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ON THE GROUP THEORETICAL APPROACH TO THE CONSPIRACY PROBLEM
FOR ARBITRARY MASSES, II

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A B S T R A C T

Using the techniques developed in the first part of this work, we build a large class of families of conspiring Regge trajectories, which satisfy all the constraints of analyticity and factorization in a region of the momentum transfer complex plane including the origin and the pseudothresholds. Equal and unequal mass reactions are treated in a unified way. Particles with arbitrary spins and parities are considered. The formalism automatically suggests a kind of L-S coupling scheme and gives rise to the right pseudothreshold factors. The pseudothreshold and the small momentum transfer behaviours of the Regge residues are given for all the daughter trajectories. The spin dependence of the high energy forward scattering amplitude is studied in detail.

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1. INTRODUCTION

In this paper we give some development of a general formalism ¹⁾⁻⁴⁾, which permits to build a large class of families of conspiring Regge trajectories, which satisfy the factorization constraints for an arbitrary number of two-body reactions involving particles with arbitrary masses and spins. The contribution of one of these families of trajectories satisfies, above a fixed finite energy, all the constraints which can be derived from the analytic properties of the amplitude. The method works for any complex value of the square W of the four-momentum transfer ⁵⁾. However, in order to simplify the formalism, we consider only values of W belonging to the complex plane cut from the smallest threshold to $+\infty$.

The general features of the method and the application to unequal mass scattering near $W=0$ have been given in the first part of this work ²⁾, which we shall indicate by I ⁶⁾. Here we exploit this formalism more completely in order to treat also the phenomena which occur at the pseudothresholds. This development is necessary in order to treat the coupling of Regge poles with pairs of equal mass particles, as in this case a pseudothreshold coincides with the point $W=0$. We obtain a unified treatment of the different mass configurations, in which the continuous transition from the unequal to the equal mass case is exhibited.

The main defect of the set of solutions that we describe in the present paper is still the fact that the Regge trajectories belonging to a Lorentz family are displaced by integers, also for $W \neq 0$. We believe that this is an unphysical feature. In fact, it is possible ⁷⁾ to use the Bethe-Salpeter model to obtain solutions with non-parallel trajectories, which satisfy all the required properties and have also a strong dynamical justification. Similar results have been obtained using Feynman graph techniques ⁸⁾. However, it seems reasonable to assume that the solutions with parallel trajectories provide a good approximation to the physical solutions,

at least for small values of W . When compared with the Bethe-Salpeter method, our "on-shell method" has the advantage of giving more directly explicit formulae for the residues and for the high energy asymptotic behaviours.

Other group theoretical methods, which permit to build general solutions of the conspiracy problem, have been given by Bitar and Tindle ⁹⁾⁻¹¹⁾ and by Klein ¹²⁾. The properties of the families of Regge trajectories have also been studied by many authors ¹³⁾⁻³⁵⁾ by means of more direct analytic methods. One of the advantages of the techniques developed here is that they can be generalized ³⁶⁾ and used for the treatment of the kinematic details of the multi-Regge model.

We do not claim to give here the most general solution with parallel trajectories. However, many features of the behaviour of our residues near $W=0$ have been proved to be necessary consequences of analyticity and factorization ¹³⁾⁻³⁵⁾.

In Section 2 we define some suitable boosts, we describe the amplitude by means of a function defined on a subgroup of the complex Lorentz group and we study the analyticity and covariance properties of this function. In Section 3 we expand this function in terms of representations of the Lorentz group, taking into account the covariance conditions. The corresponding pole contribution (Lorentz pole) is computed explicitly in Section 4 and it is decomposed into a family of Regge pole contributions. In Section 5 we study in detail the behaviour of the Regge residues. In Section 6 we derive some properties of the asymptotic contribution of a Lorentz pole to the forward scattering amplitudes.

2. THE AMPLITUDE AS A FUNCTION ON THE LITTLE GROUP

We consider the reaction

$$(3) + (4) \rightarrow (1c) + (2c), \quad (2.1)$$

where $(1c)$ denotes the antiparticle corresponding to the particle (1).

In order to apply the method described in I, we have to define the boosts b_{iW}^p [see Eq. (I.3.16)] in a suitable way. We put for W negative

$$\begin{cases} b_{1W}^p = d_W^p v_{AW} b_{1W} u_{1W}, \\ b_{3W}^p = d_W^p v_{AW} b_{3W} u_{3W}, \end{cases} \quad (2.2)$$

where d_W^p is defined by Eq. (I.2.36) and b_{1W} , b_{3W} are defined by Eqs. (I.4.1) and (I.4.2). The element $v_{AW} \in H_-^c$ is given by

$$v_{AW} = [\mu_x(-\chi_A), \mu_x(\chi_A)], \quad (2.3)$$

where

$$\cos \chi_A = (M_3 - M_1) \Theta_A^{-1}, \quad 0 \leq \chi_A \leq \pi/2, \quad (2.4)$$

$$\Theta_A = [(M_3 - M_1)^2 - W]^{\frac{1}{2}}. \quad (2.5)$$

The elements u_{1W} , $u_{3W} \in H_+^c$ are defined by

$$u_{3W} = z_x^{-1} a_z(\gamma_A) z_x, \quad (2.6)$$

$$u_{1W} = t u_z(\pi) u_{3W} [t u_z(\pi)]^{-1}, \quad (2.7)$$

where r_x is defined by Eq. (I.2.8) and

$$\exp \gamma_A = \Theta_A (2\pi)^{-1}. \quad (2.8)$$

We shall use in the following also the element c_{AW} defined by

$$c_{AW} = d_W^\pi v_{AW} b_{AW} u_{3W}, \quad (2.9)$$

where

$$b_{AW} = a_z(\zeta_A), \quad (2.10)$$

$$\exp \zeta_A = (-W)^{\frac{1}{2}} (M_3 - M_1 + \Theta_A)^{-1}. \quad (2.11)$$

All the elements defined above can be continued analytically in the complex W plane cut along the positive part of the real axis. It can be shown, by direct computation, that the elements b_{1W}^p , b_{3W}^p and c_{AW} can be continued analytically in the complex W plane cut from $(M_3 + M_1)^2$ to $+\infty$.

In the equal mass case ($M_1 = M_3$) we have

$$\Theta_A = (-W)^{\frac{1}{2}}, \quad b_{AW} = e, \quad v_{AW} = c_{AW} = r_x,$$

$$u_{3W} = r_x^{-1} (d_W^\pi)^{-1} r_x, \quad (2.12)$$

and we obtain the boosts used in Ref. 4).

By means of the substitutions

$$1 \rightarrow 4, \quad 3 \rightarrow 2, \quad A \rightarrow B, \quad (2.13)$$

we define also the elements b_{2W}^p , b_{4W}^p and c_{BW} , which have similar properties.

From Eq. (2.2) we see that the covariance group $K_W^{(A)cp}$, defined by Eq. (I.3.21), if W is not positive or zero, can be written in the form

$$K_W^{(A)cp} = d_W^\pi v_{AW} [b_{1W}^\pi H_+^c b_{1W}^{\pi-1} \cap b_{3W}^\pi H_+^c b_{3W}^{\pi-1}] (d_W^\pi v_{AW})^{-1}. \quad (2.14)$$

It is easy to show that the group in the square bracket is simply the group K^c defined in the Appendix D of I. Therefore, we have

$$K_W^{(A)cp} = d_W^\pi v_{AW} K^c (d_W^\pi v_{AW})^{-1}. \quad (2.15)$$

This means that every element $k \in K_W^{(A)cp}$, can be written in the form

$$k = d_W^\pi v_{AW} k' (d_W^\pi v_{AW})^{-1}, \quad k' \in K^c. \quad (2.16)$$

Using the fact that the elements b_{iW} and b_{AW} commute with all the elements of K^c , after some calculation we obtain, for $k \in K_W^{(A)cp}$ and W not positive or zero

$$\begin{aligned} (t u_z(\pi))^{-1} (b_{1W}^\pi)^{-1} k b_{1W}^\pi t u_z(\pi) &= \\ = (b_{3W}^\pi)^{-1} k b_{3W}^\pi &= (c_{AW})^{-1} k c_{AW} = \\ = (u_{3W})^{-1} k' u_{3W}, \end{aligned}$$

(2.17)

where k and k' are connected by Eq. (2.16). Similar identities can be obtained performing the substitutions (2.13).

The connection between the function T^W defined by Eq. (I.3.15) and the function T^{PW} defined by Eq. (I.3.18) can be found by means of the covariance and invariance conditions (I.3.10) and (I.3.11). The result is

$$T_{m_1 m_2 m_3 m_4}^W(h) = \sum_{m'_1 m'_2 m'_3 m'_4} (-1)^{m_1 - m'_1 + m_4 - m'_4} R_{m_2 m'_2}^{j_2}(\mu_2 w).$$

$$\cdot R_{m_4 m'_4}^{j_4}(\mu_2 w) T_{m'_1 m'_2 m'_3 m'_4}^{PW}(d_w^\tau v_{Bw} h (d_w^\tau v_{Aw})^{-1}).$$

$$\cdot R_{m'_1 m_1}^{j_1}(\mu_3^{-1} w) R_{m'_3 m_3}^{j_3}(\mu_3^{-1} w), \quad h \in H_-^c.$$

(2.18)

We have used Eq. (2.17) and the identity

$$R_{m m'}^{\Pi j}(\mu) = (-1)^{m - m'} R_{-m', -m}^{\bar{\Pi} j}(\mu^{-1}).$$

(2.19)

If we define the new functions

$$\mathcal{C}_{j_B m_B j_A m_A}^W(h) = \sum_{m_1 m_2 m_3 m_4} (-1)^{j_1 - m_1 + j_4 - m_4} C(j_1, j_3, j_A; m_1, m_3, m_A).$$

$$\cdot C(j_4, j_2, j_B; m_4, m_2, m_B) T_{m_1 m_2 m_3 m_4}^W(h), \quad h \in H_-^c,$$

(2.20)

and

$$\mathcal{C}_{j_B m_B j_A m_A}^{PW}(h) = \sum_{m_1 m_2 m_3 m_4} (-1)^{j_1 - m_1 + j_4 - m_4} C(j_1, j_3, j_A; m_1, m_3, m_A).$$

$$\cdot C(j_4, j_2, j_B; m_4, m_2, m_B) T_{m_1 m_2 m_3 m_4}^{PW}(h), \quad h \in H_w^{cP},$$

(2.21)

using the well-known connection between the Clebsch-Gordan coefficients and the rotation matrices ³⁷⁾, we may write Eq. (2.18) in the form

$$\begin{aligned} \mathcal{G}_{i_B m_B i_A m_A}^W(h) &= \sum_{m'_B m'_A} R_{m_B m'_B}^{i_B}(\mu_{2W}) \cdot \\ &\cdot \mathcal{G}_{i_B m'_B i_A m'_A}^W(d_W^\tau v_{BW} h (d_W^\tau v_{AW})^{-1}) R_{m'_A m_A}^{i_A}(\mu_{3W}^{-1}). \end{aligned} \quad (2.22)$$

Using Eq. (2.17) and the identity

$$R_{m m'}^{\pi j}(t h t) = R_{m m'}^{\pi j}(h), \quad (2.23)$$

we can write the covariance conditions (I.3.20) and (I.3.22) as

$$\begin{aligned} T_{m_1 m_2 m_3 m_4}^W(h k) &= \sum_{m'_1 m'_3} T_{m'_1 m_2 m'_3 m_4}^W(h) (-1)^{m_1 - m'_1} \cdot \\ &\cdot R_{m'_1 m_1}^{\pi_1 j_1}((l_{3W}^\tau)^{-1} k l_{3W}^\tau) R_{m'_3 m_3}^{\pi_3 j_3}((l_{3W}^\tau)^{-1} k l_{3W}^\tau), \\ h &\in H_W^{c\tau}, \quad k \in K_W^{(A)c\tau}, \end{aligned} \quad (2.24)$$

$$\begin{aligned} T_{m_1 m_2 m_3 m_4}^W(k h) &= \sum_{m'_2 m'_4} (-1)^{m_4 - m'_4} R_{m_2 m'_2}^{\pi_2 j_2}((l_{2W}^\tau)^{-1} k l_{2W}^\tau) \cdot \\ &\cdot R_{m_4 m'_4}^{\pi_4 j_4}((l_{2W}^\tau)^{-1} k l_{2W}^\tau) T_{m_1 m'_2 m_3 m'_4}^W(h), \\ h &\in H_W^{c\tau}, \quad k \in K_W^{(B)c\tau}. \end{aligned} \quad (2.25)$$

If we use the function defined by Eq. (2.21), the covariance conditions take the form

$$\begin{aligned} \tau_{j_B m_B j_A m_A}^{rW}(h k) &= \\ &= \sum_{m'_A} \tau_{j_B m_B j_A m'_A}^{rW}(h) R_{m'_A m_A}^{\bar{\pi}_1 \pi_3, j_A} ((l_{3W}^r)^{-1} k l_{3W}^r), \end{aligned} \quad (2.26)$$

$$\begin{aligned} \tau_{j_B m_B j_A m_A}^{rW}(k h) &= \\ &= \sum_{m'_B} R_{m_B m'_B}^{\bar{\pi}_4 \pi_2, j_B} ((l_{2W}^r)^{-1} k l_{2W}^r) \tau_{j_B m'_B j_A m_A}^{rW}(h). \end{aligned} \quad (2.27)$$

In all the practically interesting cases, when two particles have equal masses, they belong to the same multiplet of the internal symmetry group. Then we have to take into account the symmetry properties of the M function with respect to the exchange of the variables which describe these two particles. As explained in Ref. 38), if the particles (1) and (3) belong to the same multiplet we have

$$M_{m_1 m_2 m_3 m_4}(a_1, a_2, a_3, a_4) = \xi (-1)^{2j_1} M_{m_3 m_2 m_1 m_4}(a_3, a_2, a_1, a_4), \quad (2.28)$$

where ξ is the factor which appears when we permute the internal quantum numbers and is given by ³⁸⁾

$$\begin{cases} \xi = G (-1)^I & \text{for } B = Y = 0, \\ \xi = (-1)^{I - \frac{1}{2}Y} & \text{for } B \neq 0 \text{ or } Y \neq 0. \end{cases}$$

I, G, B, Y are respectively the isotopic spin, the G parity, the baryon number and the hypercharge of the exchanged object.

From the definition (2.2), using Eq. (2.12), after some calculation, we obtain

$$\begin{cases} b_{1W}^{\pi} = u_z(\pi) b_{3W}^{\pi} u_x(-\pi), \\ b_{3W}^{\pi} = u_z(\pi) b_{1W}^{\pi} u_x(-\pi), \quad M_1 = M_3. \end{cases} \quad (2.30)$$

From the definition (I.3.18) of the function T^{PW} , and from Eq. (2.28) we obtain the following symmetry condition

$$T_{m_1, m_2, m_3, m_4}^{\pi W}(h) = \xi T_{-m_3, m_2, -m_1, m_4}^{\pi W}(h u_z(\pi)). \quad (2.31)$$

If we use the identity ³⁷⁾

$$C(j_1, j_3, j_A; m_1, m_3, m_A) = C(j_3, j_1, j_A; -m_3, -m_1, -m_A), \quad (2.32)$$

and the definition (2.21), we may write Eq. (2.31) in the following equivalent form

$$\tau_{j_B m_B j_A m_A}^{\pi W}(h) = \xi (-1)^{m_A} \tau_{j_B m_B j_A, -m_A}^{\pi W}(h u_z(\pi)). \quad (2.33)$$

In a similar way, if the particles (2) and (4) belong to the same multiplet, we obtain

$$\tau_{j_B m_B j_A m_A}^{\pi W}(h) = \xi (-1)^{m_B} \tau_{j_B, -m_B j_A m_A}^{\pi W}(u_z(-\pi) h). \quad (2.34)$$

3. USE OF THE REPRESENTATIONS OF THE LORENTZ GROUP

Following the procedure developed in I, we extend the functions defined in the preceding Section from the subgroup where they are defined to the whole complex Lorentz group $\tilde{A}^c$. Then we expand these functions in terms of the matrix elements of the irreducible representations of $\tilde{A}$ analytically continued in the complex group $\tilde{A}^c$ (see Appendix E of I). We shall use the following adaptation of the expansion formula given in Ref. 38).

$$\begin{aligned}
 & \mathcal{G}_{j_B m_B j_A m_A}^{\tau W}(a) = (8i\pi^4)^{-1} \cdot \\
 & \cdot \sum_{\sigma} \int_{C_W} \sum_{j m j' m'} \mathcal{F}_{j_B m_B j_A m_A}^{j m j' m'}(\tau, W, \sigma, \tau, \lambda) \cdot \\
 & \cdot \tilde{\mathcal{D}}_{j m j' m'}^{\sigma \tau 0 \lambda} (c_{BW}^{-1} a c_{AW}) (-\lambda^2) d\lambda + (4i\pi^4)^{-1} \cdot \\
 & \cdot \sum_{\tau} \sum_{M>0} \int_{C_W} \sum_{j m q} \sum_{j' m' q'} \mathcal{F}_{j_B m_B j_A m_A}^{j m q j' m' q'}(\tau, W, \tau, M, \lambda) \cdot \\
 & \cdot \tilde{\mathcal{D}}_{j m q j' m' q'}^{\tau M \lambda} (c_{BW}^{-1} a c_{AW}) (M^2 - \lambda^2) d\lambda .
 \end{aligned} \tag{3.1}$$

The difference between this formula and the formula given in Ref. 38) is due to the fact that, at the present stage, we have assumed no covariance condition. As in Ref. 38), we assume that $\mathcal{F}$ is a meromorphic function of λ , such that the integrand functions which appear in Eq. (3.1) are symmetric with respect to the transformation

$$\lambda \rightarrow -\lambda, \quad \tau \rightarrow (-1)^{2M} \tau = \bar{\tau} . \tag{3.2}$$

Also the integration path C_W is assumed to be symmetric for $\lambda \rightarrow -\lambda$. It goes from $\lambda = -i\infty$ to $\lambda = +i\infty$ and moves, when W varies, in such a way that it is never crossed by the poles of the functions $\mathcal{F}$.

We perform the extension of the amplitude to the whole Lorentz group in such a way that the covariance conditions (2.26) and (2.27) still hold in general for the extended function. If we require that the expression (3.1) satisfies these conditions, we obtain, using Eq. (2.17),

$$\begin{aligned} \sum_{j''m''} \tilde{D}_{j'm'j''m''}^{\sigma\tau 0\lambda}(\hat{k}) F_{j_B m_B j_A m_A}^{j m j'' m''}(r, W, \sigma, \tau, \lambda) = \\ = \sum_{m'_A} F_{j_B m_B j_A m'_A}^{j m j' m'}(r, W, \sigma, \tau, \lambda) R_{m'_A m_A}^{\bar{\pi}_1 \pi_3, j_A}(\hat{k}), \end{aligned} \quad (3.3)$$

$$\begin{aligned} \sum_{j''m''q''} \tilde{D}_{j'm'q'j''m''q''}^{\tau M \lambda}(\hat{k}) F_{j_B m_B j_A m_A}^{j m q j'' m'' q''}(r, W, \tau, M, \lambda) = \\ = \sum_{m'_A} F_{j_B m_B j_A m'_A}^{j m q j' m' q'}(r, W, \tau, M, \lambda) R_{m'_A m_A}^{\bar{\pi}_1 \pi_3, j_A}(\hat{k}), \end{aligned} \quad (3.4)$$

where

$$\hat{k} = u_{3w}^{-1} k' u_{3w}, \quad k' \in K^c, \quad (3.5)$$

and

$$\begin{aligned} \sum_{j''m''} F_{j_B m_B j_A m_A}^{j'' m'' j' m'}(r, W, \sigma, \tau, \lambda) \tilde{D}_{j''m''j m}^{\sigma\tau 0\lambda}(\hat{k}) = \\ = \sum_{m'_B} R_{m_B m'_B}^{\bar{\pi}_4 \pi_2, j_B}(\hat{k}) F_{j_B m'_B j_A m_A}^{j m j' m'}(r, W, \sigma, \tau, \lambda), \end{aligned} \quad (3.6)$$

$$\begin{aligned} & \sum_{j''m''q''} \mathcal{F}_{j_B m_B j_A m_A}^{j''m''q''j'm'q'}(\nu, W, \tau, M, \lambda) \tilde{\mathcal{D}}_{j''m''q''jmq}^{\tau M \lambda}(\hat{k}) = \\ & = \sum_{m'_B} R_{m_B m'_B}^{\pi_4 \pi_2, j_B}(\hat{k}) \mathcal{F}_{j_B m'_B j_A m_A}^{jmqj'm'q'}(\nu, W, \tau, M, \lambda), \end{aligned} \quad (3.7)$$

where

$$\hat{k} = u_{2w}^{-1} k' u_{2w}, \quad k' \in K^c. \quad (3.8)$$

From Eqs. (3.5) and (3.8), we see that $\hat{k} \in H_+^c$. Therefore, we may use the following identities, which can be obtained from the definition of the representations of $\tilde{A}$ given in Ref. 38):

$$\tilde{\mathcal{D}}_{jmj'm'}^{\sigma \tau 0 \lambda}(\hat{k}) = \delta_{jj'} R_{mm'}^{\sigma(-1)^j, j}(\hat{k}), \quad (3.9)$$

$$\tilde{\mathcal{D}}_{jm q j'm'q'}^{\tau M \lambda}(\hat{k}) = \delta_{jj'} \sum_{\pi} \varphi_{qq'}^{\pi j} R_{mm'}^{\pi j}(\hat{k}), \quad \hat{k} \in H_+^c, \quad (3.10)$$

where

$$\begin{cases} \sum_{\pi} \varphi_{qq'}^{\pi j} = \delta_{qq'}, \\ \sum_{\pi} \pi \varphi_{qq'}^{\pi j} = q^{2M} (-1)^{j+M} \delta_{q, -q'}. \end{cases} \quad (3.11)$$

If we define the new functions

$$\begin{aligned} & \varphi_{\hat{j}_B \hat{j}_B \hat{m}_B \hat{j}_A \hat{j}_A \hat{m}_A}^{j j'}(\nu, W, \sigma, \tau, \lambda) = \sum_{m m' m_A m_B} (-1)^{j_A + m_A + j_B + m_B} C(j_A, j', \hat{j}_A; m_A, m', \hat{m}_A) \cdot \\ & \cdot C(j_B, j, \hat{j}_B; m_B, m, \hat{m}_B) \mathcal{F}_{j_B, -m_B, j_A, -m_A}^{jmqj'm'}(\nu, W, \sigma, \tau, \lambda), \end{aligned} \quad (3.12)$$

$$\varphi_{\hat{j}_B \hat{j}_B \hat{m}_B \hat{j}_A \hat{j}_A \hat{m}_A}^{j q j' q'}(\tau, W, \tau, M, \lambda) = \sum_{m m' m_A m_B} (-1)^{j_A + m_A + j_B + m_B} C(j_A, j', \hat{j}_A; m_A, m', \hat{m}_A) \cdot$$

$$\cdot C(j_B, j, \hat{j}_B; m_B, m, \hat{m}_B) \varphi_{\hat{j}_B, -m_B, \hat{j}_A, -m_A}^{j m q j' m' q'}(\tau, W, \tau, M, \lambda),$$

(3.13)

using the identities (2.19), (3.9) and (3.10), we may write the conditions (3.3), (3.4), (3.6) and (3.7) in the simpler form

$$\sum_{\hat{m}'_A} R_{\hat{m}_A \hat{m}'_A}^{\pi \hat{j}_A}(\hat{k}) \varphi_{\hat{j}_B \hat{j}_B \hat{m}_B \hat{j}_A \hat{j}_A \hat{m}'_A}^{j j'}(\tau, W, \sigma, \tau, \lambda) =$$

$$= \varphi_{\hat{j}_B \hat{j}_B \hat{m}_B \hat{j}_A \hat{j}_A \hat{m}_A}^{j j'}(\tau, W, \sigma, \tau, \lambda), \quad \pi = \sigma (-1)^j \pi_1 \bar{\pi}_3,$$

(3.14)

$$\sum_{\pi \hat{m}'_A q''} \varphi_{q' q''}^{\pi j'} R_{\hat{m}_A \hat{m}'_A}^{\pi \pi_1 \bar{\pi}_3, \hat{j}_A}(\hat{k}) \varphi_{\hat{j}_B \hat{j}_B \hat{m}_B \hat{j}_A \hat{j}_A \hat{m}'_A}^{j q j' q''}(\tau, W, \tau, M, \lambda) =$$

$$= \varphi_{\hat{j}_B \hat{j}_B \hat{m}_B \hat{j}_A \hat{j}_A \hat{m}_A}^{j q j' q'}(\tau, W, \tau, M, \lambda),$$

(3.15)

$$\sum_{\hat{m}'_B} \varphi_{\hat{j}_B \hat{j}_B \hat{m}'_B \hat{j}_A \hat{j}_A \hat{m}_A}^{j j'}(\tau, W, \sigma, \tau, \lambda) R_{\hat{m}'_B \hat{m}_B}^{\pi \hat{j}_B}(\hat{k}) =$$

$$\varphi_{\hat{j}_B \hat{j}_B \hat{m}_B \hat{j}_A \hat{j}_A \hat{m}_A}^{j j'}(\tau, W, \sigma, \tau, \lambda), \quad \pi = \sigma (-1)^j \pi_4 \bar{\pi}_2,$$

(3.16)

$$\sum_{\pi \hat{m}'_B q''} \varphi_{\hat{j}_B \hat{j}_B \hat{m}'_B \hat{j}_A \hat{j}_A \hat{m}_A}^{\hat{j} q'' \hat{j} q'}(\nu, W, \tau, M, \lambda) \varphi_{q'' q}^{\pi \hat{j}} R_{\hat{m}'_B \hat{m}_B}^{\pi \pi_4 \bar{\pi}_2, \hat{j}_B}(\hat{k}) =$$

$$= \varphi_{\hat{j}_B \hat{j}_B \hat{m}_B \hat{j}_A \hat{j}_A \hat{m}_A}^{\hat{j} q \hat{j} q'}(\nu, W, \tau, M, \lambda). \quad (3.17)$$

Now we want to satisfy these conditions, at least for W not positive or zero. Then we may use the following substitutions

$$\varphi_{\hat{j}_B \hat{j}_B \hat{m}_B \hat{j}_A \hat{j}_A \hat{m}_A}^{\hat{j} \hat{j}'}(\nu, W, \sigma, \tau, \lambda) = \sum_{\hat{m}'_A \hat{m}'_B} \Theta_A^{\hat{j}_A} \Theta_B^{\hat{j}_B} R_{\hat{m}'_A \hat{m}'_A}^{\hat{j}_A}(\mu_{3W}^{-1}).$$

$$\cdot \varphi_{\hat{j}_B \hat{j}_B \hat{m}'_B \hat{j}_A \hat{j}_A \hat{m}'_A}^{\hat{j} \hat{j}'}(\nu, W, \sigma, \tau, \lambda) R_{\hat{m}'_B \hat{m}_B}^{\hat{j}_B}(\mu_{2W}), \quad (3.18)$$

$$\varphi_{\hat{j}_B \hat{j}_B \hat{m}_B \hat{j}_A \hat{j}_A \hat{m}_A}^{\hat{j} q \hat{j} q'}(\nu, W, \tau, M, \lambda) = \sum_{\hat{m}'_A \hat{m}'_B} \Theta_A^{\hat{j}_A} \Theta_B^{\hat{j}_B} R_{\hat{m}'_A \hat{m}'_A}^{\hat{j}_A}(\mu_{3W}^{-1}).$$

$$\cdot \varphi_{\hat{j}_B \hat{j}_B \hat{m}'_B \hat{j}_A \hat{j}_A \hat{m}'_A}^{\hat{j} q \hat{j} q'}(\nu, W, \tau, M, \lambda) R_{\hat{m}'_B \hat{m}_B}^{\hat{j}_B}(\mu_{2W}), \quad (3.19)$$

where Θ_A is given by Eq. (2.5) and Θ_B can be obtained by means of the substitutions (2.13). We see that the functions φ satisfy four conditions similar to Eqs. (3.14)-(3.17), but with the substitution $\hat{k} \rightarrow k' \in K^C$. Note that K^C does not depend on W .

It is sufficient to impose the covariance with respect to the real group $K = K^C \cap \tilde{A}$, which contains all the elements of the forms $s'u_z(\mu)$, and $u_z(\mu)$. If we take $k' = u_z(\mu)$, we see immediately that the functions φ vanish if $\hat{m}_A \neq 0$ or $\hat{m}_B \neq 0$.

Then we may directly drop these indices. We see also that $\hat{j}_A$ and $\hat{j}_B$ must be integral. If we take $k' = s'$ [see Eq. (I.2.23)] and note that

$$R_{00}^{\pi \hat{j}}(s') = \pi (-1)^{\hat{j}}, \quad (3.20)$$

from the conditions analogous to Eqs. (3.14) and (3.16) we have

$$\begin{aligned} \psi_{\hat{j}_B \hat{j}_B \hat{j}_A \hat{j}_A}^{\hat{j} \hat{j}'}(\nu, W, \sigma, \tau, \lambda) &= 0, \\ \text{if } \sigma &\neq \pi_1 \overline{\pi}_3 (-1)^{\hat{j} + \hat{j}_A}, \text{ or if } \sigma \neq \pi_4 \overline{\pi}_2 (-1)^{\hat{j} + \hat{j}_B}. \end{aligned} \quad (3.21)$$

If we write the condition analogous to Eq. (3.15) for $k' = s'$, and use Eqs. (3.11) and (3.20), we obtain

$$\begin{aligned} \psi_{\hat{j}_B \hat{j}_B \hat{j}_A \hat{j}_A}^{\hat{j}, q, \hat{j}', -q'}(\nu, W, \tau, M, \lambda) &= \\ &= \pi_1 \overline{\pi}_3 (q')^{2M} (-1)^{\hat{j} + \hat{j}_A - M} \psi_{\hat{j}_B \hat{j}_B \hat{j}_A \hat{j}_A}^{\hat{j} q \hat{j}' q'}(\nu, W, \tau, M, \lambda). \end{aligned} \quad (3.22)$$

In a similar way from the condition (3.17) we obtain

$$\begin{aligned} \psi_{\hat{j}_B \hat{j}_B \hat{j}_A \hat{j}_A}^{\hat{j}, -q, \hat{j}', q'}(\nu, W, \tau, M, \lambda) &= \\ &= \pi_4 \overline{\pi}_2 q^{2M} (-1)^{\hat{j} + \hat{j}_B + M} \psi_{\hat{j}_B \hat{j}_B \hat{j}_A \hat{j}_A}^{\hat{j} q \hat{j}' q'}(\nu, W, \tau, M, \lambda). \end{aligned} \quad (3.23)$$

Now we have to explain the origin of the factor $\bigoplus_A^{\hat{j}_A} \bigoplus_B^{\hat{j}_B}$ which appears in Eqs. (3.18) and (3.19). We have seen that the summation indices in these formulae can take only the value zero. From Eqs. (2.6), (2.8) and (I.2.8), we have

$$u_{3W}^{-1} = (a, a),$$

$$a = u_x(\frac{1}{2}\pi) \begin{pmatrix} (2\pi)^{\frac{1}{2}} \mathbb{H}_A^{-\frac{1}{2}} & 0 \\ 0 & (2\pi)^{-\frac{1}{2}} \mathbb{H}_A^{\frac{1}{2}} \end{pmatrix} u_x(-\frac{1}{2}\pi), \quad (3.24)$$

and therefore

$$\begin{aligned} R_{\hat{m}_A 0}^{\hat{j}_A}(u_{3W}^{-1}) &= \\ &= \sum_m R_{\hat{m}_A m}^{\hat{j}_A}(u_x(\frac{1}{2}\pi)) [2\pi \mathbb{H}_A^{-1}]^m R_{m 0}^{\hat{j}_A}(u_x(-\frac{1}{2}\pi)). \end{aligned} \quad (3.25)$$

From the property

$$R_{m 0}^{\hat{j}_A}(u_x(-\frac{1}{2}\pi)) = 0 \text{ if } (-1)^{\hat{j}_A - m} \neq 1, \quad (3.26)$$

we see that $R_{\hat{m}_A 0}^{\hat{j}_A}(u_{3W}^{-1})$ contains the following powers of $\mathbb{H}_A$

$$\mathbb{H}_A^{-\hat{j}_A}, \mathbb{H}_A^{-\hat{j}_A+2}, \dots, \mathbb{H}_A^{\hat{j}_A}. \quad (3.27)$$

We see that the factor $\mathbb{H}_A^{\hat{j}_A}$ is necessary in order to cancel the singularity at $W = (M_3 - M_1)^2$. In a similar way we can show that the factor $\mathbb{H}_B^{\hat{j}_B}$ is necessary in order to cancel the singularity of $R_{0\hat{m}_B}^{\hat{j}_B}(u_{2W})$ at the point $W = (M_4 - M_2)^2$.

From the Eqs. (3.12), (3.13), (3.18) and (3.19), we have

$$\mathcal{F}_{j_B m_B j_A m_A}^{j m j' m'}(\tau, W, \sigma, \tau, \lambda) = \sum_{\hat{j}_A \hat{m}_A \hat{j}_B \hat{m}_B} (-1)^{j_A - m_A + j_B - m_B}.$$

$$\cdot C(j_A, j, \hat{j}_A; -m_A, m', \hat{m}_A) C(j_B, j, \hat{j}_B; -m_B, m, \hat{m}_B) \Theta_A^{\hat{j}_A} \Theta_B^{\hat{j}_B}.$$

$$\cdot R_{\hat{m}_A 0}^{\hat{j}_A}(\mu_{3W}^{-1}) R_{0 \hat{m}_B}^{\hat{j}_B}(\mu_{2W}) \mathcal{Y}_{\hat{j}_B \hat{j}_B \hat{j}_A \hat{j}_A}^{j j'}(\tau, W, \sigma, \tau, \lambda), \quad (3.28)$$

$$\mathcal{F}_{j_B m_B j_A m_A}^{j m q j' m' q'}(\tau, W, \tau, M, \lambda) = \sum_{\hat{j}_A \hat{m}_A \hat{j}_B \hat{m}_B} (-1)^{j_A - m_A + j_B - m_B}.$$

$$\cdot C(j_A, j, \hat{j}_A; -m_A, m', \hat{m}_A) C(j_B, j, \hat{j}_B; -m_B, m, \hat{m}_B) \Theta_A^{\hat{j}_A} \Theta_B^{\hat{j}_B}.$$

$$\cdot R_{\hat{m}_A 0}^{\hat{j}_A}(\mu_{3W}^{-1}) R_{0 \hat{m}_B}^{\hat{j}_B}(\mu_{2W}) \mathcal{Y}_{\hat{j}_B \hat{j}_B \hat{j}_A \hat{j}_A}^{j q j' q'}(\tau, W, \tau, M, \lambda). \quad (3.29)$$

If the functions $\mathcal{Y}$ satisfy the conditions (3.21), (3.22) and (3.23), the expressions (3.28) and (3.29) take completely into account the covariance conditions for W non-positive or zero. We have also shown that the functions $\mathcal{F}$ given by Eqs. (3.28) and (3.29) are analytic functions of W if and only if the functions $\mathcal{Y}$ are analytic. It can be shown that continuity requires that the covariance conditions hold also for any W real below the thresholds.

If the particles (1) and (3) [or (4) and (2)] are identical, or, more exactly, they belong to the same multiplet of the internal symmetry group, we have to take into account the condition (2.33) [or (2.34)]. From Eq. (3.1), using Eq. (2.12), we see that Eq. (2.33) is equivalent to the conditions

$$\begin{aligned}
 & \mathcal{F}_{j_B m_B j_A m_A}^{j m j' m'}(\tau, W, \sigma, \tau, \lambda) = \\
 & = \tau \sum (-1)^{j' + m' + m_A} \mathcal{F}_{j_B m_B j_A -m_A}^{j, m, j', -m'}(\tau, W, \sigma, \tau, \lambda),
 \end{aligned}
 \tag{3.30}$$

$$\begin{aligned}
 & \mathcal{F}_{j_B m_B j_A m_A}^{j m q j' m' q'}(\tau, W, \tau, M, \lambda) = \\
 & = \tau \sum (-1)^{j' + m' + m_A + M} \mathcal{F}_{j_B m_B j_A -m_A}^{j, m, q, j', -m', q'}(\tau, W, \tau, M, \lambda).
 \end{aligned}
 \tag{3.31}$$

The condition (2.34) can be treated in a similar way.

Using the identities

$$R_{m m'}^j(\mu_{3W}^{-1}) = (-1)^{m - m'} R_{-m, -m'}^j(\mu_{3W}^{-1}),
 \tag{3.32}$$

and 37)

$$\begin{aligned}
 & C(j_1, j_2, j_3; -m_1, -m_2, -m_3) = \\
 & = (-1)^{j_1 + j_2 - j_3} C(j_1, j_2, j_3; m_1, m_2, m_3),
 \end{aligned}
 \tag{3.33}$$

from Eqs. (3.28), (3.29) we obtain the symmetry condition in the form

$$\left\{ \begin{aligned} & \varphi_{\hat{j}_B \hat{j}_B \hat{j}_A \hat{j}_A}^{j j'}(r, W, \sigma, \tau, \lambda) = \varphi_{\hat{j}_B \hat{j}_B \hat{j}_A \hat{j}_A}^{j q j' q'}(r, W, \tau, M, \lambda) = 0 \\ & \text{if } \tau \xi \neq (-1)^{\hat{j}_A + \hat{j}_A + M} \quad (\text{particles (1) and (3) identical}) \\ & \text{or if } \tau \xi \neq (-1)^{\hat{j}_B + \hat{j}_B + M} \quad (\text{particles (2) and (4) identical}). \end{aligned} \right. \quad (3.34)$$

The Eqs. (3.1), (3.21), (3.22), (3.23), (3.28), (3.29) and (3.34) describe completely our expansion, which takes into account the covariance conditions and the symmetry properties without introducing kinematic singularities. We see that in Eq. (3.28) and (3.29), the spin indices of the T function and the indices of the representation matrices are coupled by means of a procedure which formally resembles the LS coupling scheme. It is known, in fact, that the LS coupling scheme is the natural formalism ³⁹⁾⁻⁴¹⁾ for the treatment of the threshold and pseudothreshold constraints.

If W is not positive or zero, we may substitute Eq. (3.1) into Eq. (2.22). We obtain for the function $\mathcal{C}_{j_B m_B j_A m_A}^W(a)$ an expansion formula similar to Eq. (3.1), where the functions $\mathcal{F}$ have been replaced by the functions

$$\begin{aligned} \tilde{\mathcal{F}}_{j_B m_B j_A m_A}^{j m j' m'}(W, \sigma, \tau, \lambda) &= \sum_{m'_B m'_A} R_{m_B m'_B}^{j_B}(u_2 W) R_{m'_A m_A}^{j_A}(u_3^{-1} W) \cdot \\ &\cdot \sum_{m'' m'''} R_{m'' m}^{j}(u_2^{-1} W) R_{m' m'''}^{j'}(u_3 W) \mathcal{F}_{j_B m'_B j_A m'_A}^{j m'' j' m'''}(r, W, \sigma, \tau, \lambda), \end{aligned} \quad (3.35)$$

$$\begin{aligned} \tilde{\mathcal{F}}_{j_B m_B j_A m_A}^{j m q j' m' q'}(W, \tau, M, \lambda) &= \sum_{m'_B m'_A} R_{m_B m'_B}^{j_B}(\mu_{2W}) R_{m'_A m_A}^{j_A}(\mu_{3W}^{-1}) \cdot \\ &\cdot \sum_{m'' m'''} R_{m'' m}^{j}(\mu_{2W}^{-1}) R_{m' m'''}^{j'}(\mu_{3W}) \tilde{\mathcal{F}}_{j_B m'_B j_A m'_A}^{j m'' q j' m''' q'}(\tau, W, \tau, M, \lambda), \end{aligned} \quad (3.36)$$

and the argument of the representation matrix elements $\tilde{\mathcal{D}}$ is

$$b_{BW}^{-1} a b_{AW}. \quad (3.37)$$

Using Eqs. (3.28) and (3.29), we obtain after some calculation

$$\begin{aligned} \tilde{\mathcal{F}}_{j_B m_B j_A m_A}^{j m j' m'}(W, \sigma, \tau, \lambda) &= \sum_{\hat{j}_B \hat{j}_A} (-1)^{j_B - m_B + j_A - m_A} \cdot \\ &\cdot C(j_A, j', \hat{j}_A; -m_A, m', 0) C(j_B, j, \hat{j}_B; -m_B, m, 0) \cdot \\ &\cdot \left(\begin{matrix} \hat{j}_A \\ H_A \end{matrix} \right) \left(\begin{matrix} \hat{j}_B \\ H_B \end{matrix} \right) \varphi_{\hat{j}_B \hat{j}_B \hat{j}_A \hat{j}_A}^{j j'}(\tau, W, \sigma, \tau, \lambda), \end{aligned} \quad (3.38)$$

$$\begin{aligned} \tilde{\mathcal{F}}_{j_B m_B j_A m_A}^{j m q j' m' q'}(W, \tau, M, \lambda) &= \sum_{\hat{j}_B \hat{j}_A} (-1)^{j_B - m_B + j_A - m_A} \cdot \\ &\cdot C(j_A, j', \hat{j}_A; -m_A, m', 0) C(j_B, j, \hat{j}_B; -m_B, m, 0) \cdot \\ &\cdot \left(\begin{matrix} \hat{j}_A \\ H_A \end{matrix} \right) \left(\begin{matrix} \hat{j}_B \\ H_B \end{matrix} \right) \varphi_{\hat{j}_B \hat{j}_B \hat{j}_A \hat{j}_A}^{j q j' q'}(\tau, W, \tau, M, \lambda). \end{aligned} \quad (3.39)$$

These expressions vanish if we have not $m_A = m'$, $m_B = m$, so that we may drop the indices m and m' from the function $\tilde{F}$. From these equations we see also that the functions $\mathcal{G}$ cannot depend on the parameter p . Therefore, in the following we shall drop this argument.

We obtain in this way the expansion formula

$$\begin{aligned} \mathcal{G}_{j_B m_B j_A m_A}^W(a) &= (-1)^{j_A - m_A + j_B - m_B} \sum_{j j' \hat{j}_A \hat{j}_B} \Theta_A^{\hat{j}_A} \Theta_B^{\hat{j}_B} \cdot \\ &\cdot C(j_A, j', \hat{j}_A; -m_A, m_A, 0) C(j_B, j, \hat{j}_B; -m_B, m_B, 0) [(8i\pi^4)^{-1} \cdot \\ &\cdot \sum_{\sigma\tau} \int_{C_W} \mathcal{G}_{\hat{j}_B j_B \hat{j}_A j_A}^{j j'}(W, \sigma, \tau, \lambda) \tilde{D}_{j m_B j' m_A}^{\sigma\tau 0\lambda}(b_{BW}^{-1} a b_{AW}) (-\lambda^2) d\lambda + \\ &+ (4i\pi^4)^{-1} \sum_{\tau} \sum_{M>0} \left\{ \sum_{q q'} \mathcal{G}_{\hat{j}_B j_B \hat{j}_A j_A}^{j q j' q'}(W, \tau, M, \lambda) \cdot \right. \\ &\cdot \tilde{D}_{j m_B q j' m_A q'}^{\tau M \lambda}(b_{BW}^{-1} a b_{AW}) (M^2 - \lambda^2) d\lambda \left. \right\}. \end{aligned} \quad (3.40)$$

Note that in the equal mass case we have

$$\begin{cases} \Theta_A = \Theta_B = (-W)^{\frac{1}{2}}, \\ b_{AW} = b_{BW} = e, \end{cases} \quad (3.41)$$

and Eq. (3.40) gives the amplitude at $W \neq 0$ as a linear combination of amplitudes (in the square brackets), which are represented by means of the same expansion formula introduced in Ref. 38) for the forward amplitude. Therefore, our expansion formula takes the general form proposed by Delbourgo, Salam and Strathdee⁴²⁾.

For equal masses and $W=0$, only the term with $j_A = j_B = 0$, $j_A = j'$ and $j_B = j$ survives in Eq. (3.40). We obtain in this way exactly the expansion formula used in Ref. 38).

For unequal masses it is convenient to use the representation property in order to obtain an expansion formula for the function $\tau_{j_B m_B j_A m_A}^W(a)$, in which the argument of the representation matrix elements is simply a . We obtain in this way an expansion similar to Eq. (3.1), in which the functions $\mathcal{F}$ have been replaced by the functions

$$\hat{\mathcal{F}}_{j_B m_B j_A m_A}^{jj'}(W, \sigma, \tau, \lambda) = \sum_{j'' j'''} d_{m_B}^{0\lambda} j'' j' (-\zeta_B).$$

$$\cdot \tilde{\mathcal{F}}_{j_B m_B j_A m_A}^{j'' j'''}(W, \sigma, \tau, \lambda) d_{m_A}^{0\lambda} j' j''' (\zeta_A),$$

(3.42)

$$\hat{\mathcal{F}}_{j_B m_B j_A m_A}^{jqj'q'}(W, \tau, M, \lambda) = \sum_{j'' j'''} d_{m_B}^{qM, \lambda} j'' j' (-\zeta_B)$$

$$\cdot \tilde{\mathcal{F}}_{j_B m_B j_A m_A}^{j'' q j''' q'}(W, \tau, M, \lambda) d_{m_A}^{q'M, \lambda} j' j''' (\zeta_A),$$

(3.43)

where ζ_A is given by Eq. (2.11).

4. LORENTZ POLE CONTRIBUTIONS

It has been shown in Ref. 38) that from the $O_{3,1}$ expansion it is possible to derive a high energy asymptotic approximation of the amplitude. Each singularity in the complex λ plane which lies on the left of the integration path gives rise to an asymptotic term. Now we want to derive and to study the contribution of a pole in the λ plane (Lorentz pole). We assume that this pole appears only in the amplitude corresponding to well defined quantum numbers σ , τ and M . It has been shown in Ref. 38) that for each Lorentz pole at a point λ_0 there is another pole (mirror pole) at the point $-\lambda_0$, which has the same quantum numbers with the exception of the Lorentz signature τ , which is opposite if M is half integral.

For a Lorentz pole at the point $\lambda(W)$ with quantum numbers $M=0$, σ , τ , we consider the residue of the corresponding mirror pole defined by

$$\gamma_{\hat{\lambda}_B \hat{\lambda}_B}^{(B)j}(W) \gamma_{\hat{\lambda}_A \hat{\lambda}_A}^{(A)j'}(W) = \lim_{\lambda \rightarrow \lambda(W)} [(\lambda - \lambda(W)) (2\pi^3)^{-1} \gamma_{\hat{\lambda}_B \hat{\lambda}_B \hat{\lambda}_A \hat{\lambda}_A}^{jj'}(W, \sigma, \tau, -\lambda)]. \quad (4.1)$$

In a similar way for a Lorentz pole with quantum numbers $M > 0$, τ , we define the residue

$$\gamma_{\hat{\lambda}_B \hat{\lambda}_B}^{(B)jq}(W) \gamma_{\hat{\lambda}_A \hat{\lambda}_A}^{(A)jq'}(W) = \lim_{\lambda \rightarrow \lambda(W)} [(\lambda - \lambda(W)) \pi^{-3} \gamma_{\hat{\lambda}_B \hat{\lambda}_B \hat{\lambda}_A \hat{\lambda}_A}^{jqj'}(W, \tau, M, -\lambda)]. \quad (4.2)$$

We have assumed for the Lorentz residues a factorization property which, as we shall see, gives rise to the factorization property of the corresponding Regge residues.

The Lorentz residues have to satisfy the following conditions, which can be derived from Eqs. (3.21)-(3.23)

$$\begin{cases}
 \gamma_{\hat{j}_A \hat{j}_A}^{(A) j'}(W) = 0 \text{ if } \sigma \neq \pi_1 \bar{\pi}_3 (-1)^{\hat{j}_A + j'}, \\
 \gamma_{\hat{j}_B \hat{j}_B}^{(B) j}(W) = 0 \text{ if } \sigma \neq \pi_4 \bar{\pi}_2 (-1)^{\hat{j}_B + j}, \quad (M=0), \\
 \gamma_{\hat{j}_A \hat{j}_A}^{(A) j', -q'}(W) = \pi_1 \bar{\pi}_3 (-1)^{\hat{j}_A + j' - M} (q')^{2M} \gamma_{\hat{j}_A \hat{j}_A}^{(A) j' q'}(W), \\
 \gamma_{\hat{j}_B \hat{j}_B}^{(B) j, -q}(W) = \pi_4 \bar{\pi}_2 (-1)^{\hat{j}_B + j + M} q^{2M} \gamma_{\hat{j}_B \hat{j}_B}^{(B) j q}(W), \\
 (M > 0).
 \end{cases} \quad (4.3)$$

If the particles (1) and (3) are of the same kind we have also from Eq. (3.34)

$$\gamma_{\hat{j}_A \hat{j}_A}^{(A) j'}(W) = 0, \gamma_{\hat{j}_A \hat{j}_A}^{(A) j' q'}(W) = 0, \text{ if } \tau \xi \neq (-1)^{\hat{j}_A + j' - M}, \quad (4.4)$$

and, if the particles (2) and (4) are of the same kind, we have

$$\gamma_{\hat{j}_B \hat{j}_B}^{(B) j}(W) = 0, \gamma_{\hat{j}_B \hat{j}_B}^{(B) j q}(W) = 0, \text{ if } \tau \xi \neq (-1)^{\hat{j}_B + j - M}. \quad (4.5)$$

The contribution of a Lorentz pole can be obtained from Eqs. (3.1), (3.28) and (3.29) using the techniques developed in Ref. 38). For $M=0$ the result is

$$\mathcal{T}_{j_B m_B j_A m_A}^{\mu W}(a) \simeq -[\lambda(W)]^2 (-1)^{j_A - m_A + j_B - m_B} \sum_{\hat{j}_A \hat{j}_B \hat{j}} \gamma_{\hat{j}_B \hat{j}_B}^{(B)j}(W) \gamma_{\hat{j}_A \hat{j}_A}^{(A)j'}(W).$$

$$\cdot \sum_{\hat{m}_A \hat{m}_B m m'} C(j_A, j, \hat{j}_A; -m_A, m', \hat{m}_A) C(j_B, j, \hat{j}_B; -m_B, m, \hat{m}_B).$$

$$\cdot \Theta_A^{\hat{j}_A} \Theta_B^{\hat{j}_B} R_{\hat{m}_A 0}^{\hat{j}_A}(\mu_3^{-1} W) R_{0 \hat{m}_B}^{\hat{j}_B}(\mu_2 W) \tilde{\mathcal{A}}_{j m j' m'}^{\sigma, \tau, 0, -\lambda(W)}(c_{B W}^{-1} a c_{A W}),$$

(4.6)

and for $M > 0$ we obtain

$$\mathcal{T}_{j_B m_B j_A m_A}^{\mu W}(a) \simeq [M^2 - (\lambda(W))^2] (-1)^{j_A - m_A + j_B - m_B}.$$

$$\cdot \sum_{\hat{j}_A \hat{j}_B \hat{j} j' q'} \gamma_{\hat{j}_B \hat{j}_B}^{(B)j q}(W) \gamma_{\hat{j}_A \hat{j}_A}^{(A)j' q'}(W).$$

$$\cdot \sum_{\hat{m}_A \hat{m}_B m m'} C(j_A, j, \hat{j}_A; -m_A, m', \hat{m}_A) C(j_B, j, \hat{j}_B; -m_B, m, \hat{m}_B).$$

$$\cdot \Theta_A^{\hat{j}_A} \Theta_B^{\hat{j}_B} R_{\hat{m}_A 0}^{\hat{j}_A}(\mu_3^{-1} W) R_{0 \hat{m}_B}^{\hat{j}_B}(\mu_2 W) \tilde{\mathcal{A}}_{j m q j' m' q'}^{\bar{\tau}, M, -\lambda(W)}(c_{B W}^{-1} a c_{A W}).$$

(4.7)

These formulae hold for W belonging to the complex plane cut from the smaller threshold to $+\infty$. Now we have to show that the expressions (4.6) and (4.7) have good analytic properties. We may assume that the residues have no unwanted singularity and therefore we have to take into account only the singularities of the functions $\tilde{\mathcal{A}}(a)$. We have seen in the Appendix E of I, that these functions

have branch points for the values of the argument a such that $L_{tt}(a) = \pm 1$. Therefore our Lorentz pole contribution is singular for

$$\Xi = L_{tt}(c_{BW}^{-1} a c_{AW}) = \pm 1, \quad a \in H_W^{\text{cr}}. \quad (4.8)$$

We show in the Appendix that this equation can be satisfied only for values of the Mandelstam variable S smaller in modulus than a continuous function of W . As we are interested in the high energy region, we may disregard these singularities. Other singularities could be generated by a divergence of the infinite sums over j and j' . In order to avoid this possibility, we have to assume that the residues decrease sufficiently fast when these indices increase.

We shall use two different formulae for the contribution of a Lorentz pole to the function $\tau_{j_B m_B j_A m_A}^W(a)$. The first can be derived from the expansion formula (3.40) and has the form

$$\begin{aligned} \tau_{j_B m_B j_A m_A}^W(a) &\simeq -[\lambda(W)]^2 (-1)^{j_A - m_A + j_B - m_B} \cdot \\ &\cdot \sum_{\hat{j}_A \hat{j}_B \hat{j}} \gamma_{\hat{j}_B \hat{j}_B}^{(B)j}(W) \gamma_{\hat{j}_A \hat{j}_A}^{(A)j'}(W) \textcircled{H}_A^{\hat{j}_A} \textcircled{H}_B^{\hat{j}_B} \cdot \\ &\cdot C(j_A, j, \hat{j}_A; -m_A, m_A, 0) C(j_B, j, \hat{j}_B; -m_B, m_B, 0) \cdot \\ &\cdot \tilde{a}_{j m_B j' m_A}^{\sigma, \tau, 0, -\lambda(W)} (b_{BW}^{-1} a b_{AW}) \quad , \quad (M=0) \quad , \end{aligned} \quad (4.9)$$

$$\tau_{j_B m_B j_A m_A}^W(a) \simeq [M^2 - (\lambda(W))^2] (-1)^{j_A - m_A + j_B - m_B}.$$

$$\cdot \sum_{\hat{j}_A \hat{j}_B \hat{j} \hat{j}'} \sum_{q q'} \gamma_{\hat{j}_B \hat{j}_B}^{(B) j q}(W) \gamma_{\hat{j}_A \hat{j}_A}^{(A) j' q'}(W) \Theta_A^{\hat{j}_A} \Theta_B^{\hat{j}_B}.$$

$$\cdot C(j_A, j', \hat{j}_A; -m_A, m_A, 0) C(j_B, j, \hat{j}_B; -m_B, m_B, 0).$$

$$\cdot \tilde{a}_{j m_B q j' m_A q'}^{\tau, M, -\lambda(W)} (b_{B\omega}^{-1} a b_{A\omega}), \quad (M > 0).$$

(4.10)

These formulae hold for W negative (or zero in the equal mass case) and for the same energies for which the Eqs. (4.6) and (4.7) hold.

The second expression for the Lorentz pole contribution can be obtained using the expansion in which the functions (3.42) and (3.43) appear. It can be obtained from the expressions (4.9) and (4.10) by means of the substitutions

$$\tilde{a}_{j m_B j' m_A}^{\sigma, \tau, 0, -\lambda} (b_{B\omega}^{-1} a b_{A\omega}) \rightarrow \sum_{j'' j'''} d_{m_B j j''}^{0, -\lambda} (-\zeta_B).$$

$$\cdot \tilde{a}_{j'' m_B j''' m_A}^{\sigma, \tau, 0, -\lambda} (a) d_{m_A j'' j'}^{0, -\lambda} (\zeta_A),$$

(4.11)

$$\begin{aligned} \tilde{a}_{j m_B q j' m_A q'}^{\bar{\tau}, M, -\lambda} (b_{BW}^{-1} a b_{AW}) &\rightarrow \sum_{j'' j'''} d_{m_B j j''}^{q M, -\lambda} (-\zeta_B) \cdot \\ \cdot \tilde{a}_{j'' m_B q j''' m_A q'}^{\bar{\tau}, M, -\lambda} (a) d_{m_A j''' j'}^{q' M, -\lambda} (\zeta_A) . \end{aligned} \quad (4.12)$$

As these infinite series do not converge in the whole region of validity of Eqs. (4.9) and (4.10), the second expression for the Lorentz pole has a more limited range of validity and has to be used with some precaution.

As it holds for W negative in the internal points of the physical region, we may use it in order to decompose a Lorentz pole contribution into Regge pole contributions. Using the formula derived in Ref. 43), which connects the functions of the second kind for the groups $O(3,1)$ and $O(2,1)$, and following the methods developed in I and in Ref. 38), we may write the contribution of a Lorentz pole with $M=0$ as

$$\begin{aligned} \tau_{j_B m_B j_A m_A}^W(h) &\simeq \sum_{n=0}^{\infty} (2\ell_n(W)+1) \cotg(\pi \ell_n(W)) \cdot \\ \cdot \hat{\beta}_{j_B m_B}^{(B)n}(W) \hat{\beta}_{j_A m_A}^{(A)n}(W) \tilde{A}_{m_B m_A}^{\sigma, \tau_n, -\ell_n(W)-1}(h) , \end{aligned} \quad (4.13)$$

where

$$\ell_n(W) = \lambda(W) - n - 1, \quad (4.14)$$

$$\tau_n = \tau (-1)^n, \quad (4.15)$$

and

$$\hat{\beta}_{j_A m_A}^{(A) n}(\omega) = -2\pi [\lambda(\omega)]^2 (-1)^{j_A - m_A} \sum_{\hat{j}_A j'} \gamma_{\hat{j}_A j'}^{(A) j'}(\omega) \cdot$$

$$\cdot \Theta_A^{\hat{j}_A} C(j_A, j', \hat{j}_A; -m_A, m_A, 0) V_{j' m_A}^{0, \lambda(\omega), n}(-\zeta_A), \quad (4.16)$$

$$\hat{\beta}_{j_B m_B}^{(B) n}(\omega) = (-1)^{j_B - m_B} \sum_{\hat{j}_B j} \gamma_{\hat{j}_B j}^{(B) j}(\omega) \cdot$$

$$\cdot \Theta_B^{\hat{j}_B} C(j_B, j, \hat{j}_B; -m_B, m_B, 0) W_{j m_B}^{0, \lambda(\omega), n}(-\zeta_B). \quad (4.17)$$

In a similar way, for $M > 0$ we obtain from Eq. (4.10)

$$\tau_{j_B m_B j_A m_A}^{\omega}(h) \simeq \sum_{n=0}^{\infty} \sum_{\sigma} (2\ell_n(\omega) + 1) \cotg[\pi(\ell_n(\omega) - M)].$$

$$\cdot \hat{\beta}_{j_B m_B}^{(B) n \sigma}(\omega) \hat{\beta}_{j_A m_A}^{(A) n \sigma}(\omega) \tilde{A}_{m_B m_A}^{\sigma \tau^2, \bar{\tau}_n, -\ell_n(\omega) - 1}(h), \quad (4.18)$$

where

$$\begin{aligned}
 \hat{\beta}_{j_A m_A}^{(A) n \sigma}(\omega) &= \pi [M^2 - (\lambda(\omega))^2] (-1)^{j_A - m_A} \sum_{\hat{j}_A j' q'} \gamma_{\hat{j}_A j' q'}^{(A) j' q'}(\omega) \cdot \\
 &\cdot (q')^{\frac{1}{2}(1-\sigma\tau^2)} \Theta_A^{\hat{j}_A} C(j_A, j, \hat{j}_A; -m_A, m_A, 0) V_{j' m_A}^{q' M, \lambda(\omega), n}(-\zeta_A) = \\
 &= \pi [M^2 - (\lambda(\omega))^2] (-1)^{j_A - m_A} \sum_{\hat{j}_A j'} \gamma_{\hat{j}_A j'}^{(A) j', +1}(\omega) C(j_A, j, \hat{j}_A; -m_A, m_A, 0) \cdot \\
 &\cdot \Theta_A^{\hat{j}_A} \left[V_{j' m_A}^{M, \lambda(\omega), n}(-\zeta_A) + \sigma \pi_1 \bar{\pi}_3 (-1)^{\hat{j}_A + j' + M} V_{j' m_A}^{-M, \lambda(\omega), n}(-\zeta_A) \right], \\
 &\hspace{15em} (4.19)
 \end{aligned}$$

$$\begin{aligned}
 \hat{\beta}_{j_B m_B}^{(B) n \sigma}(\omega) &= (-1)^{j_B - m_B} \sum_{\hat{j}_B j q} \gamma_{\hat{j}_B j q}^{(B) j q}(\omega) \cdot \\
 &\cdot q^{\frac{1}{2}(1-\sigma\tau^2)} \Theta_B^{\hat{j}_B} C(j_B, j, \hat{j}_B; -m_B, m_B, 0) W_{j m_B}^{q M, \lambda(\omega), n}(-\zeta_B) = \\
 &= (-1)^{j_B - m_B} \sum_{\hat{j}_B j} \gamma_{\hat{j}_B j}^{(B) j, +1}(\omega) C(j_B, j, \hat{j}_B; -m_B, m_B, 0) \cdot \\
 &\cdot \Theta_B^{\hat{j}_B} \left[W_{j m_B}^{M, \lambda(\omega), n}(-\zeta_B) + \sigma \pi_4 \bar{\pi}_2 (-1)^{\hat{j}_B + j - M} W_{j m_B}^{-M, \lambda(\omega), n}(-\zeta_B) \right]. \\
 &\hspace{15em} (4.20)
 \end{aligned}$$

The functions $w_{jm}^{M\lambda n}(\zeta)$ and $v_{jm}^{M\lambda n}(\zeta)$ are defined by Eq. (I.5.20) and described in the Appendix F of I. We have used Eq. (4.3) in order to perform the sum over q and q' .

In conclusion, we have shown that a Lorentz pole, also in this general case, can be decomposed into a family of Regge poles with factorized residues and well defined quantum numbers. The quantum numbers are the same as those obtained in Ref. 38) in a much more special case.

5. BEHAVIOUR OF THE REGGE RESIDUES AT THE POINTS $W=0$ AND $\textcircled{H}_A=0$

In this Section we use Eqs. (4.16) and (4.19), in order to find the leading powers of W and $\textcircled{H}_A$ contained in the residues. We determine in this way the $W=0$ and pseudothreshold behaviours. We treat only the residues connected with the vertex (A). Perfectly analogous results can be derived for the residues connected with the vertex (B), using the relation between the functions $V_{jm}^{M\lambda n}(\zeta)$ and $W_{jm}^{M\lambda n}(\zeta)$ derived in the Appendix F of I.

First of all, we have to take into account that the conservation of parity gives rise to some symmetry properties of the Regge residues. With the notations we are using, we have ³⁸⁾

$$\hat{\beta}_{j_A m_A}^{(A) n \sigma}(W) = \sigma \bar{\pi}_1 \pi_3 (-1)^{j_A + m_A} \hat{\beta}_{j_A, -m_A}^{(A) n \sigma}(W). \quad (5.1)$$

It follows

$$\hat{\beta}_{j_A 0}^{(A) n \sigma}(W) \equiv 0 \quad \text{if } \sigma \neq \bar{\pi}_1 \pi_3 (-1)^{j_A}. \quad (5.2)$$

These conditions are automatically taken into account by our formalism. It can also be shown that the residues given by our formulae vanish identically only when this is required by the selection rules of the Regge pole theory.

i) Behaviour at $W=0$

If $M_1 \neq M_3$, using Eqs. (I.F.18) and (I.F.21), we easily obtain in agreement with Eq. (I.5.23), the following behaviour for the residues which do not vanish identically

$$\hat{\beta}_{j_A m_A}^{(A) n \sigma}(W) \underset{W \rightarrow 0}{\sim} (-W)^{\frac{1}{2}(-\lambda(W)+1+|M-m_A|)} \quad (5.3)$$

If $M_1 = M_3$, the behaviour of the Regge residues, which do not vanish identically, is given by

$$\hat{\beta}_{j_A m_A}^{(A) n \sigma}(W) \underset{W \rightarrow 0}{\sim} (-W)^{\frac{1}{2} d_A}, \quad (5.4)$$

where d_A is the minimum value of $\hat{j}_A$ which appears in the sums (4.16) and (4.19). We recall that for equal mass particles $\zeta_A \equiv 0$. Using the properties of the Lorentz residues, of the Clebsch-Gordan coefficients and of the functions $V_{jm}^M \lambda^n$, it is easy to show that a term in these sums does not vanish only if the following conditions are satisfied

$$\left\{ \begin{array}{l} \sigma \bar{\pi}_1 \pi_3 (-1)^{\hat{j}_A + j'} = 1 \quad \text{if } M = 0, \\ |j_A - j'| \leq \hat{j}_A \leq j_A + j', \\ (-1)^{2\hat{j}_A} = (-1)^{2(j_A - j')} = 1, \\ |m_A| \leq j', \quad j' \geq M, \\ (-1)^{j_A + j' - \hat{j}_A} = 1 \quad \text{if } m_A = 0, \end{array} \right. \quad (5.5)$$

$$\left\{ \begin{array}{l} (-1)^{j' - m_A + n} = 1 \quad \text{if } M = 0, \\ (-1)^{j' + M + n} = 1 \quad \text{if } m_A = 0, \\ \sigma \bar{\pi}_1 \pi_3 (-1)^{M - m_A + \hat{j}_A + n} \neq -1 \quad \text{if } M > 0. \end{array} \right. \quad (5.6)$$

The last set of equations holds only in the equal mass case. As it must be, the condition (5.2) can be derived from Eqs. (5.5) and (5.6). From a detailed analysis of these conditions, we obtain the Table I, which gives the values of the parameter d_A as a function of the various quantum numbers which identify the Regge residues ⁴⁴⁾.

If the particles (1) and (3) are of the same kind, we have to take into account Eq. (4.4), i.e., to add to the conditions (5.5) and (5.6) the new condition

$$\tau \xi (-1)^{\hat{j}_A + j_A - M} = 1. \quad (5.7)$$

Using Eqs. (5.5) and (5.6), we can write Eq. (5.7) in the form

$$\tau \sigma \xi (-1)^{j_A - m_A + n} = \tau_n \sigma \xi (-1)^{j_A - m_A} = 1, \quad (5.8)$$

where τ_n is the signature of the Regge pole. From the analysis of Ref. 38) we see that Eq. (5.8) is simply the selection rule of the Regge formalism due to the symmetry of the amplitude with respect to the exchange of identical particles. In conclusion, Table I holds also in this case, but a larger number of residues vanish identically, due to the selection rule (5.8). We may also use this equation to simplify Table I and we obtain in this way Table II.

The results described above are in general agreement with the results obtained by many authors by means of non-group theoretical methods ¹³⁾⁻³⁵⁾.

ii) Behaviour at a non-vanishing pseudothreshold

From Eq. (2.11) we find that at pseudothreshold

$$\exp(\zeta_A) \underset{\Theta_A \rightarrow 0}{\sim} \exp(-i \frac{1}{2} \pi) , \quad (\Im m W > 0) ,$$

$$1 + \exp(2 \zeta_A) \underset{\Theta_A \rightarrow 0}{\sim} 2 \Theta_A (M_3 - M_1)^{-1} . \quad (5.9)$$

Therefore, in order to determine the behaviour of the Regge residues given by Eqs. (4.16) and (4.19), we have to study the behaviour of the functions $V_{jm}^{M\lambda n}(-\zeta_A)$ in the limit specified by Eq. (5.9).

From Eqs. (I.F.17) and (I.F.21), we find after some calculations

$$V_{jm}^{M\lambda n}(-\zeta_A) \underset{\Theta_A \rightarrow 0}{\sim} B_{jm}^{M\lambda n} i^{m-M} [2 \Theta_A (M_3 - M_1)^{-1}]^{\ell_n - j} . \quad (5.10)$$

The coefficients $B_{jm}^{M\lambda n}$ are given by

$$B_{jm}^{M\lambda n} = \exp[i \frac{1}{2} \pi (\lambda - 1 - 2j)] (-1)^{n+1} (2j+1)! (\Delta_{Mm}^j)^{-1} \cdot \Gamma(-\lambda-j) \Gamma(j-\ell_m) \Gamma(-2\ell_m-1) [\Gamma(n-2\lambda+1) \Gamma(M-\ell_m) \Gamma(-M-\ell_m)]^{-1} , \quad (5.11)$$

where ℓ_n is given by Eq. (4.14). Note that this expression does not depend on the signs of M and m .

From Eqs. (4.16), (4.19) and (5.10), we see that the singularity of the Regge residues at the pseudothreshold depends on the greatest value of the difference $j' - j_A$, where the indices j' and j_A have to satisfy the conditions specified by Eq. (5.5) and

$$\sigma \pi_1 \bar{\pi}_3 i^{2(j_A - j')} = 1 \quad \text{if } M > 0. \quad (5.12)$$

After some calculations, we find that the behaviour of not identically vanishing Regge residues at the pseudothreshold is given by

$$\begin{cases} \hat{\beta}_{j_A m_A}^{(A) n \sigma}(W) \sim \textcircled{H}_A^{l_n(W) - j_A} & \text{if } \sigma \pi_1 \bar{\pi}_3 i^{-2j_A} = 1, \\ \hat{\beta}_{j_A m_A}^{(A) n \sigma}(W) \sim \textcircled{H}_A^{l_n(W) - j_A + 1} & \text{if } \sigma \pi_1 \bar{\pi}_3 i^{-2j_A} = -1. \end{cases} \quad (5.13)$$

These expressions are in agreement with the results of Refs. 39), 40). It is interesting to note that this pseudothreshold behaviour depends only on the Regge pole quantum numbers and neither on the Lorentz pole quantum numbers nor on the parameter n . This is in accord with the idea that the Lorentz quantum numbers describe the behaviour of the Regge poles only in a neighbourhood of $W = 0$.

It is possible to prove that the singularities at $W = 0$ and at pseudothreshold are completely described by the "kinematic factors" given by Eqs. (5.3), (5.4) and (5.13). In other words, the residues are given by the product of these factors and of functions which are analytic in the W plane cut from the threshold to $+\infty$.

In order to facilitate the comparison of our results with the results of other works, we give here the connection with the t channel helicity formalism ⁵⁾. The analytic continuation of the helicity amplitude ⁴⁵⁾ for the W channel reaction

$$(1) + (3) \rightarrow (2c) + (4c) \quad (5.14)$$

is given by

$$f_{\lambda_1 \lambda_2 \lambda_3 \lambda_4}^W(W, \theta_W) = (-1)^{\lambda_2 + \lambda_3 - j_1 + j_4} T_{-\lambda_1, \lambda_2, \lambda_3, -\lambda_4}^W(a_x(\zeta)), \quad W < 0, \quad (5.15)$$

where θ_W is the W channel scattering angle and $\cosh \zeta$ is the analytic continuation of $-\cos \theta_W$. The Eq. (5.15) can be derived comparing the boosts used in the definition of the left and the right hand sides and using the covariance relations (I.3.10) and (I.3.11) ⁴⁶⁾.

From Eqs. (2.20), (4.13) and (4.18), using the asymptotic formula ³⁸⁾

$$\begin{aligned} \tilde{A}_{mm'}^{\sigma, \tau, -\ell-1}(a_x(\zeta)) &\simeq 2^{-\ell} (-1)^{m-m'} \Gamma(2\ell+1) \cdot \\ &\cdot [\Gamma(\ell-m+1) \Gamma(\ell+m+1)]^{-1} (\cosh \zeta)^\ell, \end{aligned} \quad (5.16)$$

we see that the asymptotic contribution of a Regge pole has the form

$$\begin{aligned} f_{\lambda_1 \lambda_2 \lambda_3 \lambda_4}^W(W, \theta_W) &\simeq (-1)^{2j_3} 2^{-\ell} \cot g[\pi(\ell-\varepsilon)] \cdot \\ &\cdot (2\ell+1) \Gamma(2\ell+1) [\Gamma(\ell-m_B+1) \Gamma(\ell+m_B+1)]^{-1} \cdot \\ &\cdot \left[\sum_{j_A} C(j_1, j_3, j_A; -\lambda_1, \lambda_3, m_A) \hat{\beta}_{j_A m_A}^{(A)}(W) \right] \cdot \\ &\cdot \left[\sum_{j_B} C(j_4, j_2, j_B; -\lambda_4, \lambda_2, m_B) \hat{\beta}_{j_B m_B}^{(B)}(W) \right] (-\cos \theta_W)^\ell, \end{aligned} \quad (5.17)$$

where

$$\begin{cases} m_A = \lambda_3 - \lambda_1, \\ m_B = \lambda_2 - \lambda_4, \end{cases}$$

(5.18)

and $\xi = 0$ for boson poles, $\xi = \frac{1}{2}$ for fermion poles. Comparing Eq. (5.17) with the analogous formula obtained from any other Regge pole formalism, we can obtain, in a straightforward way, the connections between the residue functions.

6. LORENTZ POLE CONTRIBUTIONS TO THE HIGH ENERGY FORWARD SCATTERING

As it has been pointed out by Sawyer ⁴⁷⁾, one of the most interesting features of the Lorentz pole models is the high energy behaviour of the forward S channel helicity amplitudes. We consider the S channel reaction (2.1) and we always assume that particles with equal masses belong to the same multiplet. It is sometimes convenient to use, instead of the helicity amplitude, the following auxiliary function

$$\begin{aligned} \hat{f}_{j_B m_B j_A m_A}^S(S, \theta_S) = \sum_{\lambda_1 \lambda_2 \lambda_3 \lambda_4} C(j_1, j_3, j_A; -\lambda_1, \lambda_3, m_A) \cdot \\ \cdot C(j_4, j_2, j_B; -\lambda_4, \lambda_2, m_B) (-1)^{j_1 + \lambda_1 + j_4 + \lambda_4} f_{\lambda_1 \lambda_2 \lambda_3 \lambda_4}^S(S, \theta_S). \end{aligned} \quad (6.1)$$

We have to treat separately three different mass configurations.

i) UU case: $M_3 > M_1, M_2 > M_4$

The connection of the forward helicity amplitude with our function T^W is given by ⁴⁶⁾:

$$\begin{aligned} f_{\lambda_1 \lambda_2 \lambda_3 \lambda_4}^S(S, 0) = \\ = i^{2(j_2 + j_3 + \lambda_1 - \lambda_2 + \lambda_3 - \lambda_4)} T_{\lambda_1, -\lambda_2, -\lambda_3, \lambda_4}^W(\mu_\gamma(\pi)), \end{aligned} \quad (6.2)$$

or, in terms of the functions (6.1) and (2.20)

$$\begin{aligned} \hat{f}_{j_B m_B j_A m_A}^S(S, 0) = \\ = i^{2(j_A + j_B + m_A - m_B - j_1 - j_4)} \zeta_{j_B, -m_B, j_A, -m_A}^W(u_Y(\pi)). \end{aligned} \quad (6.3)$$

For $\Theta_S = 0$, the variable W depends on S and we have

$$W \underset{S \rightarrow \infty}{\simeq} (M_2^2 - M_4^2)(M_3^2 - M_1^2) S^{-1}. \quad (6.4)$$

From Eqs. (6.3), (4.3), (4.9) and (4.10), we see after some calculation that the contribution of a Lorentz pole to the function (6.1) vanishes unless the following "selection rules" are satisfied:

$$\left\{ \begin{aligned} m_A + m_B &= 0, \\ \pi_1 \bar{\pi}_2 \bar{\pi}_3 \pi_4 (-1)^{j_A + j_B} &= 1 \text{ if } m_A = m_B = 0, \\ \sigma = \pi_1 \bar{\pi}_3 (-1)^{j_A} &= \pi_4 \bar{\pi}_2 (-1)^{j_B} \text{ if } M \equiv m_A = m_B = 0. \end{aligned} \right. \quad (6.5)$$

Using the formulae 38)

$$\left\{ \begin{aligned} \tilde{a}_{j m j' m'}^{\sigma \tau 0 \lambda}(a_z(z)) &\underset{z \rightarrow \infty}{\sim} \delta_{m m'} \exp[z(-\lambda - 1 - |m|)], \\ \tilde{a}_{j m q j' m' q'}^{\tau M \lambda}(a_z(z)) &\underset{z \rightarrow \infty}{\sim} \delta_{m m'} \delta_{q q'} \exp[z(-\lambda - 1 - |m - qM|)], \end{aligned} \right. \quad (6.6)$$

and observing that

$$\exp[-\zeta_{BW} - \zeta_{AW}] \underset{W \rightarrow 0}{\sim} W^{-1} \sim S, \quad (6.7)$$

we see also that the amplitudes, which do not vanish according to the selection rules given above, have the following high energy behaviour

$$\hat{f}_{j_A m_A j_B m_B}^S(S, 0) \underset{S \rightarrow \infty}{\sim} S^{\lambda(0) - 1 - |M - |m_A||}. \quad (6.8)$$

ii) UE case: $M_3 = M_1, M_2 > M_4$

In this case Eqs. (6.2), (6.3) and (6.4) are no more valid and have to be replaced by ⁴⁶⁾

$$\begin{aligned} \hat{f}_{\lambda_1 \lambda_2 \lambda_3 \lambda_4}^S(S, 0) &= \\ &= i^{2(j_2 - j_1 - \lambda_2 - \lambda_4)} T_{-\lambda_1, -\lambda_2, \lambda_3, \lambda_4}^W(e), \end{aligned} \quad (6.9)$$

$$\begin{aligned} \hat{f}_{j_B m_B j_A m_A}^S(S, 0) &= \\ &= i^{2(m_B - j_1 - j_4 - j_B)} \tau_{j_B, -m_B, j_A, m_A}^W(e), \end{aligned} \quad (6.10)$$

$$W \underset{S \rightarrow \infty}{\simeq} -(M_4^2 - M_2^2)^2 M_1^2 S^{-2}. \quad (6.11)$$

The selection rules are still given by Eq. (6.5), but the high energy behaviour is more complicated, due to the fact that

$$\Theta_A \sim S^{-1}. \quad (6.12)$$

Observing that in this case

$$\zeta_{AW} = 0, \quad \exp(-\zeta_{BW}) \sim W^{-\frac{1}{2}} \sim S, \quad (6.13)$$

we see that the asymptotic behaviour is

$$\hat{f}_{j_B m_B j_A m_A}^S(S, 0) \underset{S \rightarrow \infty}{\sim} S^{\lambda(0) - 1 - |M - |m_A|| - d_A}, \quad (6.14)$$

where d_A is the minimum value of j_A which can appear in Eqs. (4.9) and (4.10). It is given by Table II.

iii) EE Case: $M_3 = M_1, M_2 = M_4$

This is the case treated in Ref. 38). We have

$$\hat{f}_{\lambda_1 \lambda_2 \lambda_3 \lambda_4}^S(S, 0) = i^{2(\lambda_2 - \lambda_4 - j_1 - j_2)} T_{-\lambda_1, -\lambda_2, \lambda_3, \lambda_4}^0(a_2(\zeta)), \quad (6.15)$$

$$\hat{f}_{j_B m_B j_A m_A}^S(S, 0) = i^{2(m_B - j_1 - j_4 - j_B)} \tau_{j_B, -m_B, j_A, m_A}^0(a_2(\zeta)), \quad (6.16)$$

where

$$\cosh \zeta = (S - M_1^2 - M_2^2) (2M_1 M_2)^{-1}. \quad (6.17)$$

The selection rules are ³⁸⁾

$$\left\{ \begin{array}{l} m_A + m_B = 0, \\ j_A \gg M, \quad j_B \gg M, \\ \sigma = \pi_1 \bar{\pi}_3 (-1)^{j_A} = \pi_4 \bar{\pi}_2 (-1)^{j_B} \text{ if } M = 0, \\ \pi_1 \bar{\pi}_2 \bar{\pi}_3 \pi_4 (-1)^{j_A + j_B} = 1 \text{ if } m_A = m_B = 0, \\ \tau \xi = (-1)^{j_A - M} = (-1)^{j_B - M}, \end{array} \right. \quad (6.18)$$

and the behaviour of the non-vanishing amplitudes is given just by Eq. (6.8). We see that Eq. (6.14) holds in any case if we assume that α_A vanishes in the cases UU and EE.

Similar results can be obtained if the inequalities between the masses are reversed and for the U channel reactions ⁴⁶⁾.

A P P E N D I X

POSITION OF THE SINGULARITIES OF THE REPRESENTATION MATRIX ELEMENTS

From the definition

$$S = (P^{(1)} + P^{(2)})^2 \quad (A.1)$$

we have

$$\begin{aligned} (S - M_1^2 - M_2^2)(2M_1 M_2)^{-1} &= -L_{tt}((b_{2w}^\pi)^{-1} h b_{1w}^\pi) = \\ &= -L_{tt}(b_{2w}^{-1} h' b_{1w}) = -\sinh \zeta_{2w} \sinh \zeta_{1w} + \\ &+ \cosh \zeta_{2w} \cosh \zeta_{1w} L_{tt}(h') \quad , \quad h \in H_w^{\pi c} , \end{aligned} \quad (A.2)$$

where

$$h' = (d_w^\pi v_{Bw})^{-1} h d_w^\pi v_{Aw} \in H_-^c . \quad (A.3)$$

From Eq. (4.8) we also have

$$\begin{aligned} \Xi &= L_{tt}(b_{Bw}^{-1} h' b_{Aw}) = \\ &= -\sinh \zeta_{Bw} \sinh \zeta_{Aw} + \cosh \zeta_{Bw} \cosh \zeta_{Aw} L_{tt}(h') , \end{aligned} \quad (A.4)$$

and therefore from Eq. (A.2)

$$(S - M_1^2 - M_2^2)(2M_1 M_2)^{-1} = X \Xi + Y , \quad (A.5)$$

where

$$\begin{aligned} X &= (\cosh \zeta_{2W} \cosh \zeta_{1W}) (\cosh \zeta_{AW} \cosh \zeta_{BW})^{-1} = \\ &= (4 M_1 M_2)^{-1} [(M_3 + M_1)^2 - W]^{\frac{1}{2}} [(M_2 + M_4)^2 - W]^{\frac{1}{2}}, \end{aligned}$$

(A.6)

and

$$\begin{aligned} Y &= - \sinh \zeta_{2W} \sinh \zeta_{1W} + X \sinh \zeta_{AW} \sinh \zeta_{BW} = \\ &= (4 M_1 M_2 W)^{-1} \left\{ (M_1^2 - M_3^2 + W)(M_4^2 - M_2^2 - W) - \right. \\ &\quad \left. - (M_3 - M_1)(M_4 - M_2) [(M_3 + M_1)^2 - W]^{\frac{1}{2}} [(M_2 + M_4)^2 - W]^{\frac{1}{2}} \right\}. \end{aligned}$$

(A.7)

For $\Xi = \pm 1$, we have

$$|S| \leq M_1^2 + M_2^2 + 2 M_1 M_2 (|X| + |Y|). \quad (\text{A.8})$$

As X and Y are continuous functions of W in the complex W plane cut from the thresholds to $+\infty$, the required result is proved.

TABLE I

$j_A \gg M$	$\sigma \bar{\pi}_1 \pi_3 \cdot$ $\cdot (-1)^{M-m_A+n} \neq$ $\neq -1$	$M > 0$		$\alpha_A = 0$
		$M = 0$	$(-1)^{j_A-m_A+n} = 1$	$\alpha_A = 0$
			$(-1)^{j_A-m_A+n} = -1$	$\alpha_A = 2$
	$\sigma \bar{\pi}_1 \pi_3 (-1)^{M-m_A+n} = -1$	$\alpha_A = 1$		
$j_A < M$	$\sigma \bar{\pi}_1 \pi_3 (-1)^{j_A-m_A+n} \neq -1$			$\alpha_A = M-j_A$
	$\sigma \bar{\pi}_1 \pi_3 (-1)^{j_A-m_A+n} = -1$			$\alpha_A = M-j_A+1$

Values of the exponent α_A , which appears in Eq. (5.4), expressed as a function of the parameters which identify the Regge residue.

TABLE II

$j_A \gg M$	$\tau \xi (-1)^{j_A - M} = 1$	$M > 0$		$d_A = 0$
		$M = 0$	$\sigma (-1)^{j_A} = 1$	$d_A = 0$
			$\sigma (-1)^{j_A} = -1$	$d_A = 2$
	$\tau \xi (-1)^{j_A - M} = -1$			$d_A = 1$
$j_A < M$	$\tau \xi = 1$			$d_A = M - j_A$
	$\tau \xi = -1$			$d_A = M - j_A + 1$

Values of the exponent d_A given in simplified form when the particles (1) and (3) are of the same kind. Under the same conditions, this Table gives also the exponent d_A which appears in Eq. (6.14).

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