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EXPLICIT CONSTRUCTION OF THE N - POINT FUNCTION IN THE
GENERALIZED VENEZIANO MODEL

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A B S T R A C T

A previous conjecture for constructing the N - point function in the generalized Veneziano model is proved, and the explicit solution given. In addition to having the usual properties of Veneziano-type amplitudes, the result is shown (i) to satisfy bootstrap consistency in treating all particles as bound states of others, and (ii) to yield the decay correlations for particles with spins.

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Explicit solutions of generalized Veneziano ¹⁾ amplitudes for the five, six and seven-point functions have already been given ^{2),3),4)}. Moreover, a prescription for constructing the amplitude for general N has been suggested. In this paper, we show that the conjectured prescription is indeed valid and gives also the explicit solution.

We consider only an idealized system with only bosons and without internal symmetry. Moreover, all bosons in the model belong to the same family, - they lie either on the parent trajectory which contains a spinless particle, or on its daughters.

Our first object is the N -point function in which all external lines have zero spin. As usual, the amplitude is written as a sum of $(N-1)!/2$ terms, each corresponding to a "distinct" permutation of the N external lines, cyclic and anticyclic permutations being here regarded as identical. We need consider only one such term, corresponding, say, to the permutation $(1, 2, 3, \dots, N)$.

For convenience, we define a "partition" of the diagram $(1, 2, \dots, N)$ as a separation of the N external lines, without changing their order, into two groups, each with at least two lines. Thus, for example

$$P: (i, i+1, \dots, j)(j+1, j+2, \dots, N, 1, 2, \dots, i-1); i < j < N \quad (1)$$

Each partition then corresponds to a possible Mandelstam channel and a Regge trajectory α_P . In what follows, the partition (1) will also be denoted by the indices (i, j) . Further, in analogy to Refs. 3) and 4), we introduce for each partition $P=(i, j)$ the variables $x_P = x_{i,j} = -1 - \alpha_{i,j}$, [where, $\alpha_{i,j} = \alpha_{i,j}^0 + \alpha' s_{i,j}$; $s_{i,j} = (p_i + p_{i+1} + \dots + p_j)^2$] and the corresponding "conjugate" variables $u_P = u_{i,j}$.

Two partitions $P=(i, j)$ and $\bar{P}=(\bar{i}, \bar{j})$ are said to be "dual" to each other if the two sets of numbers $(i, \dots, j)$ and $(\bar{i}, \dots, \bar{j})$ intersect but do not contain each other, i.e.,

$$(i, \dots, j) \cap (\bar{i}, \dots, \bar{j}) \neq \emptyset, (i, \dots, j), \text{ or } (\bar{i}, \dots, \bar{j}) \quad (2)$$

This is an extension of the normal concept of "duality" between, e.g., the s and t channels in the four-point function.

It is easily seen that "dual" partitions cannot have coincident poles, since these do not correspond to Feynman graphs. The conditions on the "conjugate" variables $u_{\bar{p}}$ which guarantee the absence of such coincident poles can then be written as

$$u_p = 1 - \prod_{\bar{p}} u_{\bar{p}} \quad (3)$$

where the product is taken over all $\bar{p}$ dual to p , or

$$u_{i,j} = 1 - \prod_{p,q} u_{p,q} \prod_{r,s} u_{r,s} \quad (4)$$

$$[1 \leq p < i; i \leq q < j; i < r \leq j; j < s \leq N-1]$$

Next we wish to solve Eqs. (3) or (4) in terms of the independent variables corresponding to Fig. 1a, namely $u_{1,j}$ ($j=2, \dots, N-2$). The following solution,

$$u_{p,q} = \frac{(1 - u_{1,p} u_{1,p+1} \dots u_{1,q-1})(1 - u_{1,p-1} u_{1,p} \dots u_{1,q})}{(1 - u_{1,p-1} u_{1,p} \dots u_{1,q-1})(1 - u_{1,p} u_{1,p+1} \dots u_{1,q})}$$

$$[p = 2, \dots, N-2; q = 3, \dots, N-1; p < q] \quad (5)$$

is readily shown by substitution to satisfy all conditions in Eq. (4) if in (5) $u_{i,i} = 0$ by definition.

Under a cyclic change of independent variables from $u_{1,j}$ ($j=2,\dots,N-2$), Fig. 1a, to $u_{2,j}$ ($j=3,\dots,N-1$), Fig. 1b, the Jacobian is found to satisfy the following recurrence relation,

$$\Delta_N = (-1) \frac{\mu_{1,2} \mu_{1,3} \cdots \mu_{1,N-3}}{1 - \mu_{1,2} \mu_{1,3} \cdots \mu_{1,N-2}} \cdot \Delta_{N-1} \quad (6)$$

- Figure 1a -

- Figure 1b -

By induction, this then gives

$$\Delta_N = \frac{\partial(\mu_{2,3}, \dots, \mu_{2,N-1})}{\partial(\mu_{1,2}, \dots, \mu_{1,N-2})} = (-1)^{N-1} \frac{\mu_{1,2}^{N-4} \mu_{1,3}^{N-5} \cdots \mu_{1,N-3}}{\prod_{i=3}^{N-2} (1 - \mu_{1,2} \mu_{1,3} \cdots \mu_{1,i})} \quad (7)$$

Using (5), Δ_N can be rewritten as

$$\Delta = (-1)^{N-1} J_2 / J_1 \quad (8)$$

where

$$J_1 = \prod_{i < j} (\mu_{i,j})^{j-i-1} \quad [i = 2, \dots, N-2; j = 3, \dots, N-1]$$

$$J_2 = \prod_{i < j} (\mu_{i,j})^{j-i-1} \quad [i = 3, \dots, N-1; j = 4, \dots, N]. \quad (9)$$

Moreover, solving (5) for $u_{p,q}$ ($p = 3, \dots, N-1$; $q = 4, \dots, N$; $p < q$; $u_{p,N} = u_{1,p-1}$) in terms of $u_{2,j}$ ($j = 3, \dots, N-1$), one obtains

$$\mu_{p,q} = \frac{(1 - \mu_{2,p} \mu_{2,p+1} \dots \mu_{2,q-1})(1 - \mu_{2,p-1} \mu_{2,p} \dots \mu_{2,q})}{(1 - \mu_{2,p-1} \mu_{2,p} \dots \mu_{2,q-1})(1 - \mu_{2,p} \mu_{2,p+1} \dots \mu_{2,q})} \quad (10)$$

which is the same as (5) after a cyclic permutation of the N external lines.

We now define

$$B_N = \int_0^1 \prod_{j=2}^{N-2} du_{1,j} \left(\frac{1}{J_1} \right) \prod_P (\mu_P)^{x_P} \quad (11)$$

where the last product is to be taken over all partitions P .

Under a change of the integration variables in (11) from $u_{1,j}$ to $u_{2,j}$, one obtains by (8)

$$B_N = \int_0^1 \prod_{j=3}^{N-1} du_{2,j} \left(\frac{1}{J_2} \right) \prod_P (\mu_P)^{x_P} \quad (12)$$

with μ_P given by (10). This means that B_N is invariant under a cyclic permutation of the N external lines, (crossing symmetry).

Using exactly the same arguments as Refs. 2), 3) and 4), one shows that B_N , as given by (11), is analytic in $x_{1,j}$ except for simple poles at $x_{1,j} = -1, -2, \dots$. By the cyclic invariance just proved above, this is extended to x_P for all partitions P , (analyticity and duality). Moreover, because of the conditions (3) there are no unwanted coincident poles.

Consider next the pole at $\alpha_{1,j} = -1 + x_{1,j} = 0$. The residue is given by:

$$\text{Res.}_{\alpha_{1,j}=0} B_N(x_p) = Q(\mu_{1,j}; x_p, P \neq (1,j)) \Big|_{\mu_{1,j}=0} \quad (13)$$

where

$$Q = \int_0^1 \prod_{\substack{k=2 \\ k \neq j}}^{N-2} d\mu_{1,k} \left(\frac{1}{J_1} \right) \prod_{P \neq (1,j)} (\mu_P)^{x_P} \quad (14)$$

On putting $\mu_{1,j} = 0$, one notices by (5) that $\mu_{\tilde{P}} = 1$ for all $\tilde{P}$ dual to $(1,j)$. One obtains then,

$$J_1 \rightarrow \prod_{p < q} (\mu_{p,q})^{q-p-1} \prod_{r < s} (\mu_{r,s})^{s-r-1} \\ [p = 2, \dots, j-1; q = 3, \dots, j; r = j+1, \dots, N-2; s = j+2, \dots, N-1] \quad (15)$$

Moreover, in (11)

$$\prod_{P \neq (1,j)} (\mu_P)^{x_P} \rightarrow \prod_{P'} (\mu_{P'})^{x_{P'}} \cdot \prod_{P''} (\mu_{P''})^{x_{P''}} \quad (16)$$

where the products run over $P' = (i', j') \neq (1, j)$ and $P'' = (i'', j'') \neq (1, j)$ such that

$$P': (i', \dots, j') \subset (1, \dots, j) \\ P'': (i'', \dots, j'') \cap (1, \dots, j) = \emptyset, \sigma(1, \dots, j) \quad (17)$$

- Figure 2 a -

- Figure 2 b -

- Figure 2 c -

In other words, P' and P'' run over all partitions respectively of the diagrams in Figs. 2b and 2c. Thus,

$$\text{Res.}_{\alpha_{1,j}=0} B_N(x_P) = B_{j+1}(x_{P'}) \cdot B_{N-j+1}(x_{P''}). \quad (18)$$

corresponding exactly to the reduction in Fig. 2a (bootstrap consistency). We note also that there is no dependence in (18) on $x_{\tilde{P}}$ for any $\tilde{P}$ dual to $(1,j)$. This absence of correlation between the two parts of the diagram is necessary for interpreting the bound state at $\alpha_{1,j}=0$ as a spinless particle.

Similarly, for the pole at $\alpha_{1,j} = l$, the residue is given by

$$\text{Res.}_{\alpha_{1,j}=l} B_N(x_P) = \left[\frac{1}{l!} \cdot \frac{\partial^l}{\partial \mu_{1,j}} Q \right]_{\mu_{1,j}=0} \quad (19)$$

In order to interpret this pole as a particle of spin l , the residue should be a polynomial of total degree l in the variables $x_{\tilde{P}}$ for all $\tilde{P}$ dual to $(1,j)$. This is readily seen to be indeed the case since the dependence on $x_{\tilde{P}}$ arises only from

$$\left[\frac{\partial^l}{\partial \mu_{1,j}} \prod_{\tilde{P}} (\mu_{\tilde{P}})^{x_{\tilde{P}}} \right]_{\mu_{1,j}=0} \quad (20)$$

and $\mu_{\tilde{P}} = 1$ when $\mu_{1,j} = 0$ 5).

As explained in Ref. 3), bootstrap consistency requires that (19) be interpreted as the $(j+1)$ -point function corresponding to Fig. 2b with an external line having spin l . The decay correlations of the spinning particle into $(N-j)$ spinless ones are also in (19) explicitly given. It is clear then that by repeated application of this reduction technique, one has a consistent scheme for constructing

the N - point function with any number of spinning particles having various decay modes, the decay correlations being also explicitly given.

Finally, Regge asymptotic behaviour of B_N as given by (11) or (12) can be proved by a straightforward extension of the proof given by Bardakçi and Ruegg²⁾ for the Reggeization of the five-point function. To do this, we first make a change of integration variables in (12) as follows:

$$u_{2,j} = \exp \left\{ - \frac{\ell_2 \ell_3 \cdots \ell_{j-1}}{x_{2,3} x_{3,4} \cdots x_{j-1,j}} \right\}$$

$$[j = 3, 4, \dots, N-1]$$
(21)

In the limit when $x_{i,i+1} \rightarrow \infty$ ($i=2, \dots, N-2$) at fixed values of $x_{1,i}$ ($i=2, \dots, N-2$) and $K_j = (x_{j-1,j} x_{j,j+1}) / (x_{j-1,j+1})$ ($j=3, \dots, N-2$), namely the multi-Regge limit corresponding to Fig. 1a with 1 and N as incoming particles, one can write (12) as,

$$B_N \rightarrow \prod_{i=2}^{N-2} (x_{i,i+1})^{-x_{1,i}-1} \int_0^\infty \prod_{i=2}^{N-2} d\ell_i \lim H_N(\ell)$$
(22)

where we have assumed that one can interchange the limit and the integral, as was proved for $N=5$ ²⁾. It can then be shown that the following recurrence relation is valid;

$$\frac{\lim H_N(\ell)}{\lim H_{N-1}(\ell)} = (\ell_{N-2})^{x_{1,N-2}} \exp \left[-\ell_{N-2} - \sum_{i=2}^{N-3} \frac{\ell_i \ell_{i+1} \cdots \ell_{N-2}}{K_{i+1} K_{i+2} \cdots K_{N-2}} \right]$$
(23)

From this one can obtain the Regge residue by induction using the known result for $N=5$ and 6 ^{2),3)}. The actual Regge limit for $\text{Re } \alpha_{1,j} \rightarrow +\infty$ can be obtained as in Bardakçi and Ruegg²⁾ by a rotation of axes, (Reggeization).

Our proof that B_N , as defined in (11), have all the required properties is now complete.

We note that an alternative form of B_N , suggested by Hopkinson and Plahte, is available ⁵⁾ which gives B_N as an infinite series over B 's of lower order and which may be more convenient for practical applications. The integral form (11) can also be derived using a different technique developed by Bardakçi and Ruegg ⁶⁾.

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