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ON-MASS SHELL CURRENT ALGEBRA AND PION-NUCLEON SCATTERING

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A B S T R A C T

PCAC, current algebra and a weak form of the isobaric model of the "current-nucleon amplitude" are used to calculate the subtraction functions to fixed t dispersion relations for the invariant amplitudes describing pion-nucleon scattering, the scattered particles not being taken off their respective mass shells. The expressions so obtained are projected into partial waves and compared with the experimental information in the low energy region (pion kinetic energy up to 500 MeV). The results are satisfactory.

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A. INTRODUCTION

A great deal of attention has been given in the past few years to soft pion (and other soft pion particle) calculations ¹⁾. Their relative success has called for extensions of the used ideas and techniques to real particles ²⁾. A method is developed in the present work that allows to use the information contained in the current algebra commutators without the need of dealing with soft mesons ³⁾. The constraints imposed by PCAC (Adler's ⁴⁾ self-consistency condition) and PCAC plus current algebra (Weinberg's ⁵⁾ low energy theorems) are maintained as automatically satisfied conditions on eventual off-shell extrapolations of the amplitudes, even after some approximations are made in the calculation.

The subtractions to fixed momentum transfer (t) dispersion relations for the invariant amplitudes describing pion-nucleon scattering are calculated as functions of t . The complete real part of the pion-nucleon elastic scattering amplitude is then constructed by means of once-subtracted dispersion relations, fed with the experimental imaginary parts. The full knowledge of the subtraction functions unfortunately requires some model dependent considerations on the "current-nucleon" transition amplitude, but a somehow "minimal" use of this dependence is required.

The calculation being done with all particles on-shell, the need of arguing about the behaviour of the amplitudes as functions of the squared four-momenta of the mesons is avoided, but the formalism allows an explicit understanding of how the classical off-shell results are obtained from on-shell ones.

In Section B the on-shell pion-nucleon scattering amplitude is decomposed into an equal time commutator and a "current-nucleon" scattering contribution which are described in a more extensive way, respectively in subsections B.1 and B.2. In Section C, the details of the calculation are given. Section D contains the comparison of theory and experiment. Our conclusions are drawn in Section E.

B. DECOMPOSITION OF THE PION NUCLEON AMPLITUDE

The invariant amplitude $T_{ba}(p', q; p, k)$ describing the process nucleon $(p) + \text{pion } a(k) \rightarrow \text{nucleon } (p') + \text{pion } b(q)$, where p, k, p' and q are the corresponding four-momenta and a and b are Hermitian isotopic spin indices of the pions, might be written with the use of the LSZ formalism in terms of the source $j(x)$ of the pion field $\varphi(x)$, explicitly:

$$\begin{aligned} (2\pi)^4 \delta^4(p' + q - p - k) T_{ba}(p', q; p, k) &= \\ &= -i \int d^4x d^4y e^{iqx - ik_y} K_x K_y \langle p' | \theta(x_0 - y_0) [\varphi_b(x), \varphi_a(y)] | p \rangle = \\ &= -i \int d^4x d^4y e^{iqx - ik_y} \left\{ \langle p' | \theta(x_0 - y_0) [j_b(x), j_a(y)] | p \rangle + \langle p' | (D.C.) | p \rangle \right\} \end{aligned} \quad (1)$$

where "D.C." represents the "degenerate components" (or sea-gull terms): a set of matrix elements of equal time commutators originated in the process of bringing the Klein-Gordon operators $K_{x,y}$ inside the retarded commutator.

The PCAC equation for the pion field ⁶⁾

$$\mu^2 F_\pi \varphi^a(x) = \partial^\mu A_\mu^a(x) \quad (2)$$

where A_μ is the weak axial vector current and μ and F_π (~ 98 MeV) are the pion mass and decay constant respectively, may be used to write the source $j(x)$ in (1) as:

$$j^a(x) = (\square + \mu^2) \varphi^a(x) = \frac{\partial^\mu A_\mu^a(x)}{F_\pi} + \square \varphi^a(x) \quad (3)$$

On substituting (3) into (1), the derivatives acting on the pseudoscalar current and the pion field can be shifted, by partial integration, to operate on the step function and the wave exponentials, a by now common technique in PCAC problems. Dropping the surface terms, the modification that takes place under the integral sign in this process is:

$$\theta(x_0 - y_0) [j_b(x), j_a(y)] + \text{D.C.} \rightarrow \theta(x_0 - y_0) [i_b(x), i_a^*(y)] + \text{E.T.C.} + \text{D.C.} \quad (4)$$

where

$$i_b \equiv \frac{i}{F_\pi} q_\mu A_b^\mu + q^2 \varphi_b \quad (5a)$$

$$i_a^* \equiv -\frac{i}{F_\pi} k_\nu A_a^\nu + k^2 \varphi_a \quad (5b)$$

and "E.T.C." represents a set of equal time commutators, to be added to the earlier found set of sea-gulls. Taking into account (3) and using the canonical requirement $[\varphi(x), \varphi(0)] \delta(x_0) = 0$, the degenerate components in (1) and the equal time commutators in (4) add to

$$\begin{aligned} \text{ETC} + \text{DC} = & -\frac{1}{2F_\pi^2} \frac{q^\mu k^\mu}{2} \langle p' | [A_0^b(x), A_\mu^a(y)] \delta(x_0 - y_0) | p \rangle \\ & + \frac{i}{F_\pi} (q^2 + k^2 - \mu^2) \langle p' | [A_0^b(x), \varphi^a(y)] \delta(x_0 - y_0) | p \rangle \end{aligned} \quad (6)$$

so that the full pi-nucleon amplitude is decomposed into the contribution of some equal time commutators (6) and the retarded commutator of the pseudoscalar objects i_b and i_a^* [Eqs. (4) and (5)]. When looked at on the pion mass shell, Eq. (3) does not contain any physical information, and the decomposition we made has an identical character (except for the fact that not any axial current would satisfy the canonical requirement $[\partial A, \partial A] \delta = 0$). However, given

the derivative expression of the source (3), the amplitude satisfies Adler's rule (it vanishes, except for poles, when the four-momentum of either pion is taken to zero, the other pion remaining on-shell), as it can directly be checked on the explicit expressions (4) to (6) ⁷⁾.

Additional physical information can be introduced by assuming a definite form of the ETC in (6). The customary choice is:

$$[A_0^b(x), A_\mu^a(y)] \delta(x_0 - y_0) = i \epsilon_{bac} V_\mu^c(x) \delta^4(x - y) \quad (7a)$$

$$[A_0^b(x), \partial^\mu A_\mu^a(y)] \delta(x_0 - y_0) = i \delta_{ba} S(x) \delta^4(x - y) \quad (7b)$$

where V_μ and S are vector and scalar currents respectively, (7a) being the usual current algebra ⁸⁾ or quark model ⁹⁾ commutator and (7b) following from the σ model ¹⁰⁾ or the quark model.

The known low energy theorems ^{5), 11)} are obtained by setting $t = (q - k)^2 = 0$ and letting $q^2 \rightarrow 0$, $k^2 \rightarrow 0$ in (4) and (6) and assuming the form (2) of the pion field to be a good guess in such an extrapolation procedure (PCAC). The retarded commutator in (4) automatically drops in that limit, since it is $O(q^2, k^2, q \cdot k)$, and reasons can be given ⁵⁾ to drop the second ETC in Eq. (6).

The decomposition of the pi-nucleon amplitude (Fig. 1) into an equal time commutator contribution and a "current-nucleon" contribution [the retarded commutator in the right-hand side of (4) formally corresponds to a first order transition amplitude $p \rightarrow p'$ under the external "currents" i_b and i_a^*] can be made, as above, without going off the pion mass shell. In this paper, we discuss the possibility of estimating the on-shell pion-nucleon amplitude through the above decomposition, thereby extending the current algebra results to physical pions and above threshold. It is to be stressed that the PCAC equation, in this on-shell formulation, has up to now been only used in an identical sense.

The advantages of directly working on the mass shell of the pions are obvious: the problem of guessing how the amplitude varies when going on and off the shell is not present, the generalization to heavier pseudoscalar mesons or lighter targets is straightforward. On the other hand, it is interesting to seek an on-shell understanding of some of the delphic features (using Weinberg's expression) of the classical off-shell approach ¹²⁾.

B.1 The equal time commutators

The contributions of the vector (7a) and scalar (7b) current matrix elements to the invariant pion-nucleon scattering amplitudes defined in the usual manner:

$$T_{ba} = \bar{u}(p') \left\{ \delta_{ba} \left(-A^+ + \frac{q+k}{2} B^+ \right) + \frac{1}{2} [\tau^b \tau^a] \left(-A^- + \frac{q+k}{2} B^- \right) \right\} u(p) \quad (8)$$

can be expressed in terms of the nucleon form factors of these currents [defined in the Appendix, Eqs. (1) and (3)] as

$$A^+ = -F_s(t) / 2 F_\pi^2 \quad (9a)$$

$$B^+ = 0 \quad (9b)$$

$$A^- = \nu F_2^\nu(t) / 2 F_\pi^2 \quad (9c)$$

$$B^- = -(F_1^\nu(t) + F_2^\nu(t)) / 2 F_\pi^2 \quad (9d)$$

The conserved vector isovector current hypothesis (CVC) ¹³⁾ allows to feed the vector contribution with experimental data coming from electromagnetic form factor measurements ¹⁴⁾, but

there is unfortunately not a general principle like CVC allowing us to find an independent experimental evidence on the scalar current form factor. We shall take for $F_s(t)$ the simple ansatz

$$F_s(t) = F_s(0) \frac{m_\sigma^2}{m_\sigma^2 - t} \quad (10)$$

where m_σ (~ 710 MeV) is the mass of a σ (or ϵ), $I=0$, $\pi\pi$ resonance. $F_s(0)$ will be the only left free parameter in our calculation.

B.2 The current nucleon amplitude

Once the contribution of the equal time commutators is given in terms of form factors, there remains the task of calculating the "current nucleon scattering" term, that corresponds to the last blob in Fig. 1. Calling it t_{ab} , we have

$$(2\pi)^4 \delta^4 t_{ab} = -i \int d^4x d^4y e^{iqx - iky} \langle p' | \theta(x-y) [\dot{\psi}(x), \dot{\psi}_a^*(y)] | p \rangle \quad (11)$$

with the definitions (5) of i_b and i_a^* . There is not a model independent information on the current nucleon amplitude ¹⁵⁾ and several authors, in on-shell ¹⁶⁾ or off-shell ^{17), 18)} approaches have estimated it, as a function of energy and momentum transfer, by means of an isobar model of its relevant singularities. In the present paper, we use the isobar estimate of t_{ab} only at the lower limit of the direct physical cut $[\text{fixed } s = (m + \mu)^2, t \leq 0]$. This avoids introducing into the isobar model width (and inelasticity) phenomenological functions, in trying to reproduce the shape of the experiment imaginary part ¹⁹⁾. The complete pion-nucleon amplitude as a function of s and t is then constructed by means of once-subtracted fixed t dispersion relations for the invariant amplitudes (8), the subtractions $[\text{at } s = (m + \mu)^2, \text{ any } t]$ being given by the sum of the ETC contributions (9) and the isobar

estimate of the current-nucleon amplitude (see below). One subtraction is known to be needed in the A^+ dispersion relation, unnecessary in the B^+ case and possibly not needed for A^- and B^- ²⁰⁾. It is however obvious that a subtraction is welcome even if unnecessary since it improves the rate of convergence of the dispersion integral.

On saturating t_{ab} , only s and u channel resonances (and the nucleon itself) are taken into account. The reasons not to include t channel resonances are the following:

- a) in the t channel isospin antisymmetric amplitudes: if explicitly calculated, a ρ pole contribution to t_{ab} is seen to be small enough to be safely neglected ¹⁸⁾. This is due to the fact that, to the extent to which one can speak of ρ dominance of the electromagnetic form factors and of the low energy isospin antisymmetric t channel forces in π -nucleon scattering, Eqs. (9c), (9d) not only provide the right analytic structure and "shape" in the t variable, but also the right normalization (K.S.R.F. relation ²¹⁾), so that the ρ contribution eventually "left" in t_{ab} is small;
- b) in the isospin symmetric amplitudes: there is not a relation corresponding to the KSRF one that would allow one to expect that, at least at very low energies, the t channel isospin symmetric contributions to pion-nucleon scattering are taken into account by the ETC term. However, we have assumed for it a simple σ pole form, and, in contradistinction to the previous case, left the over-all normalization free. If the current-nucleon amplitude has also a contribution from such a σ pole, this would only modify in practice the left-free parameter of Eq. (9a).

It is important to stress the difference between our resonance saturation of the threshold current nucleon amplitude and a similar procedure as directly applied to, say, the full

pion-nucleon amplitude. This second possibility would not automatically lead to the current algebra and Adler's rule constraints on eventual off-shell extrapolations of the full amplitude that are known to be approximately satisfied in nature. In either case one would of course have the same analytic structure of the full pion-nucleon amplitude (poles at the resonances, if for the sake of argument we refer to the narrow width approximation) but, due to the different Lorentz structure of pion matrix elements and current matrix elements, extra constant non-polar contributions appear in the current nucleon case [see for instance, the nucleon contribution, Eqs. (12) below]. These explicit extra constants are such as to compensate the resonance (or baryon pole) contributions down to the PCAC + current algebra result in the off-shell limit. In a dispersion theoretic understanding of the full pion-nucleon amplitude they would be connected to distant (slowly energy dependent) singularities.

C. DETAILS OF THE CALCULATION

1) The nucleon vector form factors in the ETC contributions, Eqs. (9c), (9d), as stated earlier, are taken from the fit of Ref. 14). Up to the not very high maximum momentum transfers ($t_{\max} \simeq -7 \text{ GeV}^2$) with which we shall be concerned, different fits to the electron-nucleon data do not differ much.

2) The contribution of the single nucleon intermediate state to the current nucleon amplitude, Eq. (11), can be written in terms of the strong pion-nucleon coupling constant G , and the first axial current form factor $G_A(q^2)$, defined in the Appendix, Eq. (2). Explicitly:

$$A^+ = - \left(\frac{G_R(\mu^2)}{F_R} \right)^2 \left(\frac{G F_R}{G_R(\mu^2)} + \frac{m}{2} \right) \quad (12a)$$

$$A^- = 0 \quad (12b)$$

$$B^+ = G^2 \left(\frac{1}{s-m^2+i\epsilon} - \frac{1}{u-m^2-i\epsilon} \right) \quad (12c)$$

$$B^- = G^2 \left(\frac{1}{s-m^2+i\epsilon} + \frac{1}{u-m^2-i\epsilon} \right) + \frac{G_A(\mu^2)}{2F_\pi^2} \quad (12d)$$

where the Goldberger-Treiman relation (an identity on the pion mass shell):

$$\frac{2\mu^2 F_\pi G_A(q^2)}{q^2 - \mu^2} = 2m G_A(q^2) + \frac{q^2}{2m} H_A(q^2) \quad (13)$$

following from Eq. (3), has been used to partially recombine the axial current form factors into the strong pion-nucleon coupling constant. The value of $G_A(\mu^2)$ is not experimentally known [although $G_A(q^2)$ with q^2 sensitively different from zero might in principle be measured in muon capture processes]. We take the approximation $G_A(\mu^2) \simeq G_A(0)$ ²²⁾, since this latter quantity is measured in β decay. Note that this does not mean that we obtained Eqs. (12) off the pion mass shell.

The analytic structure of the nucleon contribution in Eqs. (12) (simple poles at $s=m^2$ and $u=m^2$) is of course the same as in the pion-nucleon amplitude as directly calculated with, say, ps-ps or pv-pv coupling, but they differ by the presence of constant terms in A^+ and B^- . These extra terms are such as to make the nucleon contribution to the amplitude very small at threshold (of the order of the deviation from an "exact" Goldberger-Treiman relation and actually vanishing in the massless meson limit ²³⁾). It is a well-known fact ²⁴⁾ that the usual Born approximation to the invariant amplitudes, when projected into partial waves, gives a very big contribution to the S waves, which is difficult to

compensate with other contributions down to the experimental values. There are several ways to overcome this difficulty ²⁵⁾, the expressions (12), that refer to invariant amplitudes and have been obtained for physical mass pions and nucleons, present an alternative one.

3) The resonance contribution to the subtractions. In the next paragraph, an attempt will be made to predict the low energy (laboratory pion kinetic energy ranging from 0 to 500 MeV) pion-nucleon S and P partial waves. Only the $P_{33}(1236)$ and $P_{11}(1466)$ resonances (the CERN "experimental" phase shift analysis ²⁶⁾ is thereafter referred to, for the sake of definiteness) lie within this region. However, the S waves are also resonant not far above the mentioned upper energy, and we shall consider the corresponding states, i.e., $S_{31}(1635)$ and $S_{11}(1548?, 1709)$, in the calculation of the threshold subtractions. The influence of these resonances is expected to be considerable in their own waves, even below the resonant energies. For the P_{33} resonance, we choose the set of nucleon-axial current isobar form factors that lead Raman ¹⁷⁾ to the best agreement with experiment at threshold, in an off the pion shell approach. The form factors and propagators associated to the rest of the considered resonances, all of which have spin $\frac{1}{2}$, are chosen in complete analogy to the nucleon form factors. The propagator of the isospin $\frac{3}{2}$ isobar is taken to be:

$$P_{\mu\nu} = \left(g_{\mu\nu} - \frac{2p_\mu p_\nu}{3M^2} - \frac{\gamma_\mu \gamma_\nu}{3} - \frac{p_\mu \gamma_\nu - \gamma_\mu p_\nu}{3M} \right) \frac{\not{p} + M}{p^2 - M^2} \quad (14)$$

It is to be reminded that the problem of introducing width and inelasticity "functions" into an isobaric calculation in such a way as to reproduce the shape of the experimentally measured imaginary part is skipped by using the isobar model only at threshold.

The explicit form of the contributions of the isobars to the subtraction are given in the Appendix. The calculation would in principle require full knowledge of the form factors describing the nucleon-current-isobar vertices, but the use of generalized Goldberger-

Treiman relations like (13), obtained for each of the resonances by sandwiching the source equation (3) between a nucleon and an isobar [Appendix, Eq. (11)] allows to recast, as in the nucleon case, the corresponding contributions in terms of strong pi-nucleon resonance coupling constants G and some weak axial form factors [more compactly, the $K(0)$ functions measuring the deviation from pure pion pole dominance [Appendix, Eqs. (5), (7), (9)].

The weak properties of the resonances are not known: an experimental knowledge of their $K(0)$ values is not available. Moreover, the current form factors and the strong coupling constants of the isobars might in general be functions of the invariant square mass of the exchanged resonance (s or u in the corresponding channel). The calculation shows that the contributions of all but the P_{33} resonance to the subtractions are small. Thus, for the rest of the resonances, the results are not sensitive to small variations in their G or $K(0)$ values. We shall therefore work in the approximation in which the coupling constants at threshold (that we are using) coincide with their measure as given by the resonance elastic width [$G^2(1466) \simeq 67.3$, $G^2(1548) \simeq 0.86$, $G^2(1709) \simeq 4.2$, $G^2(1635) \simeq 0.95$]. As far as the $K(0)$ are concerned we assume exact pion pole dominance of the divergence of the axial current at $q^2=0$, it is to say we take $K(0)=1$ ²⁷⁾. This is the only point where explicit use of PCAC in a non-identical sense is made. In the contribution of the P_{33} resonance, with our choice of form factors, $K(0)$ does not appear [Appendix, Eq. (5)] but the results are more sensitive to the value of the strong coupling constant G . We choose to determine the used threshold value of G by requiring that the contribution of the P_{33} resonance to B^+ at threshold (the amplitude for which the dispersion relations are most rapidly converging) is the same when calculated with the isobaric model, or by means of a dispersion relation in terms of the experimental imaginary part of the amplitude. This provides the sum rule

$$\text{Re } B_{\text{isob.}}^+(s=(m+\mu)^2, t=0) = \frac{32}{3} \int_{\mu}^{\infty} \frac{dw' (2m-E')}{(w'^2-\mu^2) |\vec{q}'|^2} \text{Im } f_{33}(w') \quad (15)$$

where the expression for the left-hand side can be taken from the Appendix, Eq. (5c). By feeding the integral with the CERN phase shifts ²⁶⁾ and taking $M = 1235.8$ MeV, we get $G_{33} \simeq 2.0$, some 7% below the value one would get with the method used for the rest of the resonances.

4) The subtracted dispersion integrals. The imaginary parts of the $A^\pm$, $B^\pm$ amplitudes under the dispersion relation integrals are developed in partial waves and expressed in terms of the experimental phase shifts of Ref. 26). Since we are concerned with the prediction of the low energy S and P waves, we work in the approximation in which only the resonant phase shifts in these waves (S_{11} , S_{31} , P_{11} , P_{33}) are taken into account in the evaluation of the dispersive integrals.

D. PROJECTION INTO PARTIAL WAVES AND COMPARISON WITH EXPERIMENT

In the preceding paragraphs, a method was proposed to calculate the subtractions to the $A^\pm$ and $B^\pm$ amplitudes at the lower integration limit of fixed t dispersion relations. The subtracted integrals are evaluated with the introduction of the experimental imaginary parts, and the resulting expressions for the real part of the amplitudes are projected to obtain the real parts of the first two partial waves, in the low energy region (0 to 500 MeV pion kinetic laboratory energy) where the scattering is predominantly elastic and dominated by the S and P waves. The comparison to the experiment might be directly made in the case of the isospin anti-symmetric partial waves, where no parameter has been left free. The isospin symmetric waves contain the unknown factor $F_s(0)$ of (10), related to the strength of the $J=0$ t channel interaction. The fitting method we use is most naïve: the free parameter is adjusted by sight to the S wave, where the relative contribution of this ω term is most important. This yields $F_s(0) \simeq -0.52$.

The results are given in Figs. 2 to 7, where the experimental and predicted values of $|\bar{q}|^{-l} \text{Re } f_{l\pm}^{\pm}$; $l = 0, 1$ are plotted against $|\bar{q}|$, the modulus of the centre-of-mass momentum of one of the particles. The units are chosen such that the average mass of the pion triplet equals one. The agreement is in general seen to be better in the antisymmetric waves, and one is tempted to attribute this fact to the contribution of the scalar form factor, of which the chosen narrow σ pole is probably only a very rough estimate.

The isospin antisymmetric S wave, shown in Fig. 2, is seen to be very nicely predicted. The contribution of all but the equal time commutator term is relatively very small at threshold: the ρ meson dominance of the corresponding scattering length would then not be a mere accident.

The fit to the isospin symmetric S wave is not as good (Fig. 3). A stronger "SIGMA" ($\pi\pi$, $J=0$ interaction in the crossed channel) seems to be needed at the very lowest energies. Moreover, the theoretical curve deviates from the experimental points above 300 MeV. There are, unfortunately, too many reasons that could be invoked to explain this discrepancy: the Legendre polynomial expansion of the imaginary part of the amplitudes might not converge above that energy²⁸⁾, the σ pole approximation of the scalar form factor could anew be blamed guilty and we have neglected, both in the subtractions and in the dispersive integral, the influence of the resonant D waves, that might become more important with increasing energy, etc. The approximate vanishing of the isospin symmetric scattering length (a ρ dominance or off-shell current algebra prediction) is seen to be in our on-shell approach due to compensation between different contributions (mostly σ and P_{33}), much as in partial wave dispersion relation fits²⁹⁾.

The amplitudes $f_{1-}^{\pm}$, shown in Figs. 4 and 5, are seen to be well predicted in shape but systematically displaced in absolute value. It is clear that these "small" P waves would be most influenced by the non-resonant P wave phase shifts neglected in the evaluation of the subtracted dispersion integrals.

The amplitudes $f_{1+}^{\pm}$ are seen to be well predicted (Figs. 6 and 7). The equal time commutators contribute little to them, the major terms coming from the baryon and of course the P_{33} resonance, both through their contributions to the isobaric subtraction and the dispersive integral (that provides the "shape"). Their precision can be looked at as a check of the isobaric method used to evaluate the threshold current-nucleon amplitude, at least as far as the P_{33} resonance is concerned.

E. CONCLUSIONS

A formalism combining the information contained in PCAC and current algebra has been used, along with a threshold saturation argument for an unknown current nucleon amplitude, to calculate the subtractions in once-subtracted dispersion relations for the invariant pion-nucleon amplitudes. The fact that we do not restrict ourselves to soft pion kinematics, allows a natural extension of results above threshold: the low energy pion-nucleon S and P waves are predicted in the $0 \rightarrow 500$ MeV pion kinetic laboratory energy region.

The classical current algebra plus PCAC off-shell result ^{4),5)} for the soft pion-nucleon scattering lengths is recovered in the case of the t channel isospin antisymmetric amplitude in the sense that all on-shell contributions but the axial-axial ETC are explicitly seen to be very small at this point. The result for the isospin symmetric scattering length is due to a cancellation of different contributions, mainly a σ commutator and the P_{33} resonance. This σ ETC is necessary when trying to extend the current algebra results above threshold, as pointed out earlier by several authors ^{16),17),18)}; in an on-shell approach it is moreover natural and unavoidable.

The comparison of the theory with the low energy phase shift analysis of the pion-nucleon data is satisfactory, very specially in the case of the isospin antisymmetric combination of the amplitudes where no unknown form factors (no scalar form factors) play a role.

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A P P E N D I X

FORM FACTORS OF CURRENTS, GENERALIZED GOLDBERGER-TREIMAN RELATIONS AND THE ISOBARIC SUBTRACTIONS

The nucleon form factors of vector, axial and scalar currents are defined in the following manner:

$$\langle p' | V_\mu^a(x) | p \rangle = e^{-iqx} \bar{u}' \left\{ \gamma_\mu F_1^V(q^2) - i \frac{\sigma_{\mu\nu} q_\nu}{2m} F_2^V(q^2) \right\} \frac{\tau^a}{2} u \quad (1)$$

$$\langle p' | H_\mu^a(x) | p \rangle = e^{-iqx} \bar{u}' \left\{ \gamma_\mu G_A(q^2) - \frac{q_\mu}{2m} H_A(q^2) \right\} \gamma_5 \frac{\tau^a}{2} u \quad (2)$$

$$\langle p' | S(x) | p \rangle = e^{-iqx} \bar{u} F_S(q^2) \frac{1}{2} u \quad (3)$$

where $q = p' - p$ is the four-momentum transfer.

The definition of the matrix elements of the axial current between a nucleon and a P_{33} isobar (to be labelled "1" in what follows) is taken to be, Ref. 17):

$$\begin{aligned} \langle p' | H^\mu | p_i \rangle = & \bar{u} \left\{ G_A^1 q^{\mu\nu} + H_A^1 \epsilon^{\mu\nu\lambda\sigma} (p_i + p')_\nu q_\lambda \gamma_\sigma q^\rho + \right. \\ & \left. + F_A^1 \left[(p_i + p')^\mu - q^\mu \frac{(p_i + p') \cdot q}{q^2} \right] + G_A^1 \frac{q^\mu q^\rho}{q^2} \right\} u_\rho^1 \end{aligned} \quad (4)$$

where the q dependence of the form factors and the isospin index have not been made explicit. Only the first and fourth form factors are not transversal to q . The matrix elements of an axial current between a nucleon and a spin $\frac{1}{2}$ resonance are written in complete analogy to the nucleon form factors (2), the only difference being

that a γ_5 or $(-i\gamma_5)$ factor has to be erased from the axial or pseudoscalar current expression, respectively, in the case of negative parity resonances.

The explicit form of the contribution of the isobars to the subtraction of the current nucleon amplitude at the lower limit of integration ($s = (m + \mu)^2$, fixed $t \leq 0$) of the subtracted dispersive integrals for the $A^\pm$ and $B^\pm$ amplitudes are listed below:

i) For P_{33} exchange (the index 1 labels this resonance):

$$A^+(t) = -\frac{G_1^2}{3} \left(\beta_1 - \frac{m}{3M_1^2} t + [(m+M_1)t + \alpha_1] \left[\frac{1}{(m+\mu)^2 - M_1^2} + \frac{1}{(m-\mu)^2 - M_1^2 - t} \right] \right) \quad (5a)$$

$$A^-(t) = \frac{G_1^2}{6} \left(\frac{4m^2}{3M_1^2} \left[\mu + \frac{t}{4m} \right] + [(m+M_1)t + \alpha_1] \left[\frac{1}{(m+\mu)^2 - M_1^2} - \frac{1}{(m-\mu)^2 - M_1^2 - t} \right] \right) \quad (5b)$$

$$B^+(t) = \frac{G_1^2}{3} \left(\frac{4m}{3M_1^2} \left[\mu + \frac{t}{4m} \right] + [t + \delta_1] \left[\frac{1}{(m+\mu)^2 - M_1^2} - \frac{1}{(m-\mu)^2 - M_1^2 - t} \right] \right) \quad (5c)$$

$$B^-(t) = \frac{G_1^2}{6} \left(\gamma_1 - \frac{t}{3M_1^2} + [t + \delta_1] \left[\frac{1}{(m+\mu)^2 - M_1^2} + \frac{1}{(m-\mu)^2 - M_1^2 - t} \right] \right) \quad (5d)$$

with the definitions:

$$\alpha_1 \equiv 2q_1^2 \left[(M_1 + m) + \frac{(M_1 - m)(E_1 + m)}{3(E_1 - m)} \right] \quad (6a)$$

$$\beta_1 \equiv \left[(m^2 + \mu^2)2m + 4(M_1 + m)(M_1^2 + \mu^2 + M_1 m - m^2) - 2M_1^2 m \right] / 3M_1^2 \quad (6b)$$

$$\gamma_1 \equiv 2 [M_1^2 - m^2 + 3\mu^2 + 2M_1 m] / 3M_1^2 \quad (6c)$$

$$\delta_1 \equiv 2q_1^2 \left[1 - \frac{(E_1 + m)}{3(E_1 - m)} \right] \quad (6d)$$

where E_1 and q_1 are the nucleon centre of momentum frame energy and modulus of three momentum at the resonance position, the rest of the notations being self-explanatory.

- ii) For P_{11} exchange (the index "2" thereafter labelling this resonance):

$$A^\pm(t) = G_2^2 \beta_2 \left\{ \frac{2m\mu + \mu^2}{(m+\mu)^2 - M_2^2} \pm \frac{\mu^2 - 2m\mu - t}{(m-\mu)^2 - t - M_2^2} \right\} \quad (7a)$$

$$B^\pm(t) = G_2^2 \left\{ \frac{\alpha_2(2m\mu + \mu^2) + \delta_2}{(m+\mu)^2 - M_2^2} \mp \frac{\alpha_2(\mu^2 - 2m\mu - t) + \delta_2}{(m-\mu)^2 - t - M_2^2} \right\} \quad (7b)$$

where

$$\alpha_2 \equiv [K_2(0)]^2 / (M_2 + m)^2 \quad (8a)$$

$$\beta_2 \equiv K_2(0)[2 - K_2(0)] / (M_2 + m) \quad (8b)$$

$$\gamma_2 \equiv [(K_2(0))^2(M_2 + m) + (1 - 2K_2(0))M_2 + (1 + 2K_2(0))m] / (M_2 + m) \quad (8c)$$

The quantity $K_2(0)$ will be defined below, the rest of the notation is self-explanatory.

- iii) For S_{31} and S_{11} exchange (the index "3" thereafter labelling these resonances)

$$A^{\pm}(t) = - \begin{pmatrix} 2 \\ -1 \end{pmatrix} G_3^2 \left\{ \frac{\alpha_3(2m\mu + \mu^2) + \beta_3}{(m+\mu)^2 - M_3^2} \pm \frac{\alpha_3(\mu^2 - 2m\mu - t) + \beta_3}{(m-\mu)^2 - t - M_3^2} \right\} \quad (9a)$$

$$B^{\pm}(t) = - \begin{pmatrix} 2 \\ -1 \end{pmatrix} G_3^2 \left\{ \frac{\delta_3 - \gamma_3(2m\mu + \mu^2)}{(m+\mu)^2 - M_3^2} \mp \frac{\delta_3 - \gamma_3(\mu^2 - 2m\mu - t)}{(m-\mu)^2 - t - M_3^2} \right\} \quad (9b)$$

the $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ column vector referring only to the S_{31} isobar.
The definitions of the Greek symbols in (9a) and (9b) are

$$\alpha_3 \equiv [K_3(0)]^2 (M_3 - 2m) / (M_3 - m)^2 \quad (10a)$$

$$\beta_3 \equiv (M_3 + m)(1 - K_3(0)) [M_3(K_3(0) - 1) + m] / (M_3 - m) \quad (10b)$$

$$\gamma_3 \equiv [K_3(0)]^2 / (M_3 - m)^2 \quad (10c)$$

$$\delta_3 \equiv (1 - K_3(0))^2 + 2m[K_3(0)]^2 / (M_3 - m) \quad (10c)$$

In writing the set of Eqs. (5), (7) and (9) use has been made of a generalized Goldberger-Treiman identity obtained for each resonance by sandwiching the source equation between a nucleon and the isobar. The corresponding expressions relating the current form factors to the pi-nucleon-isobar coupling constants as defined by the width of the resonances, are:

$$- \frac{\sqrt{2}\mu^2 F_\pi G_1 K_1(q^2)}{q^2 - \mu^2} = G_A^1(q^2) + \mathcal{L}_A^1(q^2) \quad (11a)$$

$$\frac{2\mu^2 F_\pi G_2 K_2(q^2)}{q^2 - \mu^2} = (M_2 + m) G_A^2(q^2) + \frac{q^2}{M_2 + m} H_A^2(q^2) \quad (11b)$$

$$\frac{\sqrt{2}\mu^2 F_\pi G_3 K_3(q^2)}{q^2 - \mu^2} = (M_3 - m) G_A^3(q^2) + \frac{q^2}{M_3 + m} H_A^3(q^2) \quad (11c)$$

The preceding expressions can be looked at as a definition of the $K_i(q^2)$ functions measuring the deviation from pure pion pole dominance $[K_i(\mu^2) = 1]$, as in the nucleon case.

REFERENCES AND FOOTNOTES

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FIG.1

FIG. 2

FIG. 3

FIG. 4

FIG.5

FIG.6

FIG. 7