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LARGE ANGLE AND LARGE MOMENTUM TRANSFER LOWER BOUNDS
FOR THE DIFFERENTIAL CROSS-SECTION IN THE LEHMANN
ELLIPSE AND THE MANDELSTAM CUT PLANE CASES

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ABSTRACT

Exact expressions for the lower bounds of the maximum of the scattering amplitude or differential cross-section between two (complementary) constant angles, as well as between $t = \text{const.}$ and $u = \text{const.}$ are derived using the Nevanlinna principle, both for finite energies as in the high energy limit. Computations were performed for two distinct assumptions about the analyticity domain of the scattering amplitude : for the shrinking ellipse of Lehmann and for the Mandelstam cosine cut complex plane.

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1. INTRODUCTION

Since the classical paper ¹⁾ of Cerulus and Martin on the lower bounds of the large angle scattering amplitude, much interest ²⁾⁻⁵⁾ was paid to this question, especially in the last time when the available high energy experimental data on the two-body large angle scattering have considerably increased. There is a strong indication that the logarithm of the large angle differential cross-section decreases as $s^{\frac{1}{2}} \ln s$ (as usual, s is the square of the c.m. total energy) which is consistent (proportional) with the asymptotic behaviour for the lower bound found by Cerulus and Martin or with that corresponding to the case in which the scattering amplitude is analytic in the entire cosine cut complex plane [case ii), see below].

On the other hand, as was shown in the interesting discussion ⁴⁾ of Chiu and Tan, the experimental study of large angle scattering could provide valuable information on the possibility of the analytical continuation of the amplitude on higher Riemann sheets as well as on the behaviour of Regge poles ^{4),5)}.

Owing to the importance of the above-mentioned problems as to the growing accuracy of the experimental data, it would be perhaps worth while to work out now the exact values of the lower bounds required (see further) by the different assumptions on the shape of the analyticity domain of the amplitude instead of resuming to the presently known asymptotic expressions. Of course, from a pure numerical point of view, we shall again not be too advanced, as the lower bounds depend explicitly on the upper bound M of the amplitude in its analyticity domain of the cosine plane. Nowadays little, if nothing, is known about M , apart from the well-known Froissart ⁶⁾ limits. Nevertheless, in analyzing the experimental data, one can take advantage of the fact that the factor $M/|f(c=1)|$ does not depend on the cosine c of the c.m. scattering angle (or on the momentum transfer squared t) but only on s .

We should like to remind the reader that the lower bound for the maximum of the modulus of the amplitude depends, apart from M , both on the angle ranges where this maximum is sought and on the shape of the cosine plane domain $\mathcal{D}$ where analyticity is assumed for the amplitude. Indeed, as is well known, an analytic function cannot be zero over a continuum without being zero everywhere (the forward scattering amplitude is surely different from zero) : of course, however, accurate experimental measurements could be, one could never assert that some analytic function $f(c)$ is rigorously zero over some angle region and the experimental study of such a problem could appear at first glance as meaning less [here $f(c)$ is either the amplitude, its real or imaginary part, the differential cross-section, or some combination of these functions with some suitable holomorphic function of cosine c].

However, making use of the Nevanlinna principle (see Appendix B), one could always find some firm, finite, lower limit for the maximum m of the modulus of $f(c)$ in the given angular range γ_1 which cannot be violated if $f(c)$ has to be analytic in some domain $\mathcal{D}$ containing the curve γ_1 .

Indeed, if we denote by $\omega(c)$ the harmonic measure of the domain $D = \mathcal{D} - \gamma_1$, i.e., a harmonic function satisfying

$$\begin{aligned}\nabla^2 \omega(x, y) &= 0 \quad \text{for } z = x + iy \in D \\ \omega(x, y) &= 0 \quad \text{for } z \in \gamma_1 \\ \omega(x, y) &= 1 \quad \text{for } z \in \Gamma\end{aligned}\tag{1.1}$$

where Γ is the exterior boundary of D , the Nevanlinna principle reads (see Appendix B) :

$$|f(c)| \leq m^{1-\omega(c)} M^{\omega(c)}\tag{1.2}$$

where M is the maximum of $|f(c)|$ on Γ (or $\mathcal{D}$). Of course, the predictions of the Nevanlinna principle (1.2) for m are strongly dependent on the shape of the domain $\mathcal{D}$ and in the present paper we shall investigate two physically important situations (assumptions) :

- i) the analyticity domain coincides with the Lehmann ellipse; as one knows, the analyticity of the amplitude inside the Lehmann ellipse follows from very general field theoretical assumptions; occasionally, poles can be, of course, removed from the function $f(c)$;
- ii) the analyticity domain of $f(c)$ coincides with the cut cosine complex plane prescribed by the Mandelstam representation.

It is of course clear (see Appendix B) that for equal M and $f(c=1)$, the lower bound for m will be stronger (higher) in the second case. In Ref. 1), for sake of simplicity, some intermediate domain was investigated; however, its boundaries being tangent to the cut of the cosine cut plane, the lower bound found by Cerulus and Martin exhibits - up to a sub-unitary factor - the same asymptotic high energy behaviour as in case ii).

In order to give to the present paper the generality of a mathematical statement, we shall refrain from making a physical assumption about the upper limit M of $f(c)$ in D , and we shall present our results in the "theory independent" form

$$\frac{m}{|f(c=0)|} \geq \left(\frac{M}{|f(c=0)|} \right)^{\frac{\omega(1)}{1-\omega(1)}} \quad (1.3)$$

where $\omega(1)$ is the value of the harmonic measure defined by (1.1) in the forward scattering point ($c=1$).

The exponent occurring in (1.3) will be calculated both for the Lehmann, i), and the Mandelstam, ii), cases, respectively in Sections 2 and 3, for every finite c.m. energy square s as well as in the high energy limit. Having in mind possible applications to Regge poles, the asymptotic expressions of $\omega(1)/(1-\omega(1))$ will be given both for the cases in which :

4.

- a) the end points of the (physical) angle region γ_1 where the bound m is defined are constant angles ($\ominus$ and $\pi - \ominus$) and;
- b) γ_1 extends between some constant $\underline{t}$ and $\underline{u}$ ($t = -t_1$ and $u = -t_1$); unlike the constant angle case where the lower bound of the cross-section tends very rapidly to zero as the energy rises, in the constant $\underline{t}$ case we shall find, of course, finite high energy limits.

For sake of simplicity, in the present paper we shall limit ourselves to the symmetrical equal mass case.

2. THE LEHMANN ELLIPSE CASE

Let us suppose that $f(c)$ is a holomorphic function in the interior of an ellipse Γ with foci at $+1$ and -1 , and whose greatest semi-axis equals $1+\varepsilon$ where

$$\varepsilon = \frac{2t_0}{s-4} \quad (2.1)$$

t_0 being either a constant or tending to a constant when s increases. From (2.1) it is clear that the ellipse Γ tends to the real segment $-1 \leq c \leq +1$ when s grows up to infinity.

To avoid supplementary, but irrelevant computation, we shall place the segment γ_1 where we would like to find the lower bound for

$$m \stackrel{\text{def.}}{=} \max_{c \in \gamma_1} |f(c)| \quad (2.2)$$

symmetrically with respect to the physical region $-1 \leq c \leq +1$; let the boundaries of γ_1 be $c = \cos\Theta$ and $c = \cos(\pi - \Theta)$, [see Fig. 1]. As was said in the Introduction, we shall first find the exact finite energy expression for the lower bound (1.3), and then we shall study its limit in the following two distinct asymptotic situations :

$$A : \text{ when } \Theta = \text{constant} \quad (2.3)$$

$$B : \text{ when } \cos\Theta = 1 - \frac{2t_1}{s-4} \equiv 1 - \varepsilon \frac{t_1}{t_0} \quad (2.4)$$

where t_1 is some positive constant (constant momentum transfer case).

We have thus to find a harmonic function $\omega(c)$, zero on γ_1 and equal to one on Γ . Since the method used here will be applied also to other problems ^{*)} which will be treated in a next paper, we shall discuss it in more detail. The reader who is interested merely in the results, could find them directly at the end of the present Section [relations (2.20), (2.24 $\odot$) and (2.24 t)].

^{*)} Concerning a mathematical procedure which ensures a stable extrapolation of the experimental data away from the physical region.

We consider first the Joukowski transformation

$$C_1 = c + i \sqrt{1-c^2} \quad (2.5)$$

which transforms the ellipse of Fig. 1 into the ring $1 \leq |C_1| \leq R_1$ (see Fig. 2), where the (outer) radius R_1 is equal to

$$\begin{aligned} R_1 &= 1 + \varepsilon + \sqrt{2\varepsilon \sqrt{1+\varepsilon/2}} = \\ &= 1 + \sqrt{2\varepsilon} + \dots \end{aligned} \quad (2.6)$$

If the square root of (2.5) is defined to be positive above its cut ($-1 \leq c \leq +1$, Fig. 1), then the upper half ellipse will map into the upper half ring. γ_1 will now be represented by two unit radius circle segments (Fig. 2) extending from $e^{i\Theta}$ to $e^{i(\pi-\Theta)}$ and $-e^{i\Theta}$ to $-e^{-i(\pi-\Theta)}$. Our intention is to map the ring from the C_1 complex plane into a ring in a suitably chosen C_2 variable, such that in this new complex plane γ_1 should coincide with the whole unit circle [and the circle $|C_2| = R_2$ with Γ , see Fig. 6]. Then this new complex plane, the harmonic function defined in (1.1), can simply be written as

$$\omega(c) = \frac{\ln |C_1|}{\ln R_2} \quad (2.7)$$

To this end we put first C_1 in the form

$$C_1 = i \exp \left\{ -i \frac{\pi u_1}{2K_1} \right\} \equiv i \exp \{ -i \pi v_1 \} \quad (2.8)$$

$$\text{with } v_1 \stackrel{\text{def}}{=} u_1 / 2K_1$$

where K_1 is some constant which will be specified later: the upper half ring transforms into the rectangle of Fig. 3. Here the peculiar size of the rectangle is necessary in order to have a subsequent simple

mapping of the C_1 plane half ring into a $z_1 = \text{sn } u_1$ plane (see Appendix A). Hence the length $2K_1$ of the base of the rectangle is by no means arbitrary, being completely determined by the ratio K'_1/K_1 of the two sizes of the rectangle or merely by the basic variable q_1 [see Eqs. (A.13b) and (A.13c)]

$$q_1 \stackrel{\text{def}}{=} \exp \{-\pi K'_1/K_1\} \quad \begin{matrix} (2.9) \\ (A.6) \end{matrix} \text{ and}$$

which is directly related to the radius R_1 ,

$$R_1 = 1/\sqrt{q_1} \quad (2.10)$$

$$\text{i.e. : } q_1 = R_1^{-2} = \exp \{-2 \text{arccosh}(1+\varepsilon)\}$$

All points of interest in the u_1 plane can readily be computed using

$$v_1 = \frac{i}{\pi} \ln \left(\frac{C_1}{i} \right) \quad , \quad u_1 = 2K_1 v_1 \quad (2.11)$$

Now, if the cut of the logarithm is taken as in Fig. 2, the lower half plane part of the ring $1 \leq |C_1| \leq R_1$ maps into the dotted rectangle of Fig. 3. Indeed, in this case the point $C_1 = -1-i0$ (the point " -1 ") has to be written as $e^{-i\pi}$ and therefore $\ln C_1/i = \ln(e^{-i\pi} e^{-i\pi/2}) = -i \frac{3\pi}{2}$, i.e., $v = +\frac{3}{2}$ (see Fig. 3).

The point $c = \cos \Theta$ maps into $v_1 = \frac{1}{2} - \Theta/2$ and $c = \cos(\pi - \Theta)$ into $v = -(\frac{1}{2} - \Theta/2)$. Special care has to be taken in the constant t case (situation B) when, simultaneously with the collapse of the v plane rectangle as the energy grows to infinity, the ends of γ_1 tend to $v = -\frac{1}{2}$ and $v = +\frac{1}{2}$. Indeed, when ε tends to zero, we have (constant t situation) :

$$2 \sin^2 \Theta/2 = \varepsilon \frac{t_1}{t_0} \quad (2.12)$$

$$\text{and hence } \Theta = \sqrt{2\varepsilon} \left(\frac{t_1}{t_0} \right)^{1/2} + \frac{\sqrt{2\varepsilon}^{3/2}}{12} \left(\frac{t_1}{t_0} \right)^{3/2} + \dots$$

8.

One can further represent the u_1 plane rectangle on the upper half plane of

$$z_1 = \operatorname{sn} u_1 \quad (2.13)$$

using Jacobi elliptic sine, as is shown in Appendix A. Here the "modulus" k_1 which fixes the end points of the cut Γ [see Fig. 4] can be computed from the basic quantity q_1 [see (2.10)] by the help of the "modular function" $\theta_2^2(q)/\theta_3^2(q)$ [see (A.13a), (A.12a) and (A.12b)].

One should point out that γ_1 does not have the same meaning as Γ_1 of Appendix A, as it does not extend between -1 and $+1$ as Γ_1 does. However, writing (see Fig. 5)

$$z_2 = z_1 / \operatorname{sn} \left(K_1 - \frac{2K_1 \Theta}{\pi} \right) \quad (2.14a)$$

and defining a new modulus k_2

$$k_2 = k_1 \cdot \operatorname{sn} \left(K_1 - \frac{2K_1 \Theta}{\pi} \right) \quad (2.14b)$$

one can again use the mapping $C(z)$ to transform now γ_1 into the (whole) unit circle of the complex plane of the function

$$C_2 = C(z_2) \equiv i \exp \left\{ -i\pi \frac{\int_0^{\operatorname{sn} u_1} \frac{dt}{\sqrt{(1-t^2)(1-k_1^2 t^2)}}}{2K_2} \right\} \quad (2.15)$$

where k_2 was defined in (2.14b) and K_2 is given by [see Eq. (A.2)]

$$K_2 = K(k_2) \equiv \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k_2^2 t^2)}} \quad (2.16a)$$

$$K'_2 = K'(k_2) \equiv \frac{1}{k_2} \int_1^{1/k_2} \frac{dt}{\sqrt{(1-t^2)(1-k_2^2 t^2)}} \quad (2.16b)$$

The radius R_2 of the circle Γ in the C_2 complex plane is

$$R_2 = \exp\left\{+\pi \frac{K_2'}{2K_2}\right\} = q_2^{-1/2} \quad (2.17)$$

where K_2, K_2' or, directly, q_2 , can be computed by means of formulae (A.7) and (A.8) from the new modulus k_2 defined in (2.14b).

The forward scattering point $c=1$ will be located between the two circles of Fig. 6 at

$$C_2 \Big|_{c=1} = C\left(1/\operatorname{sn}\left(K_1 - \frac{2K_1\Theta}{\pi}\right)\right) \quad (2.18)$$

The point $c=1$ being now a regular point of the conformal mapping, in contrast to the C_1 or u_1 previous mappings. This enables us to write the harmonic measure of D in the form (2.7), getting for $c=1$

$$\omega(1) = \frac{1}{iK_2'} \int_1^{k_2/k_1} \frac{dt}{\sqrt{(1-t^2)(1-k_2^2 t^2)}} \quad (2.19)$$

$$\left(k_2/k_1 = 1/\operatorname{sn}\left(K_1 - \frac{2K_1\Theta}{\pi}\right)\right)$$

and hence, the following expression for the Nevanlinna inequality (1.3) defining the lower bound of m :

$$m \geq |f(1)| \left(\frac{M}{|f(1)|} \right)^{\left[\frac{K'(k_2)}{\int_{k_2}^{k_1} \frac{dt}{\sqrt{(t^2-1)(1-k_2^2 t^2)}}} - 1 \right]} \quad (2.20a)$$

where the elliptic integrals in the exponential depend through k_1 and k_2 , on s and on the angle Θ . Indeed, recollecting (2.10), (A.13a) and (A.15a), we have

$$k_1 = \theta_2^2(q_1) / \theta_3^2(q_1)$$

$$k_2 = k_1 \sin\left(K_1 - \frac{2K_1\Theta}{\pi}\right) \equiv \quad (2.20b)$$

$$\equiv k_1 \frac{\theta_2(q_1) \theta_1(1/2 - \Theta/\pi; q_1)}{\theta_2(q_1) \theta_0(1/2 - \Theta/\pi; q_1)} \equiv \frac{\theta_2(q_1) \theta_2(\Theta/\pi; q_1)}{\theta_3(q_1) \theta_3(\Theta/\pi; q_1)}$$

where (2.10)

$$q_1 = \left(1 + \varepsilon + \sqrt{2\varepsilon} \sqrt{1 + \varepsilon/2}\right)^{-2} = 1 - 2\sqrt{2\varepsilon} + \dots \quad \left(\varepsilon = \frac{2t_0}{s-4}\right)$$

and where the Jacobi theta functions occurring in (2.20b) are defined in Appendix A.

As $q_1(s)$ approaches unity when the energy increases, one has to be careful to choose suitable convergent expansions for the quantities exhibited in (2.20b). As emphasized in Appendix A, it is preferable to expand in terms of

$$k' = (1 - k^2)^{1/2} \quad \text{and} \quad q' = \exp\{-\pi K'/K\}$$

rather than in k and q ; owing to (A.9), k' , as well as K and K' , can be expressed in terms of q' exactly by the same series (A.13) as k , K' and K were in terms of q .

For small ε one gets from (A.11)

$$q'_1 = \exp\{-\pi^2/2\sqrt{2\varepsilon}\} + \dots \quad (2.21)$$

and hence (A.13)

$$\begin{aligned}
 k_1'^2 &= 16q_1'(1-8q_1'-\dots) \\
 1/k_1 &\equiv [1-k_1'^2]^{-1/2} \\
 &= 1 + 8 \exp\{-\pi^2/2\sqrt{2}\sqrt{\epsilon}\} + \dots \equiv 1 + \eta
 \end{aligned}
 \tag{2.22}$$

To calculate $\text{sn}(K-2K\Theta/\pi)$ we may resort to the expansion (A.18)

$$\begin{aligned}
 \text{sn}(K_1-2K_1\Theta/\pi) &= 1 - 4q_1' [\cosh(2\pi v_1') - 1] + \dots \\
 v_1' &\equiv \frac{2K_1\Theta}{2K_1'\pi} = -\ln q_1' \cdot \frac{\Theta}{\pi^2} = \Theta/2\sqrt{2}\sqrt{\epsilon} + \dots
 \end{aligned}
 \tag{2.23}$$

Further, one has to treat the cases $\Theta = \text{const.}$ and $\cos \Theta = 1 - \epsilon t_1/t_0$ with $t_1 = \text{const.}$ separately :

- A) ($\Theta = \text{const.}$ case) : for angles Θ different from zero and $\pi/2$, $\cosh 2\pi v_1'$ grows to infinity, but $q_1' \cosh 2\pi v_1'$ still tends to zero :

$$\frac{1}{\text{sn}(K_1-2K_1\Theta/\pi)} = 1 + 2 \exp\left\{-\frac{\pi}{\sqrt{2}\epsilon} \left(\frac{\pi}{2} - \Theta\right)\right\} + \dots \equiv 1 + \zeta_v \tag{2.24\Theta}$$

- B) ($t_1 = \text{const.}$ case) : Θ tends to zero as $\sqrt{2}\sqrt{\epsilon} \cdot \sqrt{\frac{t_1}{t_0}}$ and hence

$$\begin{aligned}
 1/\text{sn}(K_1-2K_1\Theta/\pi) &= 1 + 8 \sinh^2\left(\frac{\pi}{2}\sqrt{\frac{t_1}{t_0}}\right) \exp\left\{-\frac{\pi^2}{2\sqrt{2}\epsilon}\right\} + \dots \equiv \\
 &= 1 + \zeta_t
 \end{aligned}
 \tag{2.24t}$$

As $k'_2 = [1 - k_2^2]^{\frac{1}{2}}$ also tends to zero; K'_2 from the exponent of (2.20a) can simply be replaced by $\pi/2$ as [(A.7a) and (A.9a)]

$$K'_2 = \frac{\pi}{2} \left(1 + \left(\frac{1}{2}\right)^2 k_2'^2 + \left(\frac{1}{8}\right)^2 k_2'^4 + \dots \right) \quad (2.25)$$

Further, the elliptical integral

$$\Delta u = \int_{k_2/k_2 = 1+\xi}^{k_2 = 1+\xi+\eta} \frac{dt}{\sqrt{(t^2-1)(1-k_2^2 t^2)}} \quad (2.26)$$

can be computed straightforwardly by elementary functions, as the factor $(t+1)^{\frac{1}{2}}(1+k_2 t)^{\frac{1}{2}}$ of the integrand is almost constant ($=2$) when ξ and η are small., Putting

$$x = \sqrt{\frac{1+\xi+\eta-t}{t-1}}$$

one gets

$$\Delta u = \int_0^{\sqrt{2/\xi}} \frac{dx}{1+x^2} \approx \begin{cases} 2 \exp\{-\Theta\pi/2\sqrt{\xi}\} & \text{for the } \ominus \text{ case, and} \\ \arctg\{1/\sinh(\frac{\pi}{2}\sqrt{\frac{\xi_1}{\xi_0}})\} = 2 \arctg\{\exp[-\frac{\pi}{2}\sqrt{\frac{\xi_1}{\xi_0}}]\} & \text{for the } \oplus \text{ case.} \end{cases} \quad (2.27)$$

Recollecting (2.20a), (2.25) and (2.27), we get the following two asymptotic expressions for the lower bound :

— in the $\ominus = \text{const.}$ case :

$$m \geq \left\{ f(\omega) \exp\left[-\ln\left(\frac{M}{\sqrt{f(\omega)}}\right) \cdot \left[\frac{\pi}{4} \exp(-\Theta\pi/2\sqrt{\xi}) - 1\right]\right] \right\} \quad (2.28\ominus)$$

($\xi = \frac{2t_2}{5-t_2}$)

-- in the $t_1 = \text{const.}$ case :

$$m \geq |f(\omega)| \exp \left\{ - \frac{\exp \left\{ - \frac{\pi}{4} \right\}}{(\ln 2)^2 + 4 \ln 2} \exp \left\{ - \frac{\pi}{4} \right\} \right\} \quad (2.28t)$$

As is evident from (2.28 $\oplus$), the lower limit for the constant angle case tends extremely quickly to zero as the energy increases -- much faster than in the similar situation in the Mandelstam cut plane case [compare with (3.12 $\oplus$)]. The finite limit we have obtained in the $t = \text{const.}$ case is due to the fact that the two intervals η and $\frac{\eta}{t}$ of the z_2 plane are both of the same order of magnitude; the physical significance of this fact is just obvious.

3. THE COSINE CUT PLANE DEFINED BY THE MANDELSTAM REPRESENTATION

If the analyticity domain D coincides now with the Mandelstam cut plane, the problem can readily be solved by applying the transformation $C(z)$ directly to the variable (see Fig. 7) :

$$z = 2/\omega \oplus \quad (3.1)$$

with

$$1/K = 1/\omega \oplus + 2t_0/(s-4)\omega \oplus \quad (3.2)$$

The harmonic measure reads [see (2.7) and (A.6)] :

$$\omega(c) = \frac{\ln |C(c)|}{\ln R} = \frac{1}{iK'} \left\{ \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \right\} \quad (3.3)$$

with

$$K' = \frac{1}{i} \int_0^{1/k} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

and hence, the lower bound for the maximum m of $|f(c)|$ on that part of the physical region which extends between $c = \cos \oplus$ and $c = \cos(\pi - \oplus)$ is given by

$$m \geq |f(c)| \left(\frac{M}{|f(c)|} \right)^{\frac{1}{2} \left\{ \frac{K'(k)}{\int_0^{1/k} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}} - 1 \right\}} \quad (3.4)$$

This expression is valid for every value of the c.m. energy squared, s . To work out now the two asymptotic forms - for constant $\oplus$ and for constant t_1 - in analogy with the Lehmann ellipse case, we shall write for large s :

$$\gamma(t=0) = 1 + \gamma_0$$

$$K_1 = 1 + \gamma_{0,1} + \gamma_0$$
(3.5)

In contrast to the Lehmann ellipse case, γ is infinitesimal only in the $t = \text{const.}$ case, otherwise it is finite.

A) (the $\Theta = \text{const.}$ case) :

$$\gamma_0 = 1/\cos\Theta - 1 \quad (\text{for } \Theta \in (0, \pi/2))$$

$$\gamma_0 = 2/\cos\Theta$$

$$k = \cos\Theta + O(\Theta)$$
(3.6)

In the limit $\Theta = 0$ one has

$$K'(k) = K(\sin\Theta)$$
(3.7)

where the complete elliptic integral $K(\sin\Theta)$ can be computed with the help of the series (A.7) :

$$K(\sin\Theta) = \frac{\pi}{2} \left\{ 1 + \left(\frac{1}{2}\right)^2 \sin^2\Theta + \left(\frac{1}{2}\right)^4 \sin^4\Theta + \dots \right\}$$
(3.8)

The elliptic integral occurring in (3.4) can also be computed easily in the limit $k \rightarrow 0$:

$$\Delta u \approx \frac{1}{\sqrt{1-k^2}} \frac{1}{\sqrt{2}} \int_0^{1/k} \frac{dt}{\sqrt{1-k^2 t^2}} \approx \frac{\sqrt{2}}{2k\Theta}$$
(3.9)

16.

B) (the $t_1 = \text{const.}$ case) :

Both ξ_t and η_t are infinitesimal as ε goes to zero

$$\begin{aligned}\xi_t &= \varepsilon \frac{t_1}{t_0} \\ \eta_t &= \varepsilon\end{aligned}\tag{3.10}$$

and using the same arguments as at the end of the previous Section, one gets

$$\begin{aligned}K' &\rightarrow \pi/2 \\ \Delta u &\rightarrow \text{arctg} \sqrt{\frac{\varepsilon}{t_1}}\end{aligned}\tag{3.11}$$

Thus, the desired asymptotic forms for the inequality (3.4) are :

-- in the $\Theta = \text{const.}$ case :

$$m \geq |f(\Theta)| \exp \left\{ - \ln \left(\frac{M}{|f(\Theta)|} \right) \left[\frac{\sin \Theta}{\sqrt{2\varepsilon}} K(\sin \Theta) - 1 \right] \right\}\tag{3.12(H)}$$

-- in the $t_1 = \text{const.}$ case :

$$m \geq |f(\Theta)| \exp \left\{ - \ln \left(\frac{M}{|f(\Theta)|} \right) \cdot \frac{\text{arctg} \sqrt{\frac{t_1}{2\varepsilon}}}{\text{arctg} \sqrt{\frac{t_1}{2\varepsilon}}} \right\}\tag{3.12t}$$

The lower limit for the $t = \text{const.}$ case defined by (3.12t), as well as the similar expression (2.28t) for the Lehmann ellipse case, remains, of course, finite.

4. CONCLUSIONS

The reader who is just interested in the results rather than in the details of the work, is kindly requested to read the second part of the Introduction; the relevant expressions for the lower bounds are (2.20) for the Lehmann ellipse case, and (3.4) for the Mandelstam cosine cut plane. The asymptotic expressions for finite scattering angles or for $t = t_1 = \text{const.}$ are, respectively, (2.28 $\odot$) and (2.28t), and (3.12 $\odot$) and (3.12t).

We should like to emphasize that — as one sees from (2.20) and (3.4) or from the asymptotic expressions (2.28t) and (3.12t) — the analytic properties of the lower bound are incompatible with those of the differential cross-section or scattering amplitude, as it contains a supplementary square root cut beginning at $t=0$. Therefore, even if the amplitude which actually realizes itself in nature is the smallest possible (Kinoshita hypothesis), one cannot simply identify the amplitude with the lower bound. Moreover, the observed departure of the forward scattering experimental data from the predicted lower bound is perfectly comprehensible, as the effect of the supplementary cut is greatest in the small t region. Indeed, the lower bound [see (2.28t) and (3.12t) ⁷⁾] has a square root behaviour at $t=0$ whereas this point has to be a regular point both for the amplitude and for the differential cross-section !

As one can easily see, this supplementary cut in the expression of the lower bound comes from the "spurious" cut of $C(z)$ on Γ_1 and is due to the fact that the Nevanlinna is in fact more general than is needed, in the sense that it applies also to functions not necessarily holomorphic on $\tilde{\Sigma}_1$, where the maximum m is defined.

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A P P E N D I X A

As the mapping $C(z)$ is met also in other problems ⁸⁾, we shall study it in more detail in this Appendix. $C(z)$ is defined to map the z complex plane cut along Γ_1 ($-1 \leq z \leq 1$) and Γ_R [$-\infty < z \leq -1/k$ and $+1/k \leq z < \infty$, where k is a positive number smaller than one] into the ring $1 \leq |C(z)| \leq R$. The function $C(z)$ can also be used to transform the circle $|z| \leq 1/\sqrt{k}$ cut along Γ_1 , into the ring $1 \leq |C(z)| \leq \sqrt{R}$: indeed, in both the z and $C(z)$ complex planes, Γ_R is the mirror image of Γ_1 on the respective circles $\Gamma_{\sqrt{R}}$ ($|z| = 1/\sqrt{k}$ and $|C(z)| = \sqrt{R}$). Then the mapping property quoted above follows from the Schwartz reflection principle (see Figs. A.1 - A.3).

It has to be stressed that in the physical problems in which we are interested, Γ_1 is not really a cut of the amplitude, (Γ_1 is placed in the physical region of the cosine plane !) but is merely the (virtual) boundary of the region to which we would like to extrapolate the physical data ⁸⁾ measured on Γ_1 or, as in the present paper, is a part of the boundary of the region where we would like to apply the Nevanlinna principle. Therefore, both the amplitude and the differential cross-section can be continued analytically through Γ_1 ; in the $C(z)$ plane, this means that the analyticity region extends out to the circle Γ_R' of radius $1/R$, which is the mirror image on the unit circle Γ_1 of the (actual !) cut Γ_R .

Let us fix our attention to the upper z half plane and let us perform the mapping (see Fig. A.2).

$$u(z) = \int_C \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \quad (\text{A.1})$$

the cuts of the square root being taken along Γ_1 and Γ_R (of the z plane) and let the square root be positive above the origin (see Fig. A.4).

If z is small and increases along the real axis, u increases from zero to the positive constant

$$K(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \quad (\text{A.2})$$

when z reaches the point $z=1$.

Further, if $1 < z < 1/k$, one can write

$$u(z) = \int_0^z \dots = K(k) + \int_1^z \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \quad (\text{A.3})$$

where, since the square root is negative imaginary for $1 < t < 1/k$, the second integral is a pure positive imaginary function, increasing with z . For $z=1/k$, u reaches the value $K+iK'$ with

$$iK'(k) = \int_1^{1/k} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \quad (\text{A.4})$$

If z proceeds further, along the upper side of Γ_R (path No.1 in Fig. A.4), the real part of u will decrease monotonically, as the square root is now negative. For $z=\infty$, u equals iK' . Performing the same study also in the negative direction, one finds that the whole upper half plane maps into the rectangle of Fig. A.2.

The final step consists in transforming the u plane rectangle into the upper half ring of the $C(z)$ complex plane by

$$C(z) = i \exp\left\{-i\pi \frac{u(z)}{2K}\right\} \quad (\text{A.5})$$

The point $u=0$ transforms into $C=+i$, $u=K$ into $C=+1$ and $u=K+iK'$ into $C=\exp(i\pi K'/2K)$ which defines the value of the radius R . R can be written

$$R = q^{-1/2} \quad \text{with} \quad q \equiv \exp\left\{-\pi K'/K\right\} \quad (\text{A.6})$$

*
* *
*

The next discussion is important if one is interested in asymptotic expansions : K , K' and q depend on the "modulus" k , and we have for small k the following expansions :

$$K(k) = \frac{\pi}{2} \left\{ 1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 k^4 + \dots + \left(\frac{(2n-1)!!}{2^n n!}\right)^2 k^{2n} + \dots \right\} = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right) \quad (\text{A.7a})$$

or, if k is close to unity, as is the case throughout the present paper, one has to expand $K(k)$ in the form

$$\begin{aligned} K(k) = & \ln \frac{4}{k'} + \left(\frac{1}{2}\right)^2 \left(\ln \frac{4}{k'} - \frac{2}{1 \cdot 2}\right) k'^2 + \\ & + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 \left(\ln \frac{4}{k'} - \frac{2}{1 \cdot 2} - \frac{2}{3 \cdot 4}\right) k'^4 + \\ & + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 \left(\ln \frac{4}{k'} - \frac{2}{1 \cdot 2} - \frac{2}{3 \cdot 4} - \frac{2}{5 \cdot 6}\right) k'^6 + \dots \end{aligned} \quad (\text{A.7b})$$

where the "comodulus" k' is defined to be

$$k' = \sqrt{1 - k^2} \quad (\text{A.8})$$

i.e., if $k = \sin \alpha$, then $k' = \cos \alpha$.

After some simple manipulations with the integrals (A.2) and (A.4), one gets :

$$K'(k) = K(k') \quad (\text{A.9a})$$

$$K(k) = K'(k') \quad (\text{A.9b})$$

As (A.8) is involutive, the effect of (A.9a) and (A.9b) will be that one can simply compute K' replacing k by k' and k' by k in the expansions (A.7a) and (A.7b). [Actually to find (A.7b), one starts from the equivalent expansion for $K'(k)$ and then uses (A.9b).]

Further,

$$q = K + 2K^5 + 15K^9 + 450K^{13} + 1401K^{17} + \\ + 20940K^{21} + 268800K^{25} + \dots \quad (\text{A.10})$$

where

$$2K = \frac{1-\sqrt{K'}}{1+\sqrt{K'}} = \frac{1-\sqrt{\cos\alpha}}{1+\sqrt{\cos\alpha}}, \quad (K=\cos\alpha, K'=\cos\alpha)$$

If k is close to one ($\alpha \geq 75^\circ$) it is difficult, if not impossible, to use the series (A.10) and therefore [using (A.9) together with (A.10)] one starts computing $q' \equiv \exp(-\pi K/K')$; q is related to q' by

$$\ln q - \ln q' = \pi^2 \quad (\text{A.11})$$

One can, conversely, express k, K and K' in terms of q using Jacobi θ functions

$$\begin{aligned} \theta_1(v, q) &= \sum_{n=-\infty}^{+\infty} (-1)^n q^{n(n-1)/2} e^{i\pi n v} = 2q^{1/4} (\sin \pi v - q^2 \sin 3\pi v + q^6 \sin 5\pi v - q^{12} \sin 7\pi v + \dots) \\ \theta_2(v, q) &= \sum_{n=-\infty}^{+\infty} q^{n(n-1)/2} e^{i\pi n v} = 2q^{1/4} (\cos \pi v + q^2 \cos 3\pi v + q^6 \cos 5\pi v + q^{12} \cos 7\pi v + \dots) \\ \theta_3(v, q) &= \sum_{n=-\infty}^{+\infty} q^{n^2/2} e^{i\pi n v} = 1 + 2q \cos 2\pi v + 2q^4 \cos 4\pi v + 2q^9 \cos 6\pi v + 2q^{16} \cos 8\pi v + \dots \\ \theta_4(v, q) &= \sum_{n=-\infty}^{+\infty} (-1)^n q^{n^2/2} e^{i\pi n v} = 1 - 2q \cos 2\pi v + 2q^4 \cos 4\pi v - 2q^9 \cos 6\pi v + 2q^{16} \cos 8\pi v - \dots \end{aligned} \quad (\text{A.12a})$$

computed at $v=0$. Thus one obtains for k, K and K'

$$k = \frac{\theta_2^2(q)}{\theta_3^2(q)} = \frac{4q^{1/2} (1 + q^2 + q^6 + q^{12} + q^{20} + \dots)^2}{(1 + 2q + 2q^4 + 2q^9 + 2q^{16} + \dots)^2} \quad (\text{A.13a})$$

$$K = \frac{\pi}{2} \theta_3^2(q) = \frac{\pi}{2} (1 + 2q + 2q^4 + 2q^9 + 2q^{16} + \dots)^2 \quad (\text{A.13b})$$

$$K' = -\frac{K}{\pi} \ln q = -\frac{\ln q}{2} (1 + 2q + 2q^4 + 2q^9 + 2q^{16} + \dots)^2 \quad (\text{A.13c})$$

Here, by definition, $\theta_i(q)$ equals $\theta_i(v, q)|_{v=0}$:

$$\begin{aligned}\theta_2(q) &= 2q^{1/2}(1 + q^2 + q^6 + q^{12} + q^{20} + q^{30} + q^{42} + \dots) \\ \theta_3(q) &= 1 + 2q + 2q^4 + 2q^9 + 2q^{16} + 2q^{25} + 2q^{36} + \dots \\ \theta_0(q) &= 1 - 2q + 2q^4 - 2q^9 + 2q^{16} - 2q^{25} + 2q^{36} - \dots\end{aligned}\tag{A.12b}$$

If k is close to unity, as is the case in the present paper, $q \rightarrow 1$ and $q' \rightarrow 0$, and therefore one has to use similar series in q' to compute first the $(\dots)'$ quantities : $[k' = \theta_2^2(q')/\theta_3^2(q'), \text{ and so on}]$.

Hence, using (A.11), we find for small spacings between the cuts of the z plane (i.e., $1/k \rightarrow 1$, see Fig. A.1) the following important relation between k and the radius R (Fig. A.3) :

$$1 - k^2 \approx 16 \exp\{-\pi^2/24kR\} \quad , \text{ for } k, R \rightarrow 1 \tag{A.14}$$

Further terms in (A.14) can easily be obtained from (A.13a), or from (A.6) combined with (A.7) and (A.9).

Let us return to the integral (A.1) defining the function $u(z)$. If, after reaching the point $z = 1/k$, z moves now below the cut Γ_R (see Fig. A.4, path No.2), as the square root is positive here, the real part of u will increase rather than decrease, as was the case with path No.1.

It is then clear that the lower half z plane will be mapped into the rectangle $K \leq \operatorname{Re} u \leq 3K$, $0 \leq \operatorname{Im} u \leq K'$, i.e., the whole z complex plane will map into the rectangle outlined by dark lines in Fig. A.5.

If z (Fig. A.4, path No.3) crosses the Γ_R cut and returns along the real axis on the second Riemann sheet of the square root, as its values are here opposite to those on the first Riemann sheet, the imaginary part of $u(z)$ still increases (see Fig. A.5). As one can see

without difficulty, the whole z complex plane - the second Riemann sheet of the square root - maps again into the u plane, namely into the rectangle $-K \leq \operatorname{Re} u \leq 3K$, $K' \leq \operatorname{Im} u \leq 2K'$. Further, one would find new representations of the z complex plane in the rectangle situated above, as well as at the right and left sides of the former rectangle. This demonstrates the double periodicity (periods: $4K$ and iK') of the inverse function

$$z = \operatorname{sn} u$$

Unlike the usual sine function which is simple periodic, the Jacobi elliptical sine has two co-functions

$$\operatorname{cn} u = \sqrt{1 - \operatorname{sn}^2 u} \quad \text{and} \quad \operatorname{dn} u = \sqrt{1 - k^2 \operatorname{sn}^2 u}$$

All three functions are meromorphic and can be expressed with the Jacobi θ functions ($v = u/2K$):

$$\begin{aligned} \operatorname{sn}(u, k) &= \frac{\theta_3(q)}{\theta_2(q)} \cdot \frac{\theta_1(v, q)}{\theta_0(v, q)} = \\ &= \frac{1 + 2q + 2q^3 + 2q^5 + \dots}{1 + q^2 + q^6 + q^{10} + \dots} \left(\frac{\sin \pi v - q^2 \sin 3\pi v + q^6 \sin 5\pi v - q^{10} \sin 7\pi v + \dots}{(1 - 2q \cos 2\pi v + 2q^4 \cos 4\pi v - 2q^9 \cos 6\pi v + \dots)} \right) \end{aligned} \quad (\text{A.15a})$$

$$\begin{aligned} \operatorname{cn}(u, k) &= \frac{\theta_2(q)}{\theta_3(q)} \frac{\theta_2(v, q)}{\theta_0(v, q)} = \\ &= \frac{1 - 2q + 2q^3 - 2q^5 + \dots}{1 + q^2 + q^6 + q^{10} + \dots} \left(\frac{\cos \pi v + q^2 \cos 3\pi v + q^6 \cos 5\pi v + q^{10} \cos 7\pi v + \dots}{1 - 2q \cos 2\pi v + 2q^4 \cos 4\pi v - 2q^9 \cos 6\pi v + \dots} \right) \end{aligned} \quad (\text{A.15b})$$

$$\begin{aligned} \operatorname{dn}(u, k) &= \frac{\theta_0(q)}{\theta_3(q)} \frac{\theta_3(v, q)}{\theta_0(v, q)} = \\ &= \frac{1 - 2q + 2q^3 - 2q^5 + \dots}{1 + 2q + 2q^3 + 2q^5 + \dots} \left(\frac{1 + 2q \cos 2\pi v + 2q^4 \cos 4\pi v + 2q^9 \cos 6\pi v + \dots}{1 - 2q \cos 2\pi v + 2q^4 \cos 4\pi v - 2q^9 \cos 6\pi v + \dots} \right) \end{aligned} \quad (\text{A.15c})$$

These expansions are also true for imaginary arguments iu , and one can use this fact to find asymptotic expansions when k is close to unity. Indeed, one may resort to the Jacobi imaginary transformation equations

$$\begin{aligned}
 \operatorname{sn}(u, k) &= \frac{-i \operatorname{sn}(iu, k')}{\operatorname{cn}(iu, k')} & \operatorname{sn}(K-u, k) &= \frac{\operatorname{cn}(u, k)}{\operatorname{dn}(u, k)} \\
 \operatorname{cn}(u, k) &= \frac{1}{\operatorname{cn}(iu, k')} & \operatorname{cn}(K-u, k) &= \frac{k' \operatorname{sn}(u, k)}{\operatorname{dn}(u, k)} \\
 \operatorname{dn}(u, k) &= \frac{\operatorname{dn}(iu, k')}{\operatorname{cn}(iu, k')} & \operatorname{dn}(K-u, k) &= \frac{k'}{\operatorname{dn}(u, k)}
 \end{aligned} \tag{A.16}$$

Confining our attention to $\operatorname{sn}(u, k)$ we may obtain from (A.15) and (A.16), written out for ready use ($k \rightarrow 1$):

$$\operatorname{sn}(u, k) = \frac{\Theta_3(q')}{\Theta_0(q')} \left\{ \frac{\sinh \pi v' - q'^2 \sinh 3\pi v' + q'^4 \sinh 5\pi v' - q'^6 \sinh 7\pi v' + \dots}{\cosh \pi v' + q'^2 \cosh 3\pi v' + q'^4 \cosh 5\pi v' + q'^6 \cosh 7\pi v' + \dots} \right\} \tag{A.17}$$

(where $v' = u/2K'$), or the following expression, important for computing the position of points situated in the very neighbourhood of the branching points of the Γ_1 cut, when the spacing between Γ_1 and Γ_R tends to zero ($k \rightarrow 1$):

$$\operatorname{sn}(K-u, k) = \frac{1}{\operatorname{dn}(iu, k')} = \frac{\Theta_3(q')}{\Theta_2(q')} \left\{ \frac{1 - 2q' \cosh 2\pi v' + 2q'^3 \cosh 4\pi v' - 2q'^5 \cosh 6\pi v' + \dots}{1 + 2q' \cosh 2\pi v' + 2q'^3 \cosh 4\pi v' + 2q'^5 \cosh 6\pi v' + \dots} \right\} \tag{A.18}$$

In all these expressions the values of $\Theta_2(q')$, $\Theta_3(q')$ and $\Theta_4(q')$ are to be computed with the expansions (A.12b), written of course as a series in q' .

A P P E N D I X B

THE PRINCIPLE OF NEVANLINNA

Let D be a simple or multiple connected domain in the complex $z = x + iy$ plane, and let its boundary be formed by two disjoint curves Γ and Γ_1 . Let $\omega(x, y)$ be a harmonic function in D [see (1.1)], whose boundary values are

$$\begin{aligned}\omega(x, y)|_{z \in \Gamma_1} &= 0 \\ \omega(x, y)|_{z \in \Gamma} &= 1\end{aligned}\tag{B.1}$$

$\omega(x, y)$ is called a harmonic measure of D .

Let us now suppose that the function $f(z)$ is holomorphic in D and that

$$\begin{aligned}|f(z)|_{z \in \Gamma_1} &\leq m \\ |f(z)|_{z \in \Gamma} &\leq M\end{aligned}\tag{B.2}$$

If the domain D does not contain any zero of $f(z)$ (this condition will be relaxed afterwards), $\ln f(z)$ will also be a holomorphic function in D , and hence, $\operatorname{Re}(\ln f(z)) \equiv \ln |f(z)|$ as well as the combination

$$F(x, y) = \ln |f(z)| - (1 - \omega(x, y)) \ln m - \omega(x, y) \ln M\tag{B.3}$$

will be harmonic in D . From (B.1) and (B.2) it follows that

$$F(x, y)|_{z \in \Gamma \cup \Gamma_1} \leq 0\tag{B.4}$$

and hence, from the maximum principle for harmonic functions, the inequality (B.4) will be satisfied anywhere in D . From that, together with (B.3), we get the Nevanlinna inequality in the form

$$|f(z)| \leq m^{1-\omega(x,y)} M^{\omega(x,y)} \quad (\text{B.5})$$

for every $z \in D$. The zero condition can be relaxed, isolating it by means of very small circles γ_c which can be added to the boundary of the domain where $F(x,y)$ remains harmonic. Indeed, if the circles γ_c are small enough, the inequality (B.4) is automatically fulfilled on γ_c as well as in their interior, owing to the fact that the term $\ln|f(z)|$ becomes strongly negative as z approaches a zero of $f(z)$.

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The c (cosine) - complex plane

FIG. 1

The $C_1 = c + i\sqrt{1-c^2}$ plane

FIG. 2

The $u (= 2vK_1)$ -complex plane

FIG. 3

The $z_1 = \text{sn } u_1$ plane

FIG. 4

the $z_2 = \frac{z_1}{\text{sn}(K_1 - \frac{2K_1\Theta}{\pi})}$ complex plane

FIG.5

the $C_2 = C(z_2)$ complex plane

FIG.6

the z -plane for the cut-cosine plane

FIG.7

FIG. A1.

$$u(z) = \int_0^z \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}; z = \operatorname{sn} u$$

$$K = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \equiv u(1); iK' = \int_1^{1/k} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \equiv u(1/k) - u(1)$$

the $u = u(z)$ - plane

FIG. A2

$$R = (\sqrt{Q})^{-1} = \exp(\pi K'/2K)$$

where $q \equiv \exp(-\pi K'/K)$

$$\text{the } C(z) = i \exp \left\{ -i \pi \frac{u(z)}{2K} \right\} \text{ - plane}$$

FIG. A3