**)}

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A B S T R A C T

The behaviour of the five point function proposed by Bardakçi and Ruegg is discussed in the kinematical region of high-energy and small momentum transfers. Both single and double Regge limits are given explicitly in terms of the hypergeometric functions. When applied to the single particle production at high energy, the formulae suggest a Regge exchange model in which the structure of the vertices is explicitly given.

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1. - INTRODUCTION

A generalized Veneziano model for the five-point function was proposed by Bardakçi and Ruegg ¹⁾ and by Virasoro ²⁾. In this paper we discuss the behaviour of the Bardakçi-Ruegg model in the high-energy region.

We consider the process

$$A + B \rightarrow 1 + 2 + 3 \quad (1)$$

which can be schematically represented by the diagram of Fig. 1 ³⁾. We investigated the contributions of the full Bardakçi-Ruegg amplitude under the assumption that the energy of the AB system is very large and that two of the momentum transfers (t_{A1} and t_{B3}) are limited.

Our results can be summarized as follows.

- (i) - In the double Regge limit (two subenergies, say 12 and 23, very large, the ratio 12·23/AB fixed), the behaviour of the amplitude depends rather sensitively on the mass of the "central" particle. If the mass of the central particle is small compared to the slope of the Regge trajectories (this is rather well satisfied for pions), then the amplitude can be approximated by the sum of two terms

$$B_4(AB, A1) B_4(23, B3 - A1) + B_4(AB, B3) B_4(12, A1 - B3) \quad (2)$$

where B_4 is a four-point function of Veneziano ⁴⁾, including signature factor ³⁾. This can be interpreted as a sum of two single Regge exchange diagrams (see Fig. 2). Only in the central region of the Dalitz plot, one recovers the currently used double Regge formula

$$B_4(12, A1) B_4(23, B3) \quad (3)$$

- (ii) - In the single Regge limit (one of the subenergies, say 23, is kept fixed, others tend to infinity), the amplitude can be represented as a product

$$B_4(A_3, A_1) [R_{23} + B_{23}] \quad (4)$$

where R_{23} is a sum over all resonances in the 23 system and B_{23} is a background term. We give the explicit expressions for R_{23} and B_{23} in terms of Gauss hypergeometric function and gamma function.

In Section 2, we formulate our problem and introduce the necessary notation. In Section 3, the calculation of the high-energy limits and the summary of the obtained formulae is presented. The discussion of the results is given in Section 4. Our conclusions are listed in the last Section.

2. - NOTATION AND BASIC FORMULAE

As pointed out by Bardakçi and Ruegg, the amplitude for the process (1) is a linear combination of the 12 diagrams which are presented in Fig. 3. These diagrams can be naturally divided in two groups, each one containing six diagrams. The first group contains the diagrams in which the particles A and B are in the neighbourhood of each other. These will be called direct diagrams [diagrams (1)-(6) of Fig. 3]. In the second group of diagrams, there is always one particle between the particles A and B. These will be called the crossed diagrams. Inside each group, the diagrams differ by permutation of particles 1, 2, 3. The contribution of a diagram (i) will be denoted by $B_5(i)$.

Bardakçi and Ruegg have given an integral representation for $B_5(i)$. As shown in the Appendix, this integral representation can be transformed into a product of Euler B functions, and a generalized hypergeometric function ⁵⁾. In particular, for the diagram 1, we obtain

$$B_5(1) = B(12, A1) B(23, B3) {}_3F_2(-AB+12+23, A1, B3; 12+A1, 23+B3) \quad (5)$$

where

$$\begin{aligned} {}_3F_2(-AB+12+23, A1, B3; 12+A1, 23+B3) = \\ = {}_3F_2(-AB+12+23, A1, B3; 12+A1, 23+B3; 1) \end{aligned} \quad (6)$$

is a generalized hypergeometric function defined by

$$\begin{aligned} {}_3F_2(a, b, c; d, e; z) = \\ = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(a)} \frac{\Gamma(b+n)}{\Gamma(b)} \frac{\Gamma(c+n)}{\Gamma(c)} \frac{\Gamma(d)}{\Gamma(d+n)} \frac{\Gamma(e)}{\Gamma(e+n)} \frac{z^n}{n!} \end{aligned} \quad (7)$$

The other diagrams can be calculated in an analogous way, by interchanging the variables.

Formula (5) seems rather convenient for the investigation of the properties of the Bardakçi-Ruegg amplitude.

3. - CALCULATION OF THE HIGH-ENERGY LIMITS

We investigated only the case when at least two of the momentum transfers, say A_1 and B_3 , are limited ⁶⁾. This condition defines a certain region in the three-particle phase space. As shown by Van Hove ⁷⁾ this region extends over the two sectors of an hexagonal plot ⁸⁾ (see Fig. 4 for details) ⁹⁾.

Our aim is now to find the contributions of the 12 diagrams of Fig. 3 to this particular region of phase space. The discussion of any other region is quite analogous.

To find the high-energy limit of diagram 1, we observe that, as shown in the Appendix, its contribution can be written in the form

$$\begin{aligned} B_5(1) = & B_4(AB, A_1) B_4(23, B_3 - A_1) {}_3F_2(A_1, 23, 12 + A_1 - B_3; 1 + A_1 - B_3, AB + A_1) \\ & + B_4(AB, B_3) B_4(12, A_1 - B_3) {}_3F_2(B_3, 12, 23 + B_3 - A_1; 1 + B_3 - A_1, AB + B_3) \end{aligned} \quad (8)$$

One can now easily find the high-energy limit by using the formulae

$${}_3F_2(a, b, \mu; c, \lambda; z) \xrightarrow{\lambda, \mu \rightarrow \infty} F(a, b; c; \frac{\mu z}{\lambda}) \quad (9a)$$

$${}_3F_2(a, \mu, \nu; c, \lambda; z) \xrightarrow{\lambda, \mu, \nu \rightarrow \infty} \phi(a, c; \frac{\mu \nu z}{\lambda}) \quad (9b)$$

Here, $F(a, b, c, z)$ is a Gauss hypergeometric function and $\phi(a, c, z)$ is a confluent hypergeometric function.

In the double Regge limit (region D_{123} on the hexagonal plot) we obtain

$$B_5(1) \rightarrow B(AB, A1) B(23, B3 - A1) \phi(A1; 1 + A1 - B3; \kappa) + \\ + B(AB, B3) B(12, A1 - B3) \phi(B3; 1 + B3 - A1; \kappa) \quad (10)$$

where

$$\kappa = \frac{(12)(23)}{AB}$$

It can be checked that formula (10) is equivalent to the one obtained by Bardakçi and Ruegg¹⁰⁾. The other diagrams can be treated in a similar way. One can show (see Ref.¹⁰⁾ for details) that only one direct diagram 1 and three crossed diagrams 7, 8 and 9 give non-vanishing contributions. The sum of all these diagrams results just in adding signature factors to formula (10)¹¹⁾.

For the single Regge limit in the region S_{23} (the energy 23 fixed), we get

$$B_5(1) \rightarrow \\ B_4(AB, A1) B_4(23, B3 - A1) F(A1, 23; 1 + A1 - B3; \frac{12}{AB}) + \\ + B_4(AB, B3) B_4(12, A1 - B3) F(B3, 23 + B3 - A1; 1 + B3 - A1; \frac{12}{AB}) \quad (11)$$

Using the well-known relations for analytic continuation of hypergeometric function, this formula can be equivalently written in the form

$$B_5(1) \rightarrow B_4(AB, A1) B_4(23, B3) F(A1, 23; B3 + 23; \frac{13}{AB}) \quad (12)$$

High-energy limits of the contributions from the other diagrams can be calculated in a similar way. The results are summarized in the Table. Contributions to single Regge limit in the region S_{12} (12 fixed) can be obtained from the Table after substitutions $A \leftrightarrow B$, $1 \leftrightarrow 3$.

4. - DISCUSSION

We would like to discuss now the general properties of contributions obtained in the preceding Section.

A - Single Regge limit

In order to get an idea about the physical meaning of the formulae given in the Table, we observe that they can be written as a sum over the resonances. This can be achieved by means of formula

$$B(a,b) F(c, b; a+b; z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{\Gamma(a)}{\Gamma(a-m)} F(c, -m; a-m; z) \frac{1}{b+m} \quad (13)$$

This formula can be proved, e.g., by expanding $F(c, -m; a-m; z)$ into power series, changing the order of summation and using the well-known expansion for $B(a,b)$ ¹²⁾.

Applying (13) to $B_5(1)$, as given in the Table, we get

$$B_5(1) = B(AB, A1) \mathcal{R}_{23} \quad (14)$$

where

$$\mathcal{R}_{23} = \sum_{m=0}^{\infty} \frac{1}{m+23} \frac{(-1)^m}{m!} \frac{\Gamma(B3)}{\Gamma(B3-m)} F(A1, -m; B3-m; \frac{13}{AB}) \quad (15)$$

The formula (14) exhibits clearly the factorization property of the Veneziano model : the amplitude factorizes into the two-body scattering amplitude and the vertex part which can be represented as a sum over resonances in the 23 system ¹³⁾. Let us note that the residues of these resonances depend on the momentum transfer $A1$. Comparing the expansion of the four-point Veneziano function into the resonances

$$B(23, B3) = \sum_{m=0}^{\infty} \frac{1}{23+m} \frac{(-1)^m}{m!} \frac{\Gamma(B3)}{\Gamma(B3-m)} \quad (16)$$

with (15), we see that the corrections are given by the function $F(A1, -m; B3-m; \frac{13}{AB})$. This is polynomial in variables $A1$ and $\frac{13}{AB}$ (14).

The last remark we would like to make is that the resonance residues

$$Z_m = \frac{(-1)^m}{m!} \frac{\Gamma(B3)}{\Gamma(B3-m)} F(A1, -m; B3-m; \frac{13}{AB}) \quad (17)$$

can be expressed in terms of the Jacobi polynomials

$$Z_m = P_m^{(A1-B3, B3-m-1)}\left(\frac{12-13}{AB}\right) \quad (18)$$

Contributions from other diagrams can be treated in a similar way. It is easily seen that $B_5(5)$, $B_5(7)$ and $B_5(11)$ also show the resonance structure in the 23 system and can therefore be represented by expressions analogous to (14). The amplitudes $B_5(8)$ and $B_5(12)$ do not contain the poles in the physical region of the 23 system. They may be represented, however, as a sum over the resonances in $B2$ and $B3$ systems, respectively. Since these resonances are not in the physical region of reaction (1), they will contribute only to the smooth background.

We think that the approach developed here may serve as a starting point for the phenomenological analysis of resonance production in high-energy scattering. In particular, for boson resonance production, the factorization property of formula (14) suggests the possibility of independent treatment of the resonance vertex (which contains only the mesons) and the baryon part of the amplitude. This may be of importance, because the Veneziano model for meson-baryon scattering is not yet fully understood.

B - Double Regge limit

The double Regge limit of $B_5(1)$, as given by formula (10) may be interpreted as a sum of two single Regge exchange diagrams (see Fig. 2)

$$B_5(1) = B(AB, A1) V(B3, 23, A1; \kappa) + B(AB, B3) V(A1, 12, B3; \kappa) \quad (19)$$

where the vertices V are given by

$$V(B3, 23, A1; \kappa) = B(23, B3 - A1) \phi(A1, 1 + A1 - B3; \kappa)$$

$$V(A1, 12, B3; \kappa) = B(12, A1 - B3) \phi(B3, 1 + B3 - A1; \kappa) \quad (19')$$

These vertices reggeize if the corresponding subenergy is large. The only feature by which formula (19) differs from the usual single Regge exchange amplitude is the κ dependence of the vertices. It appears that this κ dependence is very essential for the behaviour of the amplitude. To illustrate this point, let us consider two limiting cases $\kappa \rightarrow 0$ and $\kappa \rightarrow -\infty$.

For $\kappa \rightarrow 0$, one has $F(a, b, \kappa) \rightarrow 1$ and we obtain

$$V(B3, 23, A1; \kappa) \rightarrow B(23, B3 - A1)$$

$$V(A1, 12, B3; \kappa) \rightarrow B(12, A1 - B3) \quad (20)$$

Formulae (20) clearly exhibit the single Regge exchange features of the amplitude. We note that the V 's do not reduce (as one would expect) to $B(23, B3)$ and $B(12, A1)$. The shift in momentum transfer may be interpreted as an effect of the off-mass-shell correction to the two-body scattering in the vertex.

To see the behaviour of $B_5(1)$ for $\kappa \rightarrow -\infty$ one can use the formula

$$\phi(a; c; \kappa) \xrightarrow{\kappa \rightarrow -\infty} \frac{\Gamma(c)}{\Gamma(c-a)} (-\kappa)^{-a} \quad (21)$$

One obtains then

$$B_5(1) \xrightarrow{\kappa \rightarrow -\infty} B(12, A1) B(23, B3) \quad (22)$$

This is precisely the double Regge exchange formula without form factors ¹⁵⁾.

It is clear from these two examples that the amplitude $B_5(1)$ depends rather dramatically on κ in the double Regge limit ¹⁶⁾. Fortunately, the range of the variation of the κ is strongly limited ¹⁵⁾ by the inequality

$$|\kappa| \lesssim m_2^2 + \left(\sqrt{-t_{A1}} + \sqrt{-t_{B3}} \right)^2 \quad (23)$$

This inequality suggests that, provided the mass of the particle 2 is not very large, the formula (20) may be a reasonable approximation to exact expression (19). To check that we have calculated the functions

$$\phi(A1; 1+A1-B3; \kappa) \equiv \frac{V(B3, 23, A1; \kappa)}{B(23, B3-A1)}$$

and

$$\phi(B3; 1+B3-A1; \kappa) \equiv \frac{V(A1, 12, B3; \kappa)}{B(12, A1-B3)} \quad (24)$$

in the D_{123} region for some typical examples of single particle production at accelerator energies. We found that, for the diagrams with the pion as a central particle, the form factors (24) are equal to one within 15 % (for momentum transfers below $0.6 \text{ GeV}/c^2$). If the mass of the central particle is of the order of proton mass, much stronger deviations from unity are present.

5. - CONCLUSIONS

Our conclusions can be summarized as follows.

A - The five-point function of Bardakçi and Ruegg can be expressed in terms of Euler B functions and the generalized hypergeometric function of argument 1. In the high-energy limit the generalized hypergeometric function reduces to a linear combination of the Gauss hypergeometric functions. This provides a significant simplification in the investigation of properties of B_5 .

B - The five point function corresponding to a given diagram can be factorized into a product of a four-point function and a sum over the resonances in the corresponding subsystem of two particles. This representation suggests a model for the single resonance production processes, which is expected to be particularly useful at high total c.m. energy.

C - In the double Regge limit, the model predicts a strongly varying form factor in the central vertex. For small mass of the central particle and not too large momentum transfers, the sum of two single Regge exchange diagrams gives a better approximation to the Bardakçi-Ruegg amplitude than the double Regge exchange formula without form factors.

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APPENDIX

THE BARDAKCI-RUEGG AMPLITUDE AS A GENERALIZED HYPERGEOMETRIC FUNCTION

Formula (5) is just another way of writing the series representation of the Bardakci-Ruegg amplitude found by Hopkinson and Plahte ¹⁰⁾. In our notation, the Hopkinson-Plahte formula reads

$$B_5(1) = \sum_{k=0}^{\infty} (-1)^k \binom{AB-12-23}{k} B_4(23, B3+k) B_4(12, A1+k) \quad (A.1)$$

Using

$$B_4(a, b) = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)} \quad \text{and} \quad (-1)^k \binom{a}{k} = \frac{\Gamma(k-a)}{k! \Gamma(-a)} \quad (A.2)$$

we obtain

$$B_5(1) = \Gamma(23) \Gamma(12) \sum_{k=0}^{\infty} \frac{\Gamma(k-AB+12+23)}{k! \Gamma(-AB+12+23)} \frac{\Gamma(B3+k)}{\Gamma(23+B3+k)} \frac{\Gamma(A1+k)}{\Gamma(12+A1+k)} \quad (A.3)$$

This is obviously equivalent to (5).

This relation between the B_5 and the generalized hypergeometric functions allows one to use some known properties of ${}_3F_2$ for the investigation of B_5 . The relations between the different ${}_3F_2$ functions seem to be particularly useful. We give below some of them, which we used in this work. The full list can be found, e.g., in Ref. ¹⁷⁾.

$$\begin{aligned} \frac{{}_3F_2(a, b, c; d, e)}{\Gamma(s) \Gamma(d) \Gamma(e)} &= \frac{{}_3F_2(d-c, d-b, a; s+a, d)}{\Gamma(e-a) \Gamma(s+a) \Gamma(d)} = \\ &= \frac{{}_3F_2(s, d-a, e-a; s+b, s+c)}{\Gamma(a) \Gamma(s+b) \Gamma(s+c)} \end{aligned} \quad (A.4)$$

$${}_3F_2(a, b, c; d, e) = \frac{\Gamma(1-a)\Gamma(d)\Gamma(e)\Gamma(c-b)}{\Gamma(e-b)\Gamma(d-b)\Gamma(1+b-a)\Gamma(c)} *$$

$$* {}_3F_2(b, 1+b-d, 1+b-e; 1+b-c, 1+b-a) +$$

$$+ \frac{\Gamma(1-a)\Gamma(d)\Gamma(e)\Gamma(b-c)}{\Gamma(e-c)\Gamma(d-c)\Gamma(1+c-a)\Gamma(b)} *$$

$$* {}_3F_2(c, 1+c-e, 1+c-d; 1+c-b, 1+c-a)$$

(A.5)

Here $s \equiv d + e - a - b - c$.

Formula (8) is obtained by applying (A.5) and then (A.4) to the hypergeometric function entering formula (5).

- TABLE -

Diagram	Contribution
1	$B_4(AB, A1) B_4(23, B3) F(A1, 23 ; B3+23 ; \frac{13}{AB})$
2	$B_4(AB, A1) B_4(23, B2) F(A1, 23 ; B2+23 ; \frac{12}{AB})$
7	$B_4(-AB, A1) B_4(23, B3) F(A1, 23 ; B3+23 ; \frac{13}{AB})$
10	$B_4(-AB, A1) B_4(23, B2) F(A1, 23 ; B2+23 ; \frac{12}{AB})$
9	$B_4(-13, A1) B_4(B2, B3) F(A1, B2 ; B3+B2 ; \frac{AB}{13})$
8	$B_4(-12, A1) B_4(B2, B3) F(A1, B3 ; B2+B3 ; \frac{AB}{12})$

REFERENCES

- 1) K. Bardakçi and H. Ruegg - Phys.Letters 28B, 342 (1968).
- 2) M.A. Virasoro - Phys.Rev.Letters 22, 37 (1969).
- 3) The kinematic quantities are explained in Fig. 1. We use the simplified notation, e.g., $A1 = -\alpha_{A1}(s_{A1})$, where $s_{A1} = (p_A + p_1)^2$. All particles are treated as incoming.
- 4) G. Veneziano - Nuovo Cimento 57, 190 (1968).
- 5) Another derivation of formula (5) was given by W.J. Zakrzewski, see Ref. 11).
- 6) It can be shown that, if this condition is not satisfied, the model gives exponentially vanishing contributions. See Ref. 11).
- 7) L. Van Hove - Phys.Letters 28B, 429 (1969).
- 8) L. Van Hove - "Longitudinal Phase-Space Plots of Multiparticle Hadron Collisions at High-Energy", CERN Preprint Th. 979, (1968).
- 9) Since the two momentum transfers are limited, also the transverse momenta of two particles are limited. From the momentum conservation, we get then the limitation of the transverse momentum of the third particle. Therefore the analysis of L. Van Hove can be applied to our problem.
- 10) This formula can be also expressed in terms of the Whittaker function. This was suggested by J. Hopkinson and E. Plahte Phys.Letters 28B, 489 (1969).
- 11) W.J. Zakrzewski - "Signature in the Bardakçi-Ruegg Model", Cambridge Preprint, DAMPT 69-6.
- 12) See, e.g., I.S. Gradshteyn and I.M. Ryzhik - "Tables of Integrals, Series and Products", Academic Press, p. 950 (1965).
- 13) Formula (14) is valid not only in the high-energy limit, but also for the complete Bardakçi-Ruegg amplitude. The R_{23} is then given by

$$R_{23} = \sum_{m=0}^{\infty} \frac{1}{m+23} \frac{(-1)^m}{m!} \frac{\Gamma(B3)}{\Gamma(B3-m)} {}_3F_2(-12+AB+B3, A1-m; A1+AB, B3-m)$$

One can also express $B_5(1)$ in the form

$$B_5(1) = B_4(AB, B3) R_{12}$$

where R_{12} is a sum over the resonances in 12 system. This shows very clearly the duality property of the Ruegg-Bardakçi amplitude.

- 14) This property was proved for B_5 by Ruegg and Bardakçi, and for N point function by Chan Hong-Mo and Tsou-Sheung Tsun Phys.Letters 28B, 485 (1969).
- 15) See, e.g., Chan Hong-Mo, K. Kajantie and G. Ranft - Nuovo Cimento 49A, 157 (1967).
- 16) The discussion of κ dependence of the central vertex was given in a much more general framework (which included the case of Bardakçi-Ruegg amplitude as a special example), by Cambridge group. See, e.g., I.T. Drummond, P.V. Landshoff and W.J. Zakrzewski - "The Two-Reggeon/Particle Coupling", Cambridge Preprint, DAMPT 69-3, and the references quoted therein.
- 17) L.J. Slater - "Generalized Hypergeometric Functions", Cambridge University Press (1966).

FIGURE CAPTIONS

Figure_1 Diagram for the five-point function.

Figure_2 Diagram representation of the double Regge limit of the
Bardakçi-Ruegg amplitude.

Figure_3 12 diagrams of Bardakçi and Ruegg.

Figure_4 Kinematical regions in the Van Hove plot.

FIG.1

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FIG.2

FIG.3

FIG. 4